

# 1 Higher Order Besov Spaces

Given a function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ , for every  $h \in \mathbb{R}$ ,  $m \in \mathbb{N}$ ,  $i = 1, \dots, N$ , and  $x \in \mathbb{R}^N$ , we define inductively

$$\Delta_i^{h,1} u(x) := \Delta_i^h u(x)$$

for  $m = 1$ , where  $\Delta_i^h u(x)$  has been defined in (14.1), and

$$\Delta_i^{h,m} u(x) := \Delta_i^h \left( \Delta_i^{h,m-1} u(x) \right)$$

for  $m \geq 2$ . When  $N = 1$  we write  $\Delta_i^{h,m} u(x) := \Delta_1^{h,m} u(x)$ .

**Remark 1** Given a function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ , for  $i = 1, \dots, N$  and  $h \in \mathbb{R}$  consider the translation operator  $T_i^h$  defined by

$$T_i^h(u)(x) := u(x + he_i), \quad x \in \mathbb{R}^N.$$

Note that  $T_i^h \circ T_i^s = T_i^s \circ T_i^h$  for every  $h, s \in \mathbb{R}$ . Moreover,  $\Delta_i^h := T_i^h - I$  and for  $m \in \mathbb{N}$ , with  $m \geq 2$

$$\Delta_i^{h,m} = (T_i^h - I)^m := \underbrace{(T_i^h - I) \circ \dots \circ (T_i^h - I)}_{m \text{ times}}.$$

Here  $I$  is the identity operator. Hence, by the binomial theorem,

$$\Delta_i^{h,m} u(x) = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} u(x + jhe_i), \quad x \in \mathbb{R}^N. \quad (1)$$

In particular, setting  $y := x + mhe_i$  and  $k = m - j$ , we have

$$\begin{aligned} \Delta_i^{h,m} u(x) &= \sum_{k=0}^m (-1)^k \binom{m}{m-k} u(y - khe_i) \\ &= (-1)^m \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} u(y - khe_i) = (-1)^m \Delta_i^{-h,m} u(x + mhe_i). \end{aligned} \quad (2)$$

**Definition 2** Let  $1 \leq p, \theta \leq \infty$ ,  $s > 0$ ,  $m := [s] + 1$ , where  $[\cdot]$  is the integer part. A function  $u \in L_{\text{loc}}^1(\mathbb{R}^N)$  belongs to the Besov space  $B^{s,p,\theta}(\mathbb{R}^N)$  if

$$\|u\|_{B^{s,p,\theta}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + |u|_{B^{s,p,\theta}(\mathbb{R}^N)} < \infty,$$

where

$$|u|_{B^{s,p,\theta}(\mathbb{R}^N)} := \sum_{i=1}^N \left( \int_0^\infty \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}}$$

if  $\theta < \infty$ , and

$$|u|_{B^{s,p,\infty}(\mathbb{R}^N)} := \sum_{i=1}^N \sup_{h>0} \frac{1}{h^s} \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}$$

if  $\theta = \infty$ .

If  $\theta = p$ , we write

$$B^{s,p}(\mathbb{R}^N) := B^{s,p,p}(\mathbb{R}^N).$$

Observe that if  $0 < s < 1$ , then  $m = 1$ , and so  $\Delta_i^{h,1}u = \Delta_i^h u$ , which is consistent with Definition 14.1.

**Remark 3** Note that if  $N = 1$ , the space  $B^{1,\infty}(\mathbb{R})$  coincides with the Zygmund space  $\Lambda_1(\mathbb{R})$  introduced in Chapter 1 (see Definition 1.16).

**Remark 4** If  $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ , by a change of variables, we have that

$$\left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)} = \left\| \tilde{\Delta}_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)},$$

where

$$\tilde{\Delta}_i^{h,2} u(x) := u(x + he_i) - 2u(x) + u(x - he_i)$$

for  $h \in \mathbb{R}$ ,  $i = 1, \dots, N$ , and  $x \in \mathbb{R}^N$ .

**Exercise 5** Prove that the conclusions of Propositions 14.3 and 14.5 continue to hold for  $s \geq 1$ .

Next we show that in the definition of  $B^{s,p,\theta}$ , one can replace  $\Delta_i^{h,m}$  with  $\Delta_i^{h,n}$  for any  $n \in \mathbb{N}$  with  $n > m$ .

In what follows, given a function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  and  $h > 0$ , we set

$$f_h(x) := \frac{1}{h^N} f\left(\frac{x}{h}\right), \quad x \in \mathbb{R}^N. \quad (3)$$

**Proposition 6** Let  $1 \leq p \leq \infty$ ,  $1 \leq \theta \leq \infty$ ,  $s > 0$ . Then for every  $n \in \mathbb{N}$  with  $n \geq m := \lfloor s \rfloor + 1$ , the seminorm  $|\cdot|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(n)}$ , defined by

$$|u|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(n)} := \sum_{i=1}^N \left( \int_0^\infty \left\| \Delta_i^{h,n} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} \quad (4)$$

if  $\theta < \infty$ , and

$$|u|_{B^{s,p,\infty}(\mathbb{R}^N)}^{(n)} := \sum_{i=1}^N \sup_{h>0} \frac{1}{h^s} \left\| \Delta_i^{h,n} u \right\|_{L^p(\mathbb{R}^N)}$$

if  $\theta = \infty$ , is an equivalent seminorm in  $B^{s,p,\theta}(\mathbb{R}^N)$ .

**Proof.** It is enough to prove that the seminorms  $|\cdot|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(n)}$  and  $|\cdot|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(n+1)}$  are equivalent for some  $n \geq m := \lfloor s \rfloor + 1$ . Using the fact that

$$\left\| \Delta_i^{h,1} u \right\|_{L^p(\mathbb{R}^N)} \leq 2 \|u\|_{L^p(\mathbb{R}^N)}, \quad (5)$$

we have that

$$\left\| \Delta_i^{h,n+1} u \right\|_{L^p(\mathbb{R}^N)} = \left\| \Delta_i^h \left( \Delta_i^{h,n} u \right) \right\|_{L^p(\mathbb{R}^N)} \leq 2 \left\| \Delta_i^{h,n} u \right\|_{L^p(\mathbb{R}^N)}.$$

Thus, it remains to prove the opposite inequality.

**Step 1:** We consider first the case  $0 < s < 1$  and  $n = 1$ . Fix  $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ . For  $x \in \mathbb{R}^N$ ,  $h > 0$ , and  $i = 1, \dots, N$ , we have

$$\begin{aligned} 2(u(x + he_i) - u(x)) &= (u(x + 2he_i) - u(x)) \\ &\quad - (u(x + 2he_i) - 2u(x + he_i) + u(x)), \end{aligned}$$

and so

$$2\Delta_i^{h,1} u(x) = \Delta_i^{2h,1} u(x) - \Delta_i^{h,2} u(x). \quad (6)$$

If  $\theta < \infty$ , it follows by Minkowski's inequality that

$$\begin{aligned} 2 \left( \int_0^\infty \left\| \Delta_i^{h,1} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} &\leq \left( \int_0^\infty \left\| \Delta_i^{2h,1} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} \\ + \left( \int_0^\infty \left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} &= 2^s \left( \int_0^\infty \left\| \Delta_i^{\eta,1} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{d\eta}{\eta^{1+s\theta}} \right)^{\frac{1}{\theta}} \\ &\quad + \left( \int_0^\infty \left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}}, \end{aligned}$$

where we have made the change of variables  $h = \frac{\eta}{2}$ . The case  $\theta = \infty$  is similar. Hence,

$$(2 - 2^s) |u|_{B^{s,p,\theta}(\mathbb{R}^N)} \leq |u|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(2)}.$$

**Step 2:** Assume next that  $n \geq 2$ . We use the following identity for polynomials

$$\begin{aligned} (x-1)^n &= \frac{1}{2^n} (x^2-1)^n + (x-1)^n - \frac{1}{2^n} (x^2-1)^n \\ &= \frac{1}{2^n} (x^2-1)^n + P_{n-1}(x) (x-1)^{n+1}, \quad x \in \mathbb{R}, \end{aligned}$$

where  $P_{n-1}$  is the polynomial

$$\begin{aligned} P_{n-1}(x) &= -\frac{1}{2^n} \frac{(x+1)^n - 2^n}{x-1} = -\frac{1}{2^n} \frac{((x-1)+2)^n - 2^n}{x-1} \\ &= -\frac{1}{2^n} \sum_{j=1}^n 2^{n-j} \binom{n}{j} (x-1)^{j-1}, \quad x \in \mathbb{R}. \end{aligned}$$

By replacing  $x$  with  $T_i^h$ , we get

$$\Delta_i^{h,n} u = \frac{1}{2^n} \Delta_i^{2h,n} u + P_{n-1}(T_i^h) \Delta_i^{h,n+1} u.$$

Hence,

$$\begin{aligned}
\left\| \Delta_i^{h,n} u \right\|_{L^p(\mathbb{R}^N)} &\leq \frac{1}{2^n} \left\| \Delta_i^{2h,n} u \right\|_{L^p(\mathbb{R}^N)} + \left\| P_{n-1} (T_i^h) \Delta_i^{h,n+1} u \right\|_{L^p(\mathbb{R}^N)} \\
&\leq \frac{1}{2^n} \left\| \Delta_i^{2h,n} u \right\|_{L^p(\mathbb{R}^N)} + \frac{1}{2^n} \sum_{j=1}^n 2^{n-j} \binom{n}{j} \left\| \Delta_i^{h,j-1} \left( \Delta_i^{h,n+1} u \right) \right\|_{L^p(\mathbb{R}^N)} \\
&\leq \frac{1}{2^n} \left\| \Delta_i^{2h,n} u \right\|_{L^p(\mathbb{R}^N)} + \frac{2^n - 1}{2} \left\| \Delta_i^{h,n+1} u \right\|_{L^p(\mathbb{R}^N)},
\end{aligned}$$

where we have used  $j - 1$  times the inequality (5). If  $\theta < \infty$ , it follows by Minkowski's inequality that

$$\begin{aligned}
\left( \int_0^\infty \left\| \Delta_i^{h,n} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} &\leq \frac{1}{2^n} \left( \int_0^\infty \left\| \Delta_i^{2h,n} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} \\
&\quad + \frac{2^n - 1}{2} \left( \int_0^\infty \left\| \Delta_i^{h,n+1} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} \\
&= \frac{2^s}{2^n} \left( \int_0^\infty \left\| \Delta_i^{\eta,n} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{d\eta}{\eta^{1+s\theta}} \right)^{\frac{1}{\theta}} \\
&\quad + \frac{2^n - 1}{2} \left( \int_0^\infty \left\| \Delta_i^{h,n+1} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}},
\end{aligned}$$

where we have made the change of variables  $h = \frac{\eta}{2}$ . Therefore, we get

$$\begin{aligned}
\left( 1 - \frac{1}{2^{n-s}} \right) \left( \int_0^\infty \left\| \Delta_i^{h,n} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} \\
\leq \frac{2^n - 1}{2} \left( \int_0^\infty \left\| \Delta_i^{h,n+1} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}}.
\end{aligned}$$

Note that here it is important that  $n \geq m := \lfloor s \rfloor + 1 > s$ . The case  $\theta = \infty$  is similar. ■

## 2 Dependence of $B^{s,p,\theta}$ on $\theta$

We study the relation between different Besov spaces  $B^{s,p,\theta}(\mathbb{R}^N)$  as  $\theta$  varies.

**Theorem 7** *Let  $1 \leq p \leq \infty$ ,  $s > 0$ , and  $1 \leq \theta_1 < \theta_2 \leq \infty$ . Then there exists a constant  $C = C(N, p, \theta_1, \theta_2) > 0$  such that*

$$|u|_{B^{s,p,\theta_2}(\mathbb{R}^N)} \leq C |u|_{B^{s,p,\theta_1}(\mathbb{R}^N)}$$

for all  $u \in L_{\text{loc}}^1(\mathbb{R}^N)$ . In particular,  $B^{s,p,\theta_1}(\mathbb{R}^N) \subset B^{s,p,\theta_2}(\mathbb{R}^N)$ .

We begin with some auxiliary results. Consider a nonnegative even function  $\omega \in C_c^\infty(\mathbb{R})$  such that  $\text{supp } \omega \subset \left[-\frac{1}{m+1}, \frac{1}{m+1}\right]$  and  $\int_{\mathbb{R}} \omega(t) dt = 1$ . Given  $m \in \mathbb{N}$ , define

$$\varpi(t) := \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} \frac{1}{j} \omega\left(\frac{t}{j}\right), \quad t \in \mathbb{R}. \quad (7)$$

Note that  $\varpi \in C_c^\infty(\mathbb{R})$ ,  $\text{supp } \varpi \subset \left[-\frac{m}{m+1}, \frac{m}{m+1}\right]$ , and

$$\begin{aligned} \int_{\mathbb{R}} \varpi(t) dt &= \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} \int_{\mathbb{R}} \omega\left(\frac{t}{j}\right) \frac{dt}{j} \\ &= \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} = (-1)^{m+1}. \end{aligned} \quad (8)$$

Let

$$\psi(t) := (\varpi * \varpi)(t) = \int_{\mathbb{R}} \varpi(t-\tau) \varpi(\tau) d\tau, \quad t \in \mathbb{R}. \quad (9)$$

Then  $\psi$  is even,  $\psi \in C_c^\infty(\mathbb{R})$ , and by Fubini's theorem,

$$\int_{\mathbb{R}} \psi(t) dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \varpi(t-\tau) \varpi(\tau) d\tau dt = (-1)^{m+1} (-1)^{m+1} = 1. \quad (10)$$

**Lemma 8** *Let  $\omega$  be as above, let  $\varpi$  is the function defined in (7), and let  $u \in L^1_{\text{loc}}(\mathbb{R})$ . Then for  $h > 0$  and for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$ ,*

$$\int_{\mathbb{R}} \psi_h(y) \Delta^y u(t) dy = \int_{\mathbb{R}} (\omega * \varpi)_h(y) \Delta^{y,m} u(t) dy,$$

while for every  $n \in \mathbb{N}$ ,

$$\int_{\mathbb{R}} \frac{d^{(n)} \psi_h}{dy^n}(y) u(t+y) dy = \int_{\mathbb{R}} \frac{d^{(n)} (\omega * \varpi)_h}{dy^n}(y) \Delta^{y,m} u(t) dy.$$

**Proof.** By (9) and Fubini's theorem,

$$\int_{\mathbb{R}} \psi_h(y) \Delta^y u(t) dy = \int_{\mathbb{R}} \frac{1}{h} \varpi(\tau) \int_{\mathbb{R}} \varpi\left(\frac{y}{h} - \tau\right) \Delta^y u(t) dy d\tau.$$

By (1) and (8),

$$\begin{aligned}
& \int_{\mathbb{R}} \varpi \left( \frac{y}{h} - \tau \right) \Delta^y u(t) dy \\
&= \int_{\mathbb{R}} \varpi \left( \frac{y}{h} - \tau \right) u(t+y) dy - u(t) \int_{\mathbb{R}} \varpi \left( \frac{y}{h} - \tau \right) dy \\
&= \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} \frac{1}{j} \int_{\mathbb{R}} \omega \left( \frac{y}{jh} - \tau \right) u(t+y) dy - u(t) (-1)^{m+1} \frac{1}{h} \\
&= \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right) u(t+j\zeta) d\zeta + (-1)^m u(t) \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right) d\zeta \\
&= \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right) u(t+j\zeta) d\zeta = \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right) \Delta^{\zeta, m} u(t) d\zeta
\end{aligned}$$

and so, again by Fubini's theorem

$$\int_{\mathbb{R}} \psi_h(y) \Delta^y u(t) dy = \frac{1}{h} \int_{\mathbb{R}} (\omega * \varpi) \left( \frac{\zeta}{h} \right) \Delta^{\zeta, m} u(t) d\zeta.$$

To prove the second identity, let  $n \in \mathbb{N}$ . By Theorem C.20 and (9), we have that

$$\frac{d^n \psi_h}{dy^n}(y) = \left( \varpi * \frac{d^n \varpi_h}{dy^n} \right)(y) = \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}(y-\xi) \varpi(\xi) d\xi.$$

By the fact that  $\int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}(y-\xi) dy = 0$ , Fubini's theorem, and a change of variables, we get

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{d^{(n)} \psi_h}{dy^n}(y) u(t+y) dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}(y-\xi) \varpi(\xi) u(t+y) dy d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}(y-\xi) \varpi(\xi) [u(t+y) - u(t)] dy d\xi \\
&= \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}(\eta) \int_{\mathbb{R}} \varpi(y-\eta) \Delta^y u(t) dy d\eta.
\end{aligned}$$

Reasoning as in first part of the proof, we have that

$$\int_{\mathbb{R}} \varpi(y-\eta) \Delta^y u(t) dy = \int_{\mathbb{R}} \omega(y-\eta) \Delta^{y, m} u(t) dy,$$

and so,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{d^{(n)} \psi_h}{dy^n}(y) u(t+y) dy &= \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}(\eta) \int_{\mathbb{R}} \omega(y-\eta) \Delta^{y, m} u(t) dy d\eta \\
&= \int_{\mathbb{R}} \frac{d^{(n)} (\omega * \varpi)_h}{dy^n}(y) \Delta^{y, m} u(t) dy.
\end{aligned}$$

■

**Exercise 9** Let  $g \in C^m(\mathbb{R})$ ,  $m \in \mathbb{N}$ .

(i) Prove that for every  $y \in \mathbb{R}$  and  $h > 0$ ,

$$\Delta^{h,m}g(y) = \int_0^h \cdots \int_0^h g^{(m)}(y + t_1 + \cdots + t_m) dt_1 \cdots dt_m.$$

(ii) Prove that for every  $y \in \mathbb{R}$  and  $h > 0$ ,

$$|\Delta^{h,m}g(y)| \leq \int_0^{mh} \tau^{m-1} |g^{(m)}(y + \tau)| d\tau.$$

**Lemma 10** There exists a constant  $C > 0$  such that for every  $u \in L^p(\mathbb{R})$ ,  $1 \leq p \leq \infty$ ,  $m \in \mathbb{N}$ , and for all  $h > 0$ ,

$$\|\Delta^{h,m}u\|_{L^p(\mathbb{R})} \leq \frac{C}{h} \int_0^h \|\Delta^{\eta,m}u\|_{L^p(\mathbb{R})} d\eta.$$

**Proof.** Consider a nonnegative function  $\psi \in C_c^\infty(\mathbb{R})$  such that  $\text{supp } \psi \subset [0, 1]$  and  $\int_{\mathbb{R}} \psi(x) dx = 1$ . For  $h > 0$ , let  $u_h := u * \psi_h$ ,  $x \in \mathbb{R}$  (see (3)). By Minkowski's inequality and Remark 14.2, we have

$$\begin{aligned} \|\Delta^{h,m}u\|_{L^p(\mathbb{R})} &\leq \|\Delta^{h,m}u_h\|_{L^p(\mathbb{R})} + \|\Delta^{h,m}(u_h - u)\|_{L^p(\mathbb{R})} \\ &\leq \|\Delta^{h,m}u_h\|_{L^p(\mathbb{R})} + 2^m \|u_h - u\|_{L^p(\mathbb{R})}. \end{aligned}$$

Hence, to conclude the proof, it remains to show that

$$\|u_h - u\|_{L^p(\mathbb{R})} \leq \frac{C}{h} \int_0^h \|\Delta^{\eta,m}u\|_{L^p(\mathbb{R})} d\eta, \quad (11)$$

$$\|\Delta^{h,m}u_h\|_{L^p(\mathbb{R})} \leq \frac{C}{h} \int_0^h \|\Delta^{\eta,m}u\|_{L^p(\mathbb{R})} d\eta. \quad (12)$$

**Step 1:** Assume first that  $0 < s < 1$ . Since  $\int_{\mathbb{R}} \psi_h(x) dx = 1$ , we have

$$u_h(x) - u(x) = \int_{\mathbb{R}} \psi_h(\eta) (u(x + \eta) - u(x)) d\eta.$$

for  $x \in \mathbb{R}$ . Hence, by Corollary B.83 and the fact that  $\text{supp } \psi_h \subset [0, h]$ .

$$\begin{aligned} \|u_h - u\|_{L^p(\mathbb{R})} &\leq \int_{\mathbb{R}} \psi_h(\eta) \|\Delta^\eta u\|_{L^p(\mathbb{R})} d\eta \\ &\leq \frac{C}{h} \int_0^h \|\Delta^\eta u\|_{L^p(\mathbb{R})} d\eta. \end{aligned}$$

To prove (11), fix  $x \in \mathbb{R}$  and  $\xi > 0$  and define  $g(y) := u_h(x + y\xi)$ ,  $y \in \mathbb{R}$ . Then

$$\Delta^\xi u_h(x) = g(1) - g(0) = \int_0^1 g'(y) dy = \xi \int_0^1 u'_h(x + y\xi) dy. \quad (13)$$

Since  $\int_{\mathbb{R}} \psi'_h(\eta) d\eta = 0$ , we have

$$u'_h(x) = (u * \psi'_h)(x) = \int_{\mathbb{R}} \psi'_h(\eta) (u(x+\eta) - u(x)) d\eta,$$

and so, again by Corollary B.83, Remark and the facts that  $\text{supp } \psi_h \subset [0, h]$  and that  $\|\psi'_h\|_{\infty} \leq \frac{C}{h^2}$ ,

$$\begin{aligned} \|u'_h(\cdot + y\xi)\|_{L^p(\mathbb{R})} &= \|u'_h\|_{L^p(\mathbb{R})} \leq 2 \int_0^{\infty} |\psi'_h(\eta)| \|\Delta^{\eta} u\|_{L^p(\mathbb{R})} d\eta \\ &\leq \frac{C}{h^2} \int_0^h \|\Delta^{\eta} u\|_{L^p(\mathbb{R})} d\eta, \end{aligned}$$

which, together with (13), gives

$$\|\Delta^{\xi} u_h\|_{L^p(\mathbb{R})} \leq \frac{C\xi}{h^2} \int_0^h \|\Delta^{\eta} u\|_{L^p(\mathbb{R})} d\eta,$$

which yields (11).

**Step 2:** Next assume that  $s = 1$ . In this case we assume that the function  $\psi$  is even with  $\text{supp } \psi \subset [-1, 1]$ . Since  $\int_{\mathbb{R}} \psi_h(x) dx = 1$  and  $\psi_h$  is even, we have

$$u_h(x) - u(x) = \int_0^{\infty} \psi_h(\eta) (u(x+\eta) - 2u(x) + u(x-\eta)) d\eta$$

for  $x \in \mathbb{R}$ . Hence, by Corollary B.83, Remark 4, and the fact that  $\text{supp } \psi_h \subset [-h, h]$ ,

$$\|u_h - u\|_{L^p(\mathbb{R})} \leq \int_0^{\infty} \psi_h(\eta) \|\Delta^{\eta, 2} u\|_{L^p(\mathbb{R})} d\eta \leq \frac{C}{h} \int_0^h \|\Delta^{\eta, 2} u\|_{L^p(\mathbb{R})} d\eta, \quad (14)$$

so that (12) holds.

To prove (11), fix  $x \in \mathbb{R}$  and  $\xi > 0$ . By Exercise 9,

$$\Delta^{\xi, 2} u_h(x) = \xi^2 \int_0^1 \int_0^1 u''_h(x + (y_1 + y_2)\xi) dy_1 dy_2. \quad (15)$$

Since  $\int_{\mathbb{R}} \psi''_h(\eta) d\eta = 0$  and  $\psi''_h$  is even, we have

$$u''_h(x) = (u * \psi''_h)(x) = \int_0^{\infty} \psi''_h(\eta) (u(x+\eta) - 2u(x) + u(x-\eta)) d\eta,$$

and so, again by Corollary B.83, Remark 4, and the facts that  $\text{supp } \psi_h \subset [0, h]$  and that  $\|\psi''_h\|_{\infty} \leq \frac{C}{h^3}$ ,

$$\begin{aligned} \|u''_h(\cdot + (y_1 + y_2)\xi)\|_{L^p(\mathbb{R})} &= \|u''_h\|_{L^p(\mathbb{R})} \leq \int_0^{\infty} |\psi''_h(\eta)| \|\Delta^{\eta, 2} u\|_{L^p(\mathbb{R})} d\eta \\ &\leq \frac{C}{h^3} \int_0^h \|\Delta^{\eta, 2} u\|_{L^p(\mathbb{R})} d\eta, \end{aligned}$$

which, together with (15), gives

$$\|\Delta^{\xi,2}u_h\|_{L^p(\mathbb{R})} \leq \frac{C\xi^2}{h^3} \int_0^h \|\Delta^{\eta,2}u\|_{L^p(\mathbb{R})} d\eta,$$

which concludes the proof in this case.

**Step 3:** Finally, in the general case  $s > 0$ , we take  $\psi$  to be the function defined in (9). Using the fact that  $\int_{\mathbb{R}} \psi_h(x) dx = 1$  and the previous lemma, we have

$$\begin{aligned} u_h(x) - u(x) &= \int_{\mathbb{R}} \psi_h(\eta) \Delta^\eta u(x) d\eta \\ &= \int_{\mathbb{R}} (\omega * \varpi)_h(\eta) \Delta^{\eta,m} u(x) d\eta. \end{aligned}$$

Hence, by Corollary B.83, the facts that  $\text{supp } \omega \subset \left[0, \frac{1}{m+1}\right]$  and  $\text{supp } \varpi \subset \left[0, \frac{m}{m+1}\right]$ , and a change of variables,

$$\begin{aligned} \|u_h - u\|_{L^p(\mathbb{R})} &\leq \frac{1}{h} \int_{\mathbb{R}} (\omega * \varpi)\left(\frac{\eta}{h}\right) \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} d\eta \\ &\leq \frac{C}{h} \int_0^h \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} d\eta, \end{aligned}$$

which shows (12).

To prove (11), fix  $x \in \mathbb{R}$  and  $\xi > 0$ . By Exercise 9,

$$\Delta^{\xi,m} u_h(x) = \xi^m \int_0^1 \cdots \int_0^1 u_h^{(m)}(x + (y_1 + \cdots + y_m)\xi) dy_1 \cdots dy_m. \quad (16)$$

By Theorem C.20, the previous lemma, and the fact that  $\psi$  is even, we have

$$\begin{aligned} u_h^{(m)}(x) &= \left(u * \psi_h^{(m)}\right)(x) = \int_{\mathbb{R}} \psi_h^{(m)}(-\eta) u(x + \eta) d\eta \\ &= (-1)^m \int_{\mathbb{R}} (\omega * \varpi)_h^{(m)}(\eta) \Delta^{\eta,m} u(x) d\eta, \end{aligned}$$

and so, again by Corollary B.83, Remark and the facts that  $\text{supp } \omega \subset \left[-\frac{1}{m+1}, \frac{1}{m+1}\right]$ ,  $\text{supp } \varpi \subset \left[-\frac{m}{m+1}, \frac{m}{m+1}\right]$ , and  $\left\|(\omega * \varpi)_h^{(m)}\right\|_{\infty} \leq \frac{C}{h^{m+1}}$ ,

$$\begin{aligned} \left\|u_h^{(m)}(\cdot + (y_1 + \cdots + y_m)\xi)\right\|_{L^p(\mathbb{R})} &= \left\|u_h^{(m)}\right\|_{L^p(\mathbb{R})} \\ &\leq \xi^m \int_{\mathbb{R}} \left|(\omega * \varpi)_h^{(m)}(\eta)\right| \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} d\eta \\ &\leq \frac{C\xi^m}{h^{m+1}} \int_{-h}^h \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} d\eta \leq \frac{C\xi^m}{h^{m+1}} \int_0^h \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} d\eta, \end{aligned}$$

where in the last inequality we have used (2) and a change of variables. Together with (16), this gives

$$\|\Delta^{\xi,m} u_h\|_{L^p(\mathbb{R})} \leq \frac{C\xi^m}{h^{m+1}} \int_0^h \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} d\eta,$$

which yields (11). ■

**Remark 11** *The previous proof is due to Jonsson and Wallin [2] in the case  $0 < s \leq 1$ . It provides an alternative proof of Lemma 14.18.*

**Proof of Theorem 7.** The proof is the same of Theorem 14.17, with the only difference that we use the previous lemma in place of Lemma 14.18. ■

### 3 Dependence of $B^{s,p,\theta}$ on $s$

In this section we prove that for  $0 < t < s$ ,

$$B^{s,p,\theta_1}(\mathbb{R}^N) \subset B^{t,p,\theta_2}(\mathbb{R}^N).$$

**Theorem 12** *Let  $1 \leq p, \theta_1, \theta_2 \leq \infty$  and  $0 < t < s$ . Then there exists a constant  $C = C(p, \theta, t, s) > 0$  such that*

$$|u|_{B^{t,p,\theta_2}(\mathbb{R}^N)} \leq C |u|_{B^{s,p,\theta_1}(\mathbb{R}^N)} + C \|u\|_{L^p(\mathbb{R}^N)}$$

for all  $u \in B^{s,p,\theta_1}(\mathbb{R}^N)$ . In particular,  $B^{s,p,\theta_1}(\mathbb{R}^N) \subset B^{t,p,\theta_2}(\mathbb{R}^N)$ .

**Proof.** Let  $u \in B^{s,p,\theta_1}(\mathbb{R}^N)$ . If  $\theta_1 < \theta_2$ , then by Theorem 7,

$$|u|_{B^{t,p,\theta_2}(\mathbb{R}^N)} \leq C |u|_{B^{t,p,\theta_1}(\mathbb{R}^N)}.$$

Thus, it remains to show that

$$|u|_{B^{t,p,\min\{\theta_1,\theta_2\}}(\mathbb{R}^N)} \leq C |u|_{B^{s,p,\theta_1}(\mathbb{R}^N)} + C \|u\|_{L^p(\mathbb{R}^N)}.$$

Let  $\theta = \min\{\theta_1, \theta_2\}$  and  $m = \lfloor s \rfloor + 1$ . Reasoning as in the proof of Theorem 14.7 (see (14.9)), we have that

$$\begin{aligned} |u|_{B^{t,p,\theta}(\mathbb{R}^N)}^{(m)} &\leq \left( \int_0^1 \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+t\theta}} \right)^{\frac{1}{\theta}} \\ &\quad + C \|u\|_{L^p(\mathbb{R}^N)} \left( \int_1^\infty \frac{1}{h^{1+t\theta}} dh \right)^{\frac{1}{\theta}} =: I + II \end{aligned} \tag{17}$$

if  $1 \leq \theta < \infty$ , while

$$|u|_{B^{t,p,\infty}(\mathbb{R}^N)}^{(m)} \leq \sup_{0 < h < 1} \frac{1}{h^s} \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)} + C \|u\|_{L^p(\mathbb{R}^N)}$$

if  $\theta = \infty$ . It remains to estimate  $I$  in (17). If  $\theta < \theta_1 < \infty$ , by Hölder's inequality

$$\begin{aligned} I &= \left( \int_0^1 \frac{h^{(s-t)\theta}}{h^{(s-t)\theta}} \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+t\theta}} \right)^{\frac{1}{\theta}} \\ &\leq \left( \int_0^1 \left( h^{(s-t)\theta} \right)^{\left(\frac{\theta_1}{\theta}\right)'} \frac{dh}{h} \right)^{\frac{1}{\left(\frac{\theta_1}{\theta}\right)'}} \left( \int_0^1 \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}^{\theta_1} \frac{dh}{h^{1+t\theta_1}} \right)^{\frac{1}{\theta_1}}. \end{aligned}$$

Note that the first integral on the the right-hand side is finite. If  $\theta < \theta_1 = \infty$ , then

$$I \leq \left( \sup_{0 < \eta < 1} \frac{1}{\eta^s} \left\| \Delta_i^{\eta,m} u \right\|_{L^p(\mathbb{R}^N)} \right) \left( \int_0^1 h^{s\theta} \frac{dh}{h^{1+t\theta}} \right)^{\frac{1}{\theta}},$$

which is again finite. Finally, if  $\theta = \theta_1 < \infty$ , then as in the proof of Theorem 14.7 (see (14.9)) we use the fact that  $\frac{1}{h^t} < \frac{1}{h^s}$  for  $0 < h < 1$ . The result now follows, in view of Proposition 6. This concludes the proof. ■

## 4 A Characterization of $B^{s,p,\theta}$ for $0 < s < 1$

In this section we give a characterization of  $B^{s,p,\theta}(\mathbb{R}^N)$  for  $0 < s < 1$ . We will show that a function  $v \in L^p(\mathbb{R}^N)$  belongs to  $B^{s,p,\theta}(\mathbb{R}^N)$  if and only if it is the trace of a function  $v : \mathbb{R}_+^{N+1} \rightarrow \mathbb{R}$  in some weighted Sobolev space. In the special case  $\theta = p$  and  $s = 1 - \frac{1}{p}$ , we will recover Theorem ....

In what follows we set  $M := N + 1$ , and we write

$$z = (z_1, \dots, z_{M-1}, z_M) = (z', z_M) \in \mathbb{R}^N \times \mathbb{R}.$$

With a slight abuse of notation, for every  $i = 1, \dots, N$  we also write

$$z' = (z''_i, z_i) \in \mathbb{R}^{M-2} \times \mathbb{R}.$$

Let  $\varphi \in C_c^\infty(\mathbb{R}^N)$  be such that  $\text{supp } \varphi \subset \overline{B_N(0,1)}$  and

$$\int_{\mathbb{R}^N} \varphi(z') dz' = 1.$$

Given  $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ , for  $z' \in \mathbb{R}^N$  and  $z_M > 0$  define

$$v(z) := \frac{1}{z_M^N} \int_{\mathbb{R}^N} \varphi\left(\frac{z' - y'}{z_M}\right) u(y') dy'. \quad (18)$$

Note that  $v$  is just the mollification of  $u$  with  $z_M$  in place of  $\varepsilon$ .

The proof of the following theorems will make use of the Hardy's inequality

$$\left( \int_a^b \left| \frac{1}{\tau^s} \int_a^\tau f(t) dt \right|^p dx \right)^{\frac{1}{p}} \leq \frac{1}{s - \frac{1}{p}} \left( \int_a^b \left| \frac{1}{\tau^{s-1}} f(\tau) \right|^p d\tau \right)^{\frac{1}{p}}, \quad (19)$$

where  $f : (a, b) \rightarrow \mathbb{R}$  is any measurable function,  $0 \leq a < b \leq \infty$ ,  $1 \leq p < \infty$  and  $s > \frac{1}{p}$ . For a proof see, e.g., Theorem 8 in the pdf file on "An extension of the Sobolev–Gagliardo–Nirenberg theorem".

**Theorem 13** Let  $1 \leq p, \theta \leq \infty$ ,  $0 < s < 1$ , and  $M \geq 2$ . Given  $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ , let  $v$  be the function given in (18). Then

$$\begin{aligned} \frac{s}{N} |u|_{B^{s,p,\theta}(\mathbb{R}^N)} &\leq \sum_{i=1}^M \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_i}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}} \\ &\leq \frac{C}{1+s} |u|_{B^{s,p,\theta}(\mathbb{R}^N)} \end{aligned}$$

if  $1 \leq \theta < \infty$ , while

$$\begin{aligned} \frac{s}{N} |u|_{B^{s,p,\infty}(\mathbb{R}^N)} &\leq \sum_{i=1}^M \sup_{z_M > 0} \left( z_M^{1-s} \left\| \frac{\partial v}{\partial z_i}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \\ &\leq \frac{C}{1+s} |u|_{B^{s,p,\infty}(\mathbb{R}^N)} \end{aligned}$$

if  $\theta = \infty$ , where  $C = C(M, p) > 0$ .

**Proof. Step 1:** By Theorem C.20 (where  $z_M$  plays the role of  $\varepsilon$ ), for any  $i = 1, \dots, N$  and  $z \in \mathbb{R}_+^M$  we have that

$$\begin{aligned} \frac{\partial v}{\partial z_i}(z) &= \frac{1}{z_M^M} \int_{\mathbb{R}^N} \frac{\partial \varphi}{\partial z_i} \left( \frac{z' - y'}{z_M} \right) u(y') dy' \\ &= \frac{1}{z_M^M} \int_{\mathbb{R}^N} \frac{\partial \varphi}{\partial z_i} \left( \frac{z' - y'}{z_M} \right) [u(y') - u(y''_i, z_i)] dy' \\ &= \frac{1}{z_M^M} \int_{\mathbb{R}^N} \frac{\partial \varphi}{\partial z_i} \left( \frac{y'}{z_M} \right) [u(z' - y') - u(z''_i - y''_i, z_i)] dy', \end{aligned}$$

where in the second equality we used the fact that

$$\int_{\mathbb{R}} \frac{\partial \varphi}{\partial z_i} \left( \frac{z' - y'}{z_M} \right) u(y''_i, z_i) dy_i = u(y''_i, z_i) \int_{\mathbb{R}} \frac{\partial \varphi}{\partial z_i} \left( \frac{z' - y'}{z_M} \right) dy_i = 0.$$

Since  $\text{supp } \varphi \subset \overline{B_N(0, 1)} \subset \overline{B_{M-2}(0, z_M)} \times [-z_M, z_M]$ , by Tonelli's theorem we have that

$$\left| \frac{\partial v}{\partial z_i}(z) \right| \leq \frac{C}{z_M^M} \int_{B_{M-2}(0, z_M)} \int_{-z_M}^{z_M} |u(z''_i - y''_i, z_i) - u(z''_i - y''_i, z_i)| dy_i dy''_i.$$

By taking the  $L^p(\mathbb{R}^N)$  norm in  $z'$  on both sides, using Corollary B.83, and changing variables, for all  $z_M > 0$ , we get

$$\begin{aligned} \left\| \frac{\partial v}{\partial z_i}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} &\leq \frac{C}{z_M^M} \int_{B_{M-2}(0, z_M)} \int_{-z_M}^{z_M} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)} dy_i dy''_i \quad (20) \\ &\leq \frac{C}{z_M^2} \int_{-z_M}^{z_M} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)} dy_i \leq \frac{C}{z_M^2} \int_0^{z_M} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)} dy_i, \end{aligned}$$

where in the last inequality we have used the fact that for  $y_i < 0$ ,

$$\Delta_i^{y_i} u(x) = -\Delta_i^{-y_i} u(x + y_i e_i)$$

by (2), together with the change of variables  $w = x + y_i e_i$ .

If  $\theta < \infty$ , multiply both sides of (20) by  $\frac{1}{z_M^a}$ , where

$$a := \frac{1}{\theta} - (1 - s),$$

and take the  $L^\theta((0, \infty))$  norm in  $z_M$  to obtain

$$\begin{aligned} & \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_i}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}} \\ & \leq C \left( \int_0^\infty \left( \frac{1}{z_M^{1+\frac{1}{\theta}+s}} \int_0^{z_M} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)} dy_i \right)^\theta dz_M \right)^{\frac{1}{\theta}}. \end{aligned}$$

By applying Hardy's inequality to the right-hand side, we get

$$\begin{aligned} & \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_i}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}} \\ & \leq \frac{C}{1+s} \left( \int_0^\infty \|\Delta_i^{z_M} u\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1+s\theta}} \right)^{\frac{1}{\theta}} \leq \frac{C}{1+s} |u|_{B^{s,p,\theta}(\mathbb{R}^N)}. \end{aligned}$$

On the other hand, if  $\theta = \infty$ , we have that

$$\begin{aligned} \int_0^{z_M} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)} dy_i &= \int_0^{z_M} \frac{y_i^s}{y_i^s} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)} dy_i \\ &\leq \left( \int_0^{z_M} y_i^s dy_i \right) \sup_{h>0} \frac{1}{h^s} \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)} \\ &= \frac{z_M^{1+s}}{1+s} \sup_{h>0} \frac{1}{h^s} \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)}, \end{aligned} \quad (21)$$

and so by (20),

$$z_M^{1-s} \left\| \frac{\partial v}{\partial z_i}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \leq \frac{C}{1+s} |u|_{B^{s,p,\infty}(\mathbb{R}^N)},$$

which implies that

$$\sup_{z_M > 0} \left( z_M^{1-s} \left\| \frac{\partial v}{\partial z_i}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \leq \frac{C}{1+s} |u|_{B^{s,p,\infty}(\mathbb{R}^N)}.$$

To estimate  $\frac{\partial v}{\partial z_M}$ , we write

$$\begin{aligned} & u(y') - u(z') \\ &= \sum_{i=1}^N [u(y_1, \dots, y_i, z_{i+1}, \dots, z_N) - u(y_1, \dots, y_{i-1}, z_i, \dots, z_N)]. \end{aligned}$$

Since  $\int_{\mathbb{R}^N} \varphi(z') dz' = 1$ , we have that

$$\begin{aligned} v(z', z_M) &= \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{1}{z_M^N} \varphi\left(\frac{z' - y'}{z_M}\right) \\ &\quad \times [u(y_1, \dots, y_i, z_{i+1}, \dots, z_N) - u(y_1, \dots, y_{i-1}, z_i, \dots, z_N)] dy' \\ &\quad + u(z'). \end{aligned} \quad (22)$$

By Theorem B.53 we obtain that

$$\begin{aligned} \frac{\partial v}{\partial z_M}(z', z_M) &= \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{\partial}{\partial z_M} \left( \frac{1}{z_M^N} \varphi\left(\frac{z' - y'}{z_M}\right) \right) \\ &\quad \times [u(y_1, \dots, y_i, z_{i+1}, \dots, z_N) - u(y_1, \dots, y_{i-1}, z_i, \dots, z_N)] dy'. \end{aligned}$$

In turn,

$$\begin{aligned} & \left| \frac{\partial v}{\partial z_M}(z', z_M) \right| \\ & \leq \sum_{i=1}^N \frac{C}{z_M^N} \int_{B_N(0, z_M)} |u(z_1 - y_1, \dots, z_i - y_i, z_{i+1}, \dots, z_N) \\ & \quad - u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N)| dy' \end{aligned}$$

We can now continue as before (see (20)) to conclude that

$$\left( \int_0^\infty \left\| \frac{\partial v}{\partial z_M}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}} \leq \frac{C}{1+s} |u|_{B^{s,p,\theta}(\mathbb{R}^N)}$$

if  $1 \leq \theta < \infty$ , while

$$\sup_{z_M > 0} \left( z_M^{1-s} \left\| \frac{\partial v}{\partial z_M}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \leq \frac{C}{1+s} |u|_{B^{s,p,\infty}(\mathbb{R}^N)}$$

if  $\theta = \infty$ .

**Step 2:** To prove the other inequality, assume that  $u \in C^1(\mathbb{R}^N)$ . For  $z' \in \mathbb{R}^N$ ,  $h > 0$  and  $i = 1, \dots, N$ , we have

$$\begin{aligned} \Delta_i^h u(z') &= u(z' + he'_i) - u(z') = [v(z' + he'_i, h) - v(z', h)] \\ &\quad + [u(z' + he'_i) - v(z' + he'_i, h)] + [v(z', h) - u(z')], \end{aligned} \quad (23)$$

where  $e_i = (e'_i, 0) \in \mathbb{R}^N \times \mathbb{R}$ . We now estimate the various terms in (23). By the fundamental theorem of calculus

$$\begin{aligned} v(z' + he'_i, h) - v(z', h) &= \int_0^h \frac{\partial v}{\partial z_i}(z''_i, z_i + t, h) dt, \\ v(z' + he'_i, h) - u(z' + he'_i) &= \int_0^h \frac{\partial v}{\partial z_M}(z''_i, z_i + h, t) dt, \\ v(z', h) - u(z') &= \int_0^h \frac{\partial v}{\partial z_M}(z', t) dt. \end{aligned}$$

Hence,

$$\begin{aligned} |\Delta_i^h u(z')| &\leq \int_0^h \left| \frac{\partial v}{\partial z_i}(z''_i, z_i + t, h) \right| dt \\ &\quad + \int_0^h \left( \left| \frac{\partial v}{\partial z_M}(z''_i, z_i + h, t) \right| + \left| \frac{\partial v}{\partial z_M}(z', t) \right| \right) dt. \end{aligned} \quad (24)$$

If  $1 \leq p < \infty$ , by Corollary B.83, and a change of variables,

$$\begin{aligned} \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)} &\leq \left( \int_{\mathbb{R}^N} \left( \int_0^h \left| \frac{\partial v}{\partial z_i}(z''_i, z_i + t, h) \right| dt \right)^p dz' \right)^{\frac{1}{p}} \\ &\quad + \int_0^h \left\| \frac{\partial v}{\partial z_M}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt =: I + II. \end{aligned}$$

To estimate  $I$ , note that if  $M > 2$ , then by Tonelli's theorem and Corollary B.83,

$$\begin{aligned} I &\leq \left( \int_{\mathbb{R}} \left( \int_0^h \left\| \frac{\partial v}{\partial z_i}(\cdot, z_i + t, h) \right\|_{L^p(\mathbb{R}^{M-2})} dt \right)^p dz_i \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbb{R}} h^{\frac{p}{p'}} \int_0^h \left\| \frac{\partial v}{\partial z_i}(\cdot, z_i + t, h) \right\|_{L^p(\mathbb{R}^{M-2})}^p dt dz_i \right)^{\frac{1}{p}} \\ &= h \left\| \frac{\partial v}{\partial z_i}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}, \end{aligned} \quad (25)$$

where in the second inequality we have used Hölder's inequality and in the last equality, Tonelli's theorem and a change of variables. Hence, we have proved that

$$\|\Delta_i^h u\|_{L^p(\mathbb{R}^N)} \leq h \left\| \frac{\partial v}{\partial z_i}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} + \int_0^h \left\| \frac{\partial v}{\partial z_M}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt. \quad (26)$$

Note that in view of (24), the same inequality holds if  $p = \infty$ .

If  $1 \leq \theta < \infty$ , multiply both sides of the previous inequality by  $\frac{1}{h^{\frac{1}{\theta}+s}}$  and then and take the  $L^\theta((0, \infty))$  norm in  $h$  to obtain,

$$\begin{aligned} & \left( \int_0^\infty \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} \leq \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_i}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}} \\ & \quad + \left( \int_0^\infty \left( \frac{1}{h^{\frac{1}{\theta}+s}} \int_0^h \left\| \frac{\partial v}{\partial z_M}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)}^\theta dt \right)^\theta dh \right)^{\frac{1}{\theta}} \\ & \leq \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_i}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}} \\ & \quad + \frac{1}{s} \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_M}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}}, \end{aligned}$$

where in the last inequality we have used Hardy's inequality. It follows that

$$\begin{aligned} |u|_{B^{s,p,\theta}(\mathbb{R}^N)} & \leq \sum_{i=1}^N \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_i}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}} + \\ & \quad + \frac{N}{s} \left( \int_0^\infty \left\| \frac{\partial v}{\partial z_M}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}} \right)^{\frac{1}{\theta}}. \end{aligned}$$

On the other hand, if  $\theta = \infty$ , multiply both sides of (26) by  $\frac{1}{h^s}$  and proceed as in (21) to obtain

$$\begin{aligned} \sup_{h>0} \frac{1}{h^s} \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)} & \leq \sup_{h>0} h^{1-s} \left\| \frac{\partial v}{\partial z_i}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} \\ & \quad + \frac{1}{s} \sup_{h>0} h^{1-s} \left\| \frac{\partial v}{\partial z_M}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

It follows that

$$\begin{aligned} |u|_{B^{s,p,\infty}(\mathbb{R}^N)} & \leq \sum_{i=1}^N \sup_{h>0} h^{1-s} \left\| \frac{\partial v}{\partial z_i}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} + \\ & \quad + \frac{N}{s} \sup_{h>0} h^{1-s} \left\| \frac{\partial v}{\partial z_M}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

To remove the extra assumption that  $u \in C^1(\mathbb{R}^N)$ , we use a mollification argument. We omit the details. This concludes the proof.  $\blacksquare$

## 5 A Characterization of $B^{1,p,\theta}$

In this section we prove the analog of Theorem 13 for  $B^{1,p,\theta}$ . In what follows, we assume that the function  $\varphi$  in the previous section is assumed to be even in

each variable  $z_i$ .

**Theorem 14** *Let  $1 \leq p, \theta \leq \infty$  and  $M \geq 2$ . Given  $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ , let  $v$  be the function given in (18), where  $\varphi$  is even in each variable  $z_i$ ,  $i = 1, \dots, N$ . Then*

$$\begin{aligned} \frac{1}{C} |u|_{B^{1,p,\theta}(\mathbb{R}^N)} &\leq \sum_{i,j=1}^M \left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta}} \right)^{\frac{1}{\theta}} \\ &\leq C |u|_{B^{1,p,\theta}(\mathbb{R}^N)} \end{aligned}$$

if  $1 \leq \theta < \infty$ , while

$$\begin{aligned} \frac{1}{C} |u|_{B^{1,p,\infty}(\mathbb{R}^N)} &\leq \sum_{i,j=1}^M \sup_{z_M > 0} \left( z_M \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \\ &\leq C |u|_{B^{1,p,\infty}(\mathbb{R}^N)} \end{aligned}$$

if  $\theta = \infty$ , where  $C = C(M, p) > 0$ .

**Proof. Step 1:** By Theorem C.20, for every  $i, j = 1, \dots, N$ , we have that

$$\begin{aligned} \frac{\partial^2 v}{\partial z_i \partial z_j}(z) &= \frac{1}{z_M^{M+1}} \int_{\mathbb{R}^N} \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \left( \frac{y'}{z_M} \right) u(z' - y') dy' \\ &= \frac{1}{z_M^{M+1}} \int_{\mathbb{R}^{M-2}} \int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \left( \frac{y'}{z_M} \right) [u(z''_i - y''_i, z_i - y_i) + u(z''_i - y''_i, z_i + y_i)] dy' dy''_i \\ &= \frac{1}{z_M^{M+1}} \int_{\mathbb{R}^{M-2}} \int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \left( \frac{y'}{z_M} \right) \tilde{\Delta}_i^{y_i, 2} u(z''_i - y''_i, z_i) dy_i dy''_i, \end{aligned} \tag{27}$$

where in the second equality we have used the fact that  $\varphi$  is even in the  $z_i$  variable and in the last equality the fact that

$$\begin{aligned} &2 \int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \left( \frac{y'}{z_M} \right) u(z''_i - y''_i, z_i) dy_i \\ &= 2u(z''_i - y''_i, z_i) \int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \left( \frac{y''_i, y_i}{z_M} \right) dy_i = 0. \end{aligned}$$

Since  $\varphi \in C_c^\infty(B(0, 1))$ , we have that

$$\left| \frac{\partial^2 v}{\partial z_i \partial z_j}(z) \right| \leq \frac{c}{z_M^{M+1}} \int_{B_{M-2}(0, z_M)} \int_0^{z_M} \left| \tilde{\Delta}_i^{y_i, 2} u(z''_i - y''_i, z_i) \right| dy_i dy''_i. \tag{28}$$

Reasoning as in (20), we get

$$\begin{aligned} \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} &\leq \frac{C}{z_M^3} \int_0^{z_M} \left\| \tilde{\Delta}_i^{y_i, 2} u \right\|_{L^p(\mathbb{R}^N)} dy_i \\ &= \frac{C}{z_M^3} \int_0^{z_M} \left\| \Delta_i^{y_i, 2} u \right\|_{L^p(\mathbb{R}^N)} dy_i. \end{aligned} \tag{29}$$

for all  $z_M > 0$ . If  $\theta < \infty$ , multiply both sides of the previous inequality by  $\frac{1}{z_M^\theta}$ , where

$$a := \frac{1}{\theta} - 1,$$

and take the  $L^\theta((0, \infty))$  norm in  $z_M$  to obtain

$$\begin{aligned} & \left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta}} \right)^{\frac{1}{\theta}} \\ & \leq C \left( \int_0^\infty \left( \frac{1}{z_M^{2+\frac{1}{\theta}}} \int_0^{z_M} \left\| \Delta_i^{y_i, 2} u \right\|_{L^p(\mathbb{R}^N)} dy_i \right)^\theta dz_M \right)^{\frac{1}{\theta}}. \end{aligned}$$

By applying Hardy's inequality to the right-hand side, we get

$$\begin{aligned} \left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta}} \right)^{\frac{1}{\theta}} & \leq \frac{C}{2} \left( \int_0^\infty \left\| \Delta_i^{z_M, 2} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1+\theta}} \right)^{\frac{1}{\theta}} \\ & \leq C |u|_{B^{s,p,\theta}(\mathbb{R}^N)}. \end{aligned}$$

On the other hand, if  $\theta = \infty$ , reasoning as in (21), we have that

$$\int_0^{z_M} \left\| \Delta_i^{y_i, 2} u \right\|_{L^p(\mathbb{R}^N)} dy_i \leq \frac{z_M^2}{2} \sup_{h>0} \frac{1}{h^s} \left\| \Delta_i^{h, 2} u \right\|_{L^p(\mathbb{R}^N)},$$

and so by (29),

$$\sup_{z_M > 0} \left( z_M \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \leq |u|_{B^{1,p,\infty}(\mathbb{R}^N)}.$$

To estimate  $\frac{\partial^2 v}{\partial z_M^2}$ , write

$$\begin{aligned} & v(z', z_M) \\ & = \frac{1}{z_M^N} \int_{\mathbb{R}^{M-2}} \int_0^\infty \varphi\left(\frac{y'}{z_M}\right) [u(z''_i - y''_i, z_i - y_i) + u(z''_i - y''_i, z_i + y_i)] dy' dy''_i \\ & = \frac{1}{z_M^N} \int_{\mathbb{R}^N} \int_0^\infty \varphi\left(\frac{y'}{z_M}\right) \tilde{\Delta}_i^{y_i, 2} u(z''_i - y''_i, z_i) dy' dy''_i + 2u(z'), \end{aligned}$$

where in the first equality we have used the fact that  $\varphi$  is even in the  $z_M$  variable and in the second the fact that  $\int_{\mathbb{R}^N} \varphi(z') dz' = 1$ . Hence, by differentiating with respect to  $z_M$  and using the fact that  $\varphi \in C_c^\infty(B(0, 1))$ , we obtain

$$\left| \frac{\partial^2 v}{\partial z_M^2}(z', z_M) \right| \leq \frac{c}{z_M^{M+1}} \sum_{i=1}^N \int_{B_{M-2}(0, z_M)} \int_0^{z_M} \left| \tilde{\Delta}_i^{y_i, 2} u(z''_i - y''_i, z_i) \right| dy'_i dy''_i.$$

We can now proceed as in (28) to prove that

$$\left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta}} \right)^{\frac{1}{\theta}} \leq C |u|_{B^{s,p,\theta}(\mathbb{R}^N)}$$

if  $\theta < \infty$ , while

$$\sup_{z_M > 0} \left( z_M \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \leq |u|_{B^{1,p,\infty}(\mathbb{R}^N)}.$$

Similar estimates hold for  $\frac{\partial^2 v}{\partial z_i \partial z_M}$ .

**Step 2:** Assume that  $u \in C^2(\mathbb{R}^N)$ . For  $z' \in \mathbb{R}^N$ ,  $h > 0$  and  $i = 1, \dots, N$ , we have

$$\begin{aligned} \Delta_i^{h,2} u(z') &= -\Delta_i^{h,2} v(z', 2h) + 2\Delta_i^{h,2} v(z', h) \\ &\quad + \Delta_M^{h,2} v(z' + 2he_i, 0) - 2\Delta_M^{h,2} v(z' + he_i, 0) + \Delta_M^{h,2} v(z', 0). \end{aligned}$$

We now estimate the various terms on the right-hand side. By Exercise 9,

$$\begin{aligned} \left| \Delta_i^{h,2} u(z') \right| &\leq \int_0^{2h} \tau \left( \left| \frac{\partial^2 v}{\partial z_i^2}(z_i'', z_i + \tau, 2h) \right| + 2 \left| \frac{\partial^2 v}{\partial z_i^2}(z_i'', z_i + \tau, h) \right| \right) d\tau \\ &\quad + \int_0^{2h} \tau \left( \left| \frac{\partial^2 v}{\partial z_M^2}(z_i'', z_i + 2h, \tau) \right| + 2 \left| \frac{\partial^2 v}{\partial z_M^2}(z_i'', z_i + h, \tau) \right| + \left| \frac{\partial^2 v}{\partial z_M^2}(z_i'', z_i, \tau) \right| \right) d\tau. \end{aligned}$$

If  $1 \leq p < \infty$ , by Corollary B.83 and a change of variables,

$$\begin{aligned} \left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)} &\leq \left( \int_{\mathbb{R}^N} \left( \int_0^{2h} \tau \left| \frac{\partial^2 v}{\partial z_i^2}(z_i'', z_i + \tau, 2h) \right| d\tau \right)^p dz' \right)^{\frac{1}{p}} \\ &\quad + 2 \left( \int_{\mathbb{R}^N} \left( \int_0^{2h} \tau \left| \frac{\partial^2 v}{\partial z_i^2}(z_i'', z_i + \tau, h) \right| d\tau \right)^p dz' \right)^{\frac{1}{p}} \\ &\quad + 4 \int_0^{2h} \tau \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} d\tau =: I + II + III. \end{aligned}$$

To estimate  $I$  and  $II$ , we reason as in (25) to obtain

$$I + II \leq 2h^2 \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, 2h) \right\|_{L^p(\mathbb{R}^N)} + 4h^2 \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}.$$

Hence,

$$\begin{aligned} \left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)} &\leq 2h^2 \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, 2h) \right\|_{L^p(\mathbb{R}^N)} + 4h^2 \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} \\ &\quad + 4 \int_0^{2h} \tau \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} d\tau. \end{aligned} \quad (30)$$

The same inequality holds if  $p = \infty$ .

If  $1 \leq \theta < \infty$ , multiply both sides of the previous inequality by  $\frac{1}{h^{\frac{1}{\theta}+1}}$ , take the  $L^\theta((0, \infty))$  norm in  $h$ , and use the change of variables  $h' = 2h$ , to obtain,

$$\begin{aligned} \left( \int_0^\infty \left\| \Delta_i^h u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+\theta}} \right)^{\frac{1}{\theta}} &\leq 5 \left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta}} \right)^{\frac{1}{\theta}} \\ &+ 4 \left( \int_0^\infty \left( \frac{1}{h^{\frac{1}{\theta}+1}} \int_0^{2h} \tau \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)}^\theta d\tau \right)^\theta dh \right)^{\frac{1}{\theta}} =: A + B. \end{aligned}$$

To estimate  $B$ , we apply Hardy's inequality and a change of variables, to obtain

$$B \leq C \left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta}} \right)^{\frac{1}{\theta}}.$$

It follows that

$$\begin{aligned} |u|_{B^{1,p,\theta}(\mathbb{R}^N)} &\leq 5 \sum_{i=1}^N \left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta}} \right)^{\frac{1}{\theta}} + \\ &+ NC \left( \int_0^\infty \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta}} \right)^{\frac{1}{\theta}}. \end{aligned}$$

if  $\theta < \infty$ , and

$$|u|_{B^{1,p,\infty}(\mathbb{R}^M)} := \sum_{i=1}^M \sup_{h>0} \frac{1}{h} \left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^M)}$$

On the other hand, if  $\theta = \infty$ , multiply both sides of (30) by  $\frac{1}{h}$  to obtain

$$\begin{aligned} \frac{1}{h} \left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)} &\leq 2h \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, 2h) \right\|_{L^p(\mathbb{R}^N)} + 4h \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} \\ &+ \frac{4}{h} \int_0^{2h} \tau \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} d\tau dt =: E + F + G. \end{aligned}$$

Since

$$\begin{aligned} G &\leq \sup_{\tau>0} \left( \tau \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} \right) \left( \frac{4}{h} \int_0^{2h} d\tau \right) \\ &= 8 \sup_{\tau>0} \left( \tau \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} \right), \end{aligned}$$

we obtain

$$\begin{aligned} \sup_{h>0} \frac{1}{h} \left\| \Delta_i^{h,2} u \right\|_{L^p(\mathbb{R}^N)} &\leq C \sup_{h>0} \left( h \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} \right) \\ &\quad + C \sup_{h>0} \left( h \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} \right). \end{aligned}$$

It follows that

$$\begin{aligned} |u|_{B^{1,p,\infty}(\mathbb{R}^N)} &\leq C \sum_{i=1}^N \sup_{h>0} \left( h \left\| \frac{\partial^2 v}{\partial z_i^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} \right) + \\ &\quad + NC \sup_{h>0} \left( h \left\| \frac{\partial^2 v}{\partial z_M^2}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} \right). \end{aligned}$$

To remove the extra assumption that  $u \in C^2(\mathbb{R}^N)$ , we use a mollification argument. We omit the details. This concludes the proof.  $\blacksquare$

## 6 A Characterization of $B^{s,p,\theta}$ for $s > 0$

In this section we extend Theorems 13 and 14 to  $B^{s,p,\theta}$  for all  $s > 0$ . Note that although Theorem 15 includes Theorems 13 and 14 as special cases, for didactical purposes we have kept  $0 < s < 1$  and  $s = 1$  separate, since the proof is more transparent in those cases. The function  $\varphi$  used in the definition (18) should now be chosen more carefully.

Define

$$\varphi(z') := \prod_{i=1}^N \psi(z_i), \quad z' \in \mathbb{R}^N, \quad (31)$$

where  $\omega$  is the function defined in (9).

**Theorem 15** *Let  $1 \leq p, \theta \leq \infty$ ,  $s > 0$ ,  $m \in \mathbb{N}$  with  $m \geq [s] + 1$ , and  $M \geq 2$ . Given  $u \in L_{\text{loc}}^1(\mathbb{R}^N)$ , let  $v$  be the function given in (18), where  $\varphi$  is given in (31). Then*

$$\begin{aligned} \frac{1}{C} |u|_{B^{s,p,\theta}(\mathbb{R}^N)} &\leq \sum_{|\alpha|=m} \left( \int_0^\infty \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\ &\leq C |u|_{B^{s,p,\theta}(\mathbb{R}^N)} \end{aligned} \quad (32)$$

if  $1 \leq \theta < \infty$ , while

$$\begin{aligned} \frac{1}{C} |u|_{B^{s,p,\infty}(\mathbb{R}^N)} &\leq \sum_{|\alpha|=m} \sup_{z_M>0} \left( z_M^{m-s} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \\ &\leq C |u|_{B^{s,p,\infty}(\mathbb{R}^N)} \end{aligned} \quad (33)$$

if  $\theta = \infty$ , where  $C = C(M, p, s) > 0$ .

We now turn to the proof of Theorem 15.

**Proof of Theorem 15.** In view of Proposition 6, it is enough to show that the inequalities (32) and (33) hold with  $|u|_{B^{s,p,\theta}(\mathbb{R}^N)}$  replaced by the seminorm  $|u|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(m)}$  defined in (4).

**Step 1:** Let  $\alpha \in \mathbb{N}_0^M$  be a multi-index of the form  $\alpha = (\alpha_1, \dots, \alpha_N, 0)$ , with  $|\alpha| = m$ , and fix  $i = 1, \dots, N$  such that  $\alpha_i > 0$ . By Theorem C.20, (31), and the fact that  $\psi$  is an even function of one variable, we have that

$$\begin{aligned} \frac{\partial^\alpha v}{\partial z^\alpha}(z) &= \frac{1}{z_M^{M+m-1}} \int_{\mathbb{R}^N} \frac{\partial^\alpha \varphi}{\partial z^\alpha} \left( \frac{z' - y'}{z_M} \right) u(y') dy' \\ &= \frac{1}{z_M^{M+m-1}} \int_{\mathbb{R}^{M-2}} \prod_{j \neq i} \psi^{(\alpha_j)} \left( \frac{z_j - y_j}{z_M} \right) \int_{\mathbb{R}} \psi^{(\alpha_i)} \left( \frac{z_i - y_i}{z_M} \right) u(y_i'', y_i) dy_i dy_i'' \\ &= \frac{(-1)^{|\alpha_i|}}{z_M^{M+m-1}} \int_{\mathbb{R}^{M-2}} \prod_{j \neq i} \psi^{(\alpha_j)} \left( \frac{z_j - y_j}{z_M} \right) \int_{\mathbb{R}} \psi^{(\alpha_i)} \left( \frac{\eta_i}{z_N} \right) u(y_i'', z_i + \eta_i) d\eta_i dy_i''. \end{aligned} \quad (34)$$

Using the notation (3), we have that

$$\frac{d^{(\alpha_i)}(\psi_{z_N})}{dt^{(\alpha_i)}}(t) = \frac{1}{z_M^{|\alpha_i|+1}} \frac{d^{(\alpha_i)}\psi}{dt^{(\alpha_i)}} \left( \frac{t}{z_N} \right), \quad t \in \mathbb{R},$$

and so, by Lemma 8,

$$\int_{\mathbb{R}} \psi^{(\alpha_i)} \left( \frac{\eta_i}{z_N} \right) u(y_i'', z_i + \eta_i) d\eta_i = \int_{\mathbb{R}} (\omega * \varpi)^{(\alpha_i)} \left( \frac{\eta_i}{z_N} \right) \Delta_i^{\eta_i, m} u(y_i'', z_i + \eta_i) d\eta_i.$$

Since  $\varpi, \psi, \omega \in C_c^\infty(B(0, 1))$ , by (34) we have that

$$\left| \frac{\partial^\alpha v}{\partial z^\alpha}(z) \right| \leq \frac{C}{z_M^{M+m-1}} \int_{B_{M-2}(0, z_M)} \int_{-z_M}^{z_M} |\Delta_i^{\eta_i, m} u(y_i'', z_i + \eta_i)| d\eta_i dy_i''. \quad (35)$$

Reasoning as in (20) (see also 2), we get

$$\left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \leq \frac{C}{z_M^{m+1}} \int_0^{z_M} \|\Delta_i^{\eta_i, m} u\|_{L^p(\mathbb{R}^N)} d\eta_i \quad (36)$$

for all  $z_M > 0$ . If  $\theta < \infty$ , multiply both sides of the previous inequality by  $\frac{1}{z_M^\theta}$ , where

$$a := \frac{1}{\theta} - (m - s),$$

and take the  $L^\theta((0, \infty))$  norm in  $z_M$  to obtain

$$\begin{aligned} &\left( \int_0^\infty \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\ &\leq C \left( \int_0^\infty \left( \frac{1}{z_M^{1+\frac{1}{\theta}+s}} \int_0^{z_M} \|\Delta_i^{\eta_i, m} u\|_{L^p(\mathbb{R}^N)} d\eta_i \right)^\theta dz_M \right)^{\frac{1}{\theta}}. \end{aligned}$$

By applying Hardy's inequality to the right-hand side, we get

$$\begin{aligned} & \left( \int_0^\infty \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\ & \leq \frac{C}{1+s} \left( \int_0^\infty \left\| \Delta_i^{z_M, m} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1+s\theta}} \right)^{\frac{1}{\theta}} \leq \frac{C}{1+s} |u|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(m)}. \end{aligned}$$

On the other hand, if  $\theta = \infty$ , reasoning as in (21), from (36), we get

$$\sup_{z_M > 0} \left( z_M^{m-s} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \leq \frac{C}{1+s} |u|_{B^{s,p,\infty}(\mathbb{R}^N)}^{(m)}.$$

To estimate  $\frac{\partial^m v}{\partial z_M^m}$ , we use (22) and (31) to write

$$\begin{aligned} v(z', z_M) &= \sum_{i=1}^N \int_{\mathbb{R}^{M-2}} \frac{1}{z_M^{N-1}} \prod_{j \neq i} \psi \left( \frac{y_j}{z_M} \right) \\ & \times \int_{\mathbb{R}} \frac{1}{z_M} \psi \left( \frac{y_i}{z_M} \right) \Delta_i^{-y_i, 1} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N) dy_i dy_i'' \\ & + u(z'). \end{aligned}$$

Since  $\psi$  is even, after a change of variables, we get

$$\begin{aligned} & \int_{\mathbb{R}} \frac{1}{z_M} \psi \left( \frac{y_i}{z_M} \right) \Delta_i^{-y_i, 1} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N) dy_i \\ & = \int_{\mathbb{R}} \frac{1}{z_M} \psi \left( \frac{\eta_i}{z_M} \right) \Delta_i^{\eta_i, 1} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N) d\eta_i \\ & = \int_{\mathbb{R}} \frac{1}{z_M} (\omega * \varpi) \left( \frac{\eta_i}{z_M} \right) \Delta_i^{\eta_i, m} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N) d\eta_i, \end{aligned}$$

where in the last equality we have used Lemma 8. Hence, after renaming  $\eta_i$  and using Fubini's theorem,

$$\begin{aligned} v(z', z_M) &= u(z') \\ & + \sum_{i=1}^N \int_{\mathbb{R}^{M-1}} \frac{1}{z_M^N} K_i \left( \frac{y'}{z_M} \right) \Delta_i^{y_i, m} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N) dy', \end{aligned}$$

where

$$K_i(z') := (\omega * \varpi)(z_i) \prod_{j \neq i} \psi(z_j), \quad z' \in \mathbb{R}^N.$$

We now differentiate the previous identity with respect to  $z_M$  and use Leibnitz

formula for the derivative of a product, to get

$$\begin{aligned}
\frac{\partial^m v}{\partial z_M^m}(z', z_M) &= \sum_{i=1}^N \int_{\mathbb{R}^{M-1}} \frac{\partial^m}{\partial z_M^m} \left( \frac{1}{z_M^N} K_i \left( \frac{y'}{z_M} \right) \right) \\
&\quad \times \Delta_i^{y_i, m} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N) dy' \\
&= \sum_{i=1}^N \sum_{n=0}^m \frac{C(M, m, n)}{z_M^{N+m-n}} \int_{\mathbb{R}^{M-1}} \frac{\partial^n}{\partial z_M^n} \left( K_i \left( \frac{y'}{z_M} \right) \right) \\
&\quad \times \Delta_i^{y_i, m} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N) dy'.
\end{aligned}$$

For  $n = 0, \dots, m$ ,  $\frac{\partial^n}{\partial z_M^n} \left( K_i \left( \frac{y'}{z_M} \right) \right)$  is given by sums of the form

$$\frac{1}{z_M^n} \left( \frac{y'}{z_M} \right)^\alpha \frac{\partial^\alpha K_i}{\partial z^\alpha} \left( \frac{y'}{z_M} \right),$$

where  $\alpha = (\alpha_1, \dots, \alpha_N, 0)$ , with  $0 \leq |\alpha| \leq n$ . It follows that

$$\begin{aligned}
\left| \frac{\partial^m v}{\partial z_M^m}(z', z_M) \right| &\leq \sum_{i=1}^N \frac{C}{z_M^{M+m-1}} \\
&\quad \times \int_{B_{M-2}(0, z_M)} \int_{-z_M}^{z_M} |\Delta_i^{y_i, m} u(z_1 - y_1, \dots, z_{i-1} - y_{i-1}, z_i, \dots, z_N)| d\eta_i dy_i''.
\end{aligned}$$

Reasoning as in (35), we have that

$$\left( \int_0^\infty \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \leq \frac{C}{1+s} |u|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(m)}$$

if  $\theta < \infty$ , while

$$\sup_{z_M > 0} \left( z_M^{m-s} \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)} \right) \leq \frac{C}{1+s} |u|_{B^{s,p,\infty}(\mathbb{R}^N)}^{(m)}.$$

Similar estimates hold for  $\frac{\partial^\alpha v}{\partial z^\alpha}$ , when  $\alpha = (\alpha_1, \dots, \alpha_N, \alpha_N)$ , with  $|\alpha| = m$ , and  $0 < \alpha_M < m$ . We omit the details.

**Step 2:** Assume that  $u \in C^m(\mathbb{R}^N)$ . For  $z' \in \mathbb{R}^N$ ,  $h > 0$  and  $i = 1, \dots, N$ , we have

$$\begin{aligned}
\Delta_i^{h,m} u(z') &= \Delta_i^{h,m} v(z', 0) = - \sum_{j=1}^m (-1)^j \binom{m}{j} \Delta_i^{h,m} v(z', jh) \\
&\quad + \sum_{j=0}^m (-1)^j \binom{m}{j} \Delta_M^{h,m} v(z' + jhe_i, 0).
\end{aligned} \tag{37}$$

To see this, we use the following identity for polynomials

$$\begin{aligned} (x-1)^m &= (x-1)^m (1 - (-1)^m (y-1)^m) + (-1)^m (x-1)^m (y-1)^m \\ &= -\sum_{j=1}^m (-1)^j \binom{m}{j} y^j (x-1)^m + \sum_{j=0}^m (-1)^j \binom{m}{j} x^j (y-1)^m \end{aligned}$$

replace  $x$  and  $y$  with  $T_i^h$  and  $T_M^h$ , respectively, and use the fact that  $\Delta_i^{h,m} = (T_i^h - I)^m$ .

We now estimate the various terms on the right-hand side of (37). By Exercise 9,

$$\begin{aligned} \left| \Delta_i^{h,m} u(z') \right| &\leq \sum_{j=1}^m \binom{m}{j} \int_0^{mh} \tau^{m-1} \left| \frac{\partial^m v}{\partial z_i^m}(z_i'', z_i + \tau, jh) \right| d\tau \\ &\quad + \sum_{j=0}^m \binom{m}{j} \int_0^{mh} \tau^{m-1} \left| \frac{\partial^m v}{\partial z_M^m}(z_i'', z_i + jh, \tau) \right| d\tau. \end{aligned}$$

If  $1 \leq p < \infty$ , by Corollary B.83 and a change of variables,

$$\begin{aligned} &\left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)} \\ &\leq C(m, p) \sum_{j=1}^m \left( \int_{\mathbb{R}^N} \left( \int_0^{mh} \tau^{m-1} \left| \frac{\partial^m v}{\partial z_i^m}(z_i'', z_i + \tau, jh) \right| d\tau \right)^p dz' \right)^{\frac{1}{p}} \\ &\quad + C(m, p) \sum_{j=0}^m \int_0^{mh} \tau^{m-1} \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} d\tau \\ &=: C(m, p) \sum_{j=1}^m I_j + C(m, p) \sum_{j=0}^m II_j. \end{aligned}$$

To estimate  $I_j$  we reason as in (25) to obtain

$$I_j \leq C(m, p) h^m \left\| \frac{\partial^m v}{\partial z_i^m}(\cdot, jh) \right\|_{L^p(\mathbb{R}^N)}.$$

Hence,

$$\begin{aligned} \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)} &\leq C(m, p) \sum_{j=1}^m h^m \left\| \frac{\partial^m v}{\partial z_i^m}(\cdot, jh) \right\|_{L^p(\mathbb{R}^N)} \\ &\quad + C(m, p) \sum_{j=0}^m \int_0^{mh} \tau^{m-1} \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} d\tau. \end{aligned} \tag{38}$$

The same inequality holds if  $p = \infty$ .

If  $1 \leq \theta < \infty$ , multiply both sides of the previous inequality by  $\frac{1}{h^{\frac{1}{\theta}+s}}$  and then take the  $L^\theta((0, \infty))$  norm in  $h$  to obtain,

$$\begin{aligned}
& \left( \int_0^\infty \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \right)^{\frac{1}{\theta}} \\
& \leq C(m, p) \sum_{j=1}^m \left( \int_0^\infty \left\| \frac{\partial^m v}{\partial z_i^m}(\cdot, jh) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\
& \quad + C(m, p) \sum_{j=0}^m \left( \int_0^\infty \left( \frac{1}{h^{\frac{1}{\theta}+s}} \int_0^{mh} \tau^{m-1} \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)}^\theta d\tau \right) dh \right)^{\frac{1}{\theta}} \\
& \leq C(m, p) \left( \int_0^\infty \left\| \frac{\partial^m v}{\partial z_i^m}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\
& \quad + \frac{C(m, p)}{s} \left( \int_0^\infty \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}},
\end{aligned}$$

where in the last inequality we have used Hardy's inequality and a change of variables. It follows that

$$\begin{aligned}
|u|_{B^{s,p,\theta}(\mathbb{R}^N)} & \leq C(m, p) \sum_{i=1}^N \left( \int_0^\infty \left\| \frac{\partial^m v}{\partial z_i^m}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} + \\
& \quad + N \frac{C(m, p)}{s} \left( \int_0^\infty \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}}.
\end{aligned}$$

On the other hand, if  $\theta = \infty$ , multiply both sides of (38) by  $\frac{1}{h^s}$  and proceed as in (21) to obtain

$$\frac{1}{h^s} \int_0^{mh} \tau^{m-1} \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, \tau) \right\|_{L^p(\mathbb{R}^N)} d\tau \leq \frac{m^s}{s} \sup_{h>0} h^{m-s} \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)},$$

It follows that

$$\begin{aligned}
|u|_{B^{s,p,\infty}(\mathbb{R}^N)} & \leq C(m, p) \sum_{i=1}^N \sup_{h>0} h^{m-s} \left\| \frac{\partial^m v}{\partial z_i^m}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)} + \\
& \quad + N \frac{C(m, p) m^s}{s} \sup_{h>0} h^{m-s} \left\| \frac{\partial^m v}{\partial z_M^m}(\cdot, h) \right\|_{L^p(\mathbb{R}^N)}.
\end{aligned}$$

To remove the extra assumption that  $u \in C^m(\mathbb{R}^N)$ , we use a mollification argument. We omit the details. This concludes the proof.  $\blacksquare$

**Corollary 16** *Let  $1 \leq p, \theta \leq \infty$ ,  $s > 1$ , and*

$$k := \begin{cases} \lfloor s \rfloor & \text{if } s \text{ is not an integer or } \theta = 1, \\ s - 1 & \text{if } s \text{ is an integer and } \theta > 1. \end{cases}$$

*Then*

$$B^{s,p,\theta}(\mathbb{R}^N) \subset W^{k,p}(\mathbb{R}^N).$$

**Proof.** Here we take  $m = \lfloor s \rfloor + 1$ .

**Step 1:** Assume first that  $s$  is not an integer or  $\theta = 1$ . Since the function  $v$  given in (18) belongs to  $C^\infty(\mathbb{R}_+^M)$ , by the fundamental theorem of calculus, for any multi-index  $\beta \in \mathbb{N}_0^M$  of the form  $\beta = (\beta_1, \dots, \beta_N, 0)$ , with  $|\beta| = k$ , and for any  $i = 1, \dots, N$  such that  $\beta_i > 0$ , we have

$$\frac{\partial^\beta v}{\partial z^\beta}(z', h) - \frac{\partial^\beta v}{\partial z^\beta}(z', \varepsilon) = \int_\varepsilon^h \frac{\partial^\alpha v}{\partial z^\alpha}(z', t) dt$$

for all  $z' \in \mathbb{R}^N$  and  $0 < \varepsilon < h < 1$ , where  $\alpha := (\beta_1, \dots, \beta_N, 1)$ . If  $1 \leq p < \infty$ , by Corollary B.83,

$$\left\| \frac{\partial^\beta v}{\partial z^\beta}(\cdot, h) - \frac{\partial^\beta v}{\partial z^\beta}(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^N)} \leq \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt.$$

The same inequality holds for  $p = \infty$ .

If  $1 \leq \theta < \infty$ , we consider two cases. If  $\theta > 1$  and  $s$  is not an integer, then  $m < s + 1$ , and so by Hölder's inequality,

$$\begin{aligned} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt &= \int_\varepsilon^h \frac{t^{\frac{1}{\theta} - (m-s)}}{t^{\frac{1}{\theta} - (m-s)}} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt \\ &\leq \left( \int_\varepsilon^h t^{\frac{\theta'}{\theta} + \theta'(s-m)} dt \right)^{\frac{1}{\theta'}} \left( \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dt}{t^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\ &\leq \frac{h^{1-(m-s)}}{[\theta'(1-(m-s))]^{\frac{1}{\theta'}}} \left( \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dt}{t^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}}, \end{aligned}$$

where we have used Hölder's inequality. If  $\theta = 1$  and  $s$  is possibly an integer, then

$$\begin{aligned} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt &= \int_\varepsilon^h \frac{t^{1-(m-s)}}{t^{1-(m-s)}} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt \\ &\leq h^{1-(m-s)} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} \frac{dt}{t^{1-(m-s)}}, \end{aligned}$$

where we have used the fact that  $1 - (m - s) \geq 0$ , since  $m = \lfloor s \rfloor + 1 \leq s + 1$ .

In both cases, we obtain

$$\left\| \frac{\partial^\beta v}{\partial z^\beta}(\cdot, h) - \frac{\partial^\beta v}{\partial z^\beta}(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^N)} \leq Ch^{1-(m-s)} \left( \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dt}{t^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}}. \quad (39)$$

Since the right-hand side goes to zero as  $h \rightarrow 0^+$ , we have that  $\left\{ \frac{\partial^\beta v}{\partial z^\beta}(\cdot, h) \right\}_h$  is a Cauchy sequence in  $L^p(\mathbb{R}^N)$ , and hence it converges in  $L^p(\mathbb{R}^N)$  to some function  $g_\beta$ . Since  $v(\cdot, h) = u * \varphi_h$ , where  $\varphi_h$  is a standard mollifier, it follows by standard interpolation results and (39) that the sequence  $\{u * \varphi_h\}$  is a Cauchy sequence in  $W^{k,p}(\mathbb{R}^N)$ . Hence,  $u$  belongs to  $W^{k,p}(\mathbb{R}^N)$ .

If  $\theta = \infty$  and  $s$  is not an integer, then

$$\begin{aligned} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt &= \int_\varepsilon^h \frac{t^{m-s}}{t^{m-s}} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt \\ &\leq \int_\varepsilon^h \frac{1}{t^{m-s}} dt \sup_{\varepsilon < t < h} \left( t^{m-s} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} \right) \\ &\leq \frac{h^{1-(m-s)}}{(1-(m-s))} \sup_{\varepsilon < t < h} \left( t^{m-s} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} \right), \end{aligned}$$

We can now continue as before to conclude that  $u$  belongs to  $W^{k,p}(\mathbb{R}^N)$ .

**Step 2:** Next we consider the case in which  $s$  is an integer and  $\theta > 1$ , so that  $k = s - 1$ . Let  $u \in B^{s,p,\theta}(\mathbb{R}^N)$ . In this case, by Theorem 12, we have that  $B^{k+1,p,\theta} \subset B^{k+s,p,\theta}$  for every  $0 < s < 1$ . Hence, by the previous part, applied to  $B^{k+s,p,\theta}$ , we have that  $u$  belongs to  $W^{k,p}(\mathbb{R}^N)$ . ■

**Remark 17** Reasoning as in Step 1 of the previous proof, we can show that if  $u \in L^1_{\text{loc}}(\mathbb{R}^N)$  is such that  $|u|_{B^{s,p,\theta}(\mathbb{R}^N)} < \infty$  for some  $1 \leq p, \theta \leq \infty$ ,  $s > 1$ , with  $s$  is not an integer or  $\theta = 1$ , then all the distributional derivatives of  $u$  of order  $k$  belong to  $L^p(\mathbb{R}^N)$ .

## 7 A Characterization of $B^{s,p,\theta}$ for $s > 1$

The next theorem allows to extend most of the results of Chapter 14 to higher order Besov spaces.

**Theorem 18** Let  $1 \leq p \leq \infty$ ,  $1 \leq \theta \leq \infty$ , let  $s > 1$ , and let  $n \in \mathbb{N}$  with  $n < s$ . Then a function  $u \in L^p(\mathbb{R}^N)$  belongs to  $B^{s,p,\theta}(\mathbb{R}^N)$  if and only if for every multi-index  $\alpha$  with  $|\alpha| = n$ , the distributional derivative  $\frac{\partial^\alpha u}{\partial x_\alpha}$  belongs to  $B^{s-n,p,\theta}(\mathbb{R}^N)$ . Moreover, the expression

$$\|u\|_{L^p(\mathbb{R}^N)} + \sum_{|\alpha|=n} \left| \frac{\partial^\alpha u}{\partial x_\alpha} \right|_{B^{s-n,p,\theta}(\mathbb{R}^N)}$$

is an equivalent norm in  $B^{s,p,\theta}(\mathbb{R}^N)$ .

**Proof.** By induction, it suffices to prove the theorem in the case  $n = 1$ .

**Step 1:** Assume that  $\frac{\partial u}{\partial x_i} \in B^{s-1,p,\theta}(\mathbb{R}^N)$  for all  $i = 1, \dots, N$ . In particular,  $u \in W^{1,p}(\mathbb{R}^N)$ , and so by (14.10),

$$\frac{1}{h} \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}$$

for every  $h > 0$ . Since  $u \in W^{1,p}(\mathbb{R}^N)$ , for every fixed  $h > 0$  the function  $v := \Delta_i^{h,m-1} u$  belongs to  $W^{1,p}(\mathbb{R}^N)$  and

$$\frac{\partial}{\partial x_i} \left( \Delta_i^{h,m-1} u \right) = \Delta_i^{h,m-1} \left( \frac{\partial u}{\partial x_i} \right). \quad (40)$$

Thus, applying the previous inequality with  $v$  in place of  $u$ , we get

$$\frac{1}{h} \|\Delta_i^{h,m} u\|_{L^p(\mathbb{R}^N)} \leq \left\| \Delta_i^{h,m-1} \left( \frac{\partial u}{\partial x_i} \right) \right\|_{L^p(\mathbb{R}^N)} \quad (41)$$

for every  $h > 0$ . If  $\theta < \infty$ , raising both sides to the power  $\theta$ , multiplying by  $\frac{1}{h^{1+(s-1)\theta}}$  and integrating over  $h$  gives

$$\int_0^\infty \|\Delta_i^{h,m} u\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \leq \int_0^\infty \left\| \Delta_i^{h,m-1} \left( \frac{\partial u}{\partial x_i} \right) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+(s-1)\theta}},$$

which implies that  $u$  belongs to  $B^{s,p,\theta}(\mathbb{R}^N)$ . If  $\theta = \infty$  the same conclusion follows by multiplying both sides of (41) by  $\frac{1}{h^{s-1}}$  and taking the supremum over  $h > 0$ .

**Step 2:** Assume that  $u$  belongs to  $B^{s,p,\theta}(\mathbb{R}^N)$ . In view of Corollary 16, we have that  $B^{s,p,\theta}(\mathbb{R}^N) \subset W^{1,p}(\mathbb{R}^N)$  and thus the distributional derivatives  $\frac{\partial u}{\partial x_i}$ ,  $i = 1, \dots, N$ , of  $u$  belong to  $L^p(\mathbb{R}^N)$ . We want to show that actually they belong to  $B^{s-1,p,\theta}(\mathbb{R}^N)$ . Fix  $i = 1, \dots, N$ , let  $m \in \mathbb{N}$  with  $m \geq [s] + 1$ , and for  $z' \in \mathbb{R}^N$  and  $z_M > 0$  define

$$w_i(z) := \frac{1}{z_M^N} \int_{\mathbb{R}^N} \varphi \left( \frac{z' - y'}{z_M} \right) \frac{\partial u}{\partial y_i}(y') dy',$$

where  $\varphi$  is given in (31). Integrating by parts, we get

$$\begin{aligned} w_i(z) &= \frac{1}{z_M^{N+1}} \int_{\mathbb{R}^N} \frac{\partial \varphi}{\partial y_i} \left( \frac{z' - y'}{z_M} \right) u(y') dy' \\ &= \frac{\partial v}{\partial z_i}(z), \end{aligned}$$

where  $v$  is the function defined in (18). Hence, for any multi-index  $\beta \in \mathbb{N}_0^M$  with  $|\beta| = m - 1$ , we have that

$$\frac{\partial^\beta w_i}{\partial z_\beta} = \frac{\partial^\alpha v}{\partial z_\alpha},$$

where  $\alpha$  is the multi-index of components

$$\alpha_j := \begin{cases} \beta_j & \text{if } j \neq i, \\ \beta_i + 1 & \text{if } j = i, \end{cases}$$

so that  $|\alpha| = m$ . It follows from Theorem 15 that

$$\begin{aligned} & \sum_{|\beta|=m-1} \left( \int_0^\infty \left\| \frac{\partial^\beta w_i}{\partial z^\beta}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-1-(s-1))}} \right)^{\frac{1}{\theta}} \\ & \leq \sum_{|\alpha|=m} \left( \int_0^\infty \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\ & \leq C |u|_{B^{s,p,\theta}(\mathbb{R}^N)} < \infty, \end{aligned}$$

which implies that  $\frac{\partial u}{\partial x_i}$  belongs to  $B^{s-1,p,\theta}(\mathbb{R}^N)$ , again by Theorem 15. ■

## References

- [1] V.I. Burenkov, *Sobolev spaces on domains*, Teubner-Texte zur Mathematik, 137. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1998.
- [2] A. Jonsson and H. Wallin, The trace to subsets of  $R^n$  of Besov spaces in the general case. *Anal. Math.* 6 (1980), no. 3, 223–254.
- [3] V.G. Maz'ja, *Sobolev spaces*, translated from the Russian by T. O. Shaposhnikova, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.
- [4] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, no. 30, Princeton University Press, Princeton, N.J, 1970.