# 1 Higher Order Besov Spaces

Given a function $u:\mathbb{R}^N\to\mathbb{R}$, for every $h\in\mathbb{R}$, $m\in\mathbb{N}$, $i=1,\ldots,N$, and $x\in\mathbb{R}^N$, we define inductively

$$\Delta_i^{h,1}u(x):=\Delta_i^h u(x)$$

for $m=1$, where $\Delta_i^h u(x)$ has been defined in (14.1), and

$$\Delta_i^{h,m}u(x):=\Delta_i^h\left(\Delta_i^{h,m-1}u(x)\right)$$

for $m\geq 2$. When $N=1$ we write $\Delta^{h,m}u(x):=\Delta_1^{h,m}u(x)$.

**Remark 1** *Given a function $u:\mathbb{R}^N\to\mathbb{R}$, for $i=1,\ldots,N$ and $h\in\mathbb{R}$ consider the translation operator $T_i^h$ defined by*

$$T_i^h(u)(x):=u(x+he_i),\quad x\in\mathbb{R}^N.$$

*Note that $T_i^h\circ T_i^s=T_i^s\circ T_i^h$ for every $h,s\in\mathbb{R}$. Moreover, $\Delta_i^h:=T_i^h-I$ and for $m\in\mathbb{N}$, with $m\geq 2$*

$$\Delta_i^{h,m}=\left(T_i^h-I\right)^m:=\underbrace{\left(T_i^h-I\right)\circ\cdots\circ\left(T_i^h-I\right)}_{m\ times}.$$

*Here $I$ is the identity operator. Hence, by the binomial theorem,*

$$\Delta_i^{h,m}u(x)=\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}u(x+jhe_i),\quad x\in\mathbb{R}^N. \tag{1}$$

*In particular, setting $y:=x+mhe_i$ and $k=m-j$, we have*

$$\begin{aligned}\Delta_i^{h,m}u(x)&=\sum_{k=0}^m(-1)^k\binom{m}{m-k}u(y-khe_i) \qquad (2)\\ &=(-1)^m\sum_{k=0}^m(-1)^{m-k}\binom{m}{k}u(y-khe_i)=(-1)^m\Delta_i^{-h,m}u(x+mhe_i).\end{aligned}$$

**Definition 2** *Let $1\leq p,\theta\leq\infty$, $s>0$, $m:=\lfloor s\rfloor+1$, where $\lfloor\cdot\rfloor$ is the integer part. A function $u\in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$ belongs to the* Besov space *$B^{s,p,\theta}\left(\mathbb{R}^N\right)$ if*

$$\|u\|_{B^{s,p,\theta}(\mathbb{R}^N)}:=\|u\|_{L^p(\mathbb{R}^N)}+|u|_{B^{s,p,\theta}(\mathbb{R}^N)}<\infty,$$

*where*

$$|u|_{B^{s,p,\theta}(\mathbb{R}^N)}:=\sum_{i=1}^N\left(\int_0^\infty\left\|\Delta_i^{h,m}u\right\|^\theta_{L^p(\mathbb{R}^N)}\frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}}$$

*if $\theta<\infty$, and*

$$|u|_{B^{s,p,\infty}(\mathbb{R}^N)}:=\sum_{i=1}^N\sup_{h>0}\frac{1}{h^s}\left\|\Delta_i^{h,m}u\right\|_{L^p(\mathbb{R}^N)}$$

*if $\theta=\infty$.*

If $\theta = p$, we write

$$B^{s,p}\left(\mathbb{R}^N\right) := B^{s,p,p}\left(\mathbb{R}^N\right).$$

Observe that if $0 < s < 1$, then $m = 1$, and so $\Delta_i^{h,1} u = \Delta_i^h u$, which is consistent with Definition 14.1.

**Remark 3** *Note that if $N = 1$, the space $B^{1,\infty}(\mathbb{R})$ coincides with the Zygmund space $\Lambda_1(\mathbb{R})$ introduced in Chapter 1 (see Definition 1.16).*

**Remark 4** *If $u \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$, by a change of variables, we have that*

$$\left\|\Delta_i^{h,2} u\right\|_{L^p(\mathbb{R}^N)} = \left\|\widetilde{\Delta}_i^{h,2} u\right\|_{L^p(\mathbb{R}^N)},$$

*where*

$$\widetilde{\Delta}_i^{h,2} u(x) := u(x + he_i) - 2u(x) + u(x + he_i)$$

*for $h \in \mathbb{R}$, $i = 1, \ldots, N$, and $x \in \mathbb{R}^N$.*

**Exercise 5** *Prove that the conclusions of Propositions 14.3 and 14.5 continue to hold for $s \geq 1$.*

Next we show that in the definition of $B^{s,p,\theta}$, one can replace $\Delta_i^{h,m}$ with $\Delta_i^{h,n}$ for any $n \in \mathbb{N}$ with $n > m$.

In what follows, given a function $f : \mathbb{R}^N \to \mathbb{R}$ and $h > 0$, we set

$$f_h(x) := \frac{1}{h^N} f\left(\frac{x}{h}\right), \quad x \in \mathbb{R}^N. \tag{3}$$

**Proposition 6** *Let $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, $s > 0$. Then for every $n \in \mathbb{N}$ with $n \geq m := \lfloor s \rfloor + 1$, the seminorm $|\cdot|^{(n)}_{B^{s,p,\theta}(\mathbb{R}^N)}$, defined by*

$$|u|^{(n)}_{B^{s,p,\theta}(\mathbb{R}^N)} := \sum_{i=1}^N \left(\int_0^\infty \left\|\Delta_i^{h,n} u\right\|^\theta_{L^p(\mathbb{R}^N)} \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} \tag{4}$$

*if $\theta < \infty$, and*

$$|u|^{(n)}_{B^{s,p,\infty}(\mathbb{R}^N)} := \sum_{i=1}^N \sup_{h>0} \frac{1}{h^s} \left\|\Delta_i^{h,n} u\right\|_{L^p(\mathbb{R}^N)}$$

*if $\theta = \infty$, is an equivalent seminorm in $B^{s,p,\theta}\left(\mathbb{R}^N\right)$.*

**Proof.** It is enough to prove that the seminorms $|\cdot|^{(n)}_{B^{s,p,\theta}(\mathbb{R}^N)}$ and $|\cdot|^{(n+1)}_{B^{s,p,\theta}(\mathbb{R}^N)}$ are equivalent for some $n \geq m := \lfloor s \rfloor + 1$. Using the fact that

$$\left\|\Delta_i^{h,1} u\right\|_{L^p(\mathbb{R}^N)} \leq 2\|u\|_{L^p(\mathbb{R}^N)}, \tag{5}$$

we have that

$$\left\|\Delta_i^{h,n+1}u\right\|_{L^p(\mathbb{R}^N)} = \left\|\Delta_i^h\left(\Delta_i^{h,n}u\right)\right\|_{L^p(\mathbb{R}^N)} \leq 2\left\|\Delta_i^{h,n}u\right\|_{L^p(\mathbb{R}^N)}.$$

Thus, it remains to prove the opposite inequality.
**Step 1:** We consider first the case $0 < s < 1$ and $n = 1$. Fix $u \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$. For $x \in \mathbb{R}^N$, $h > 0$, and $i = 1, \ldots, N$, we have

$$\begin{aligned} 2\left(u\left(x+he_i\right)-u\left(x\right)\right) &= \left(u\left(x+2he_i\right)-u\left(x\right)\right) \\ &\quad - \left(u\left(x+2he_i\right)-2u\left(x+he_i\right)+u\left(x\right)\right), \end{aligned}$$

and so

$$2\Delta_i^{h,1}u\left(x\right) = \Delta_i^{2h,1}u\left(x\right) - \Delta_i^{h,2}u\left(x\right). \tag{6}$$

If $\theta < \infty$, it follows by Minkowski's inequality that

$$\begin{aligned} 2\left(\int_0^\infty \left\|\Delta_i^{h,1}u\right\|^\theta_{L^p(\mathbb{R}^N)}\frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} &\leq \left(\int_0^\infty \left\|\Delta_i^{2h,1}u\right\|^\theta_{L^p(\mathbb{R}^N)}\frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} \\ + \left(\int_0^\infty \left\|\Delta_i^{h,2}u\right\|^\theta_{L^p(\mathbb{R}^N)}\frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} &= 2^s\left(\int_0^\infty \left\|\Delta_i^{\eta,1}u\right\|^\theta_{L^p(\mathbb{R}^N)}\frac{d\eta}{\eta^{1+s\theta}}\right)^{\frac{1}{\theta}} \\ &\quad + \left(\int_0^\infty \left\|\Delta_i^{h,2}u\right\|^\theta_{L^p(\mathbb{R}^N)}\frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}}, \end{aligned}$$

where we have made the change of variables $h = \frac{\eta}{2}$. The case $\theta = \infty$ is similar. Hence,

$$\left(2-2^s\right)\left|u\right|_{B^{s,p,\theta}(\mathbb{R}^N)} \leq \left|u\right|^{(2)}_{B^{s,p,\theta}(\mathbb{R}^N)}.$$

**Step 2:** Assume next that $n \geq 2$. We use the following identity for polynomials

$$\begin{aligned} \left(x-1\right)^n &= \frac{1}{2^n}\left(x^2-1\right)^n + \left(x-1\right)^n - \frac{1}{2^n}\left(x^2-1\right)^n \\ &= \frac{1}{2^n}\left(x^2-1\right)^n + P_{n-1}\left(x\right)\left(x-1\right)^{n+1}, \quad x \in \mathbb{R}, \end{aligned}$$

where $P_{n-1}$ is the polynomial

$$\begin{aligned} P_{n-1}\left(x\right) &= -\frac{1}{2^n}\frac{\left(x+1\right)^n-2^n}{x-1} = -\frac{1}{2^n}\frac{\left(\left(x-1\right)+2\right)^n-2^n}{x-1} \\ &= -\frac{1}{2^n}\sum_{j=1}^n 2^{n-j}\binom{n}{j}\left(x-1\right)^{j-1}, \quad x \in \mathbb{R}. \end{aligned}$$

By replacing $x$ with $T_i^h$, we get

$$\Delta_i^{h,n}u = \frac{1}{2^n}\Delta_i^{2h,n}u + P_{n-1}\left(T_i^h\right)\Delta_i^{h,n+1}u.$$

Hence,

$$\begin{aligned}
\left\|\Delta_i^{h,n}u\right\|_{L^p(\mathbb{R}^N)} &\le \frac{1}{2^n}\left\|\Delta_i^{2h,n}u\right\|_{L^p(\mathbb{R}^N)} + \left\|P_{n-1}\left(T_i^h\right)\Delta_i^{h,n+1}u\right\|_{L^p(\mathbb{R}^N)} \\
&\le \frac{1}{2^n}\left\|\Delta_i^{2h,n}u\right\|_{L^p(\mathbb{R}^N)} + \frac{1}{2^n}\sum_{j=1}^{n}2^{n-j}\binom{n}{j}\left\|\Delta_i^{h,j-1}\left(\Delta_i^{h,n+1}u\right)\right\|_{L^p(\mathbb{R}^N)} \\
&\le \frac{1}{2^n}\left\|\Delta_i^{2h,n}u\right\|_{L^p(\mathbb{R}^N)} + \frac{2^n-1}{2}\left\|\Delta_i^{h,n+1}u\right\|_{L^p(\mathbb{R}^N)},
\end{aligned}$$

where we have used $j-1$ times the inequality (5). If $\theta<\infty$, it follows by Minkowski's inequality that

$$\begin{aligned}
\left(\int_0^\infty \left\|\Delta_i^{h,n}u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} &\le \frac{1}{2^n}\left(\int_0^\infty \left\|\Delta_i^{2h,n}u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} \\
&\quad + \frac{2^n-1}{2}\left(\int_0^\infty \left\|\Delta_i^{h,n+1}u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} \\
&= \frac{2^s}{2^n}\left(\int_0^\infty \left\|\Delta_i^{\eta,n}u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{d\eta}{\eta^{1+s\theta}}\right)^{\frac{1}{\theta}} \\
&\quad + \frac{2^n-1}{2}\left(\int_0^\infty \left\|\Delta_i^{h,n+1}u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}},
\end{aligned}$$

where we have made the change of variables $h=\frac{\eta}{2}$. Therefore, we get

$$\begin{aligned}
&\left(1-\frac{1}{2^{n-s}}\right)\left(\int_0^\infty \left\|\Delta_i^{h,n}u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} \\
&\qquad \le \frac{2^n-1}{2}\left(\int_0^\infty \left\|\Delta_i^{h,n+1}u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}}.
\end{aligned}$$

Note that here it is important that $n\ge m:=\lfloor s\rfloor+1>s$. The case $\theta=\infty$ is similar. ■

# 2 Dependence of $B^{s,p,\theta}$ on $\theta$

We study the relation between different Besov spaces $B^{s,p,\theta}\left(\mathbb{R}^N\right)$ as $\theta$ varies.

**Theorem 7** *Let $1\le p\le\infty$, $s>0$, and $1\le\theta_1<\theta_2\le\infty$. Then there exists a constant $C=C\left(N,p,\theta_1,\theta_2\right)>0$ such that*

$$|u|_{B^{s,p,\theta_2}(\mathbb{R}^N)} \le C\,|u|_{B^{s,p,\theta_1}(\mathbb{R}^N)}$$

*for all $u\in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$. In particular, $B^{s,p,\theta_1}\left(\mathbb{R}^N\right)\subset B^{s,p,\theta_2}\left(\mathbb{R}^N\right)$.*

We begin with some auxiliary results. Consider a nonnegative even function $\omega \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\omega \subset \left[-\frac{1}{m+1}, \frac{1}{m+1}\right]$ and $\int_{\mathbb{R}} \omega(t)\, dt = 1$. Given $m \in \mathbb{N}$, define

$$\varpi(t) := \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} \frac{1}{j} \omega\left(\frac{t}{j}\right), \quad t \in \mathbb{R}. \tag{7}$$

Note that $\varpi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\varpi \subset \left[-\frac{m}{m+1}, \frac{m}{m+1}\right]$, and

$$\begin{aligned} \int_{\mathbb{R}} \varpi(t)\, dt &= \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} \int_{\mathbb{R}} \omega\left(\frac{t}{j}\right) \frac{dt}{j} \\ &= \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} = (-1)^{m+1}. \end{aligned} \tag{8}$$

Let

$$\psi(t) := (\varpi * \varpi)(t) = \int_{\mathbb{R}} \varpi(t-\tau)\, \varpi(\tau)\, d\tau, \quad t \in \mathbb{R}. \tag{9}$$

Then $\psi$ is even, $\psi \in C_c^\infty(\mathbb{R})$, and by Fubini's theorem,

$$\int_{\mathbb{R}} \psi(t)\, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \varpi(t-\tau)\, \varpi(\tau)\, d\tau\, dt = (-1)^{m+1} (-1)^{m+1} = 1. \tag{10}$$

**Lemma 8** *Let $\omega$ be as above, let $\varpi$ is the function defined in (7), and let $u \in L^1_{\text{loc}}(\mathbb{R})$. Then for $h > 0$ and for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$,*

$$\int_{\mathbb{R}} \psi_h(y)\, \Delta^y u(t)\, dy = \int_{\mathbb{R}} (\omega * \varpi)_h(y)\, \Delta^{y,m} u(t)\, dy,$$

*while for every $n \in \mathbb{N}$,*

$$\int_{\mathbb{R}} \frac{d^{(n)}\psi_h}{dy^n}(y)\, u(t+y)\, dy = \int_{\mathbb{R}} \frac{d^{(n)}(\omega * \varpi)_h}{dy^n}(y)\, \Delta^{y,m} u(t)\, dy.$$

**Proof.** By (9) and Fubini's theorem,

$$\int_{\mathbb{R}} \psi_h(y)\, \Delta^y u(t)\, dy = \int_{\mathbb{R}} \frac{1}{h} \varpi(\tau) \int_{\mathbb{R}} \varpi\left(\frac{y}{h} - \tau\right) \Delta^y u(t)\, dy d\tau.$$

By (1) and (8),

$$
\begin{aligned}
&\int_{\mathbb{R}} \varpi \left( \frac{y}{h} - \tau \right) \Delta^{y} u\left(t\right)\, dy \\
&= \int_{\mathbb{R}} \varpi \left( \frac{y}{h} - \tau \right) u\left(t+y\right)\, dy - u\left(t\right) \int_{\mathbb{R}} \varpi \left( \frac{y}{h} - \tau \right)\, dy \\
&= \sum_{j=1}^{m} \left(-1\right)^{m-j} \binom{m}{j} \frac{1}{j} \int_{\mathbb{R}} \omega \left( \frac{y}{jh} - \tau \right) u\left(t+y\right)\, dy - u\left(t\right) \left(-1\right)^{m+1} \frac{1}{h} \\
&= \sum_{j=1}^{m} \left(-1\right)^{m-j} \binom{m}{j} \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right) u\left(t+j\zeta\right)\, d\zeta + \left(-1\right)^{m} u\left(t\right) \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right)\, d\zeta \\
&= \sum_{j=0}^{m} \left(-1\right)^{m-j} \binom{m}{j} \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right) u\left(t+j\zeta\right)\, d\zeta = \int_{\mathbb{R}} \omega \left( \frac{\zeta}{h} - \tau \right) \Delta^{\zeta,m} u\left(t\right)\, d\zeta
\end{aligned}
$$

and so, again by Fubini's theorem

$$
\int_{\mathbb{R}} \psi_h\left(y\right) \Delta^{y} u\left(t\right)\, dy = \frac{1}{h} \int_{\mathbb{R}} \left(\omega * \varpi\right) \left( \frac{\zeta}{h} \right) \Delta^{\zeta,m} u\left(t\right)\, d\zeta.
$$

To prove the second identity, let $n \in \mathbb{N}$. By Theorem C.20 and (9), we have that

$$
\frac{d^n \psi_h}{dy^n}\left(y\right) = \left( \varpi * \frac{d^n \varpi_h}{dy^n} \right)\left(y\right) = \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}\left(y-\xi\right) \varpi\left(\xi\right)\, d\xi.
$$

By the fact that $\int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}\left(y-\xi\right)\, dy = 0$, Fubini's theorem, and a change of variables, we get

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{d^{(n)} \psi_h}{dy^n}\left(y\right) u\left(t+y\right)\, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}\left(y-\xi\right) \varpi\left(\xi\right) u\left(t+y\right)\, dy d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}\left(y-\xi\right) \varpi\left(\xi\right) \left[u\left(t+y\right) - u\left(t\right)\right]\, dy d\xi \\
&= \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}\left(\eta\right) \int_{\mathbb{R}} \varpi\left(y-\eta\right) \Delta^{y} u\left(t\right)\, dy d\eta.
\end{aligned}
$$

Reasoning as in first part of the proof, we have that

$$
\int_{\mathbb{R}} \varpi\left(y-\eta\right) \Delta^{y} u\left(t\right)\, dy = \int_{\mathbb{R}} \omega\left(y-\eta\right) \Delta^{y,m} u\left(t\right)\, dy,
$$

and so,

$$
\begin{aligned}
\int_{\mathbb{R}} \frac{d^{(n)} \psi_h}{dy^n}\left(y\right) u\left(t+y\right)\, dy &= \int_{\mathbb{R}} \frac{d^n \varpi_h}{dy^n}\left(\eta\right) \int_{\mathbb{R}} \omega\left(y-\eta\right) \Delta^{y,m} u\left(t\right)\, dy d\eta. \\
&= \int_{\mathbb{R}} \frac{d^{(n)} \left(\omega * \varpi\right)_h}{dy^n}\left(y\right) \Delta^{y,m} u\left(t\right)\, dy.
\end{aligned}
$$

■

**Exercise 9** *Let $g \in C^m(\mathbb{R})$, $m \in \mathbb{N}$.*

*(i) Prove that for every $y \in \mathbb{R}$ and $h > 0$,*

$$\Delta^{h,m} g(y) = \int_0^h \cdots \int_0^h g^{(m)}(y + t_1 + \cdots + t_m)\, dt_1 \cdots dt_m.$$

*(ii) Prove that for every $y \in \mathbb{R}$ and $h > 0$,*

$$\left|\Delta^{h,m} g(y)\right| \le \int_0^{mh} \tau^{m-1} \left|g^{(m)}(y + \tau)\right| d\tau.$$

**Lemma 10** *There exists a constant $C > 0$ such that for every $u \in L^p(\mathbb{R})$, $1 \le p \le \infty$, $m \in \mathbb{N}$, and for all $h > 0$,*

$$\left\|\Delta^{h,m} u\right\|_{L^p(\mathbb{R})} \le \frac{C}{h} \int_0^h \left\|\Delta^{\eta,m} u\right\|_{L^p(\mathbb{R})} d\eta.$$

**Proof.** Consider a nonnegative function $\psi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\psi \subset [0,1]$ and $\int_{\mathbb{R}} \psi(x)\, dx = 1$. For $h > 0$, let $u_h := u * \psi_h$, $x \in \mathbb{R}$ (see (3)). By Minkowski's inequality and Remark 14.2, we have

$$\begin{aligned}\left\|\Delta^{h,m} u\right\|_{L^p(\mathbb{R})} &\le \left\|\Delta^{h,m} u_h\right\|_{L^p(\mathbb{R})} + \left\|\Delta^{h,m}(u_h - u)\right\|_{L^p(\mathbb{R})} \\ &\le \left\|\Delta^{h,m} u_h\right\|_{L^p(\mathbb{R})} + 2^m \left\|u_h - u\right\|_{L^p(\mathbb{R})}.\end{aligned}$$

Hence, to conclude the proof, it remains to show that

$$\left\|u_h - u\right\|_{L^p(\mathbb{R})} \le \frac{C}{h} \int_0^h \left\|\Delta^{\eta,m} u\right\|_{L^p(\mathbb{R})} d\eta, \tag{11}$$

$$\left\|\Delta^{h,m} u_h\right\|_{L^p(\mathbb{R})} \le \frac{C}{h} \int_0^h \left\|\Delta^{\eta,m} u\right\|_{L^p(\mathbb{R})} d\eta. \tag{12}$$

**Step 1:** Assume first that $0 < s < 1$. Since $\int_{\mathbb{R}} \psi_h(x)\, dx = 1$, we have

$$u_h(x) - u(x) = \int_{\mathbb{R}} \psi_h(\eta)\left(u(x + \eta) - u(x)\right) d\eta.$$

for $x \in \mathbb{R}$. Hence, by Corollary B.83 and the fact that $\operatorname{supp}\psi_h \subset [0,h]$.

$$\begin{aligned}\left\|u_h - u\right\|_{L^p(\mathbb{R})} &\le \int_{\mathbb{R}} \psi_h(\eta) \left\|\Delta^\eta u\right\|_{L^p(\mathbb{R})} d\eta \\ &\le \frac{C}{h} \int_0^h \left\|\Delta^\eta u\right\|_{L^p(\mathbb{R})} d\eta.\end{aligned}$$

To prove (11), fix $x \in \mathbb{R}$ and $\xi > 0$ and define $g(y) := u_h(x + y\xi)$, $y \in \mathbb{R}$. Then

$$\Delta^\xi u_h(x) = g(1) - g(0) = \int_0^1 g'(y)\, dy = \xi \int_0^1 u_h'(x + y\xi)\, dy. \tag{13}$$

Since $\int_{\mathbb{R}} \psi_h'(\eta)\, d\eta = 0$, we have

$$u_h'(x) = (u * \psi_h')(x) = \int_{\mathbb{R}} \psi_h'(\eta) (u(x+\eta) - u(x))\, d\eta,$$

and so, again by Corollary B.83, Remark and the facts that $\operatorname{supp} \psi_h \subset [0,h]$ and that $\|\psi_h'\|_\infty \le \frac{C}{h^2}$,

$$\begin{aligned} \|u_h'(\cdot + y\xi)\|_{L^p(\mathbb{R})} = \|u_h'\|_{L^p(\mathbb{R})} &\le 2 \int_0^\infty |\psi_h'(\eta)| \|\Delta^\eta u\|_{L^p(\mathbb{R})}\, d\eta \\ &\le \frac{C}{h^2} \int_0^h \|\Delta^\eta u\|_{L^p(\mathbb{R})}\, d\eta, \end{aligned}$$

which, together with (13), gives

$$\left\|\Delta^\xi u_h\right\|_{L^p(\mathbb{R})} \le \frac{C\xi}{h^2} \int_0^h \|\Delta^\eta u\|_{L^p(\mathbb{R})}\, d\eta,$$

which yields (11).

**Step 2:** Next assume that $s = 1$. In this case we assume that the function $\psi$ is even with $\operatorname{supp} \psi \subset [-1,1]$. Since $\int_{\mathbb{R}} \psi_h(x)\, dx = 1$ and $\psi_h$ is even, we have

$$u_h(x) - u(x) = \int_0^\infty \psi_h(\eta) (u(x+\eta) - 2u(x) + u(x-\eta))\, d\eta$$

for $x \in \mathbb{R}$. Hence, by Corollary B.83, Remark 4, and the fact that $\operatorname{supp} \psi_h \subset [-h,h]$,

$$\|u_h - u\|_{L^p(\mathbb{R})} \le \int_0^\infty \psi_h(\eta) \left\|\Delta^{\eta,2} u\right\|_{L^p(\mathbb{R})} d\eta \le \frac{C}{h} \int_0^h \left\|\Delta^{\eta,2} u\right\|_{L^p(\mathbb{R})} d\eta, \quad (14)$$

so that (12) holds.

To prove (11), fix $x \in \mathbb{R}$ and $\xi > 0$. By Exercise 9,

$$\Delta^{\xi,2} u_h(x) = \xi^2 \int_0^1 \int_0^1 u_h''(x + (y_1 + y_2)\xi)\, dy_1 dy_2. \quad (15)$$

Since $\int_{\mathbb{R}} \psi_h''(\eta)\, d\eta = 0$ and $\psi_h''$ is even, we have

$$u_h''(x) = (u * \psi_h'')(x) = \int_0^\infty \psi_h''(\eta) (u(x+\eta) - 2u(x) + u(x-\eta))\, d\eta,$$

and so, again by Corollary B.83, Remark 4, and the facts that $\operatorname{supp} \psi_h \subset [0,h]$ and that $\|\psi_h''\|_\infty \le \frac{C}{h^3}$,

$$\begin{aligned} \|u_h''(\cdot + (y_1 + y_2)\xi)\|_{L^p(\mathbb{R})} = \|u_h''\|_{L^p(\mathbb{R})} &\le \int_0^\infty |\psi_h''(\eta)| \left\|\Delta^{\eta,2} u\right\|_{L^p(\mathbb{R})} d\eta \\ &\le \frac{C}{h^3} \int_0^h \left\|\Delta^{\eta,2} u\right\|_{L^p(\mathbb{R})} d\eta, \end{aligned}$$

which, together with (15), gives

$$\left\|\Delta^{\xi,2} u_h\right\|_{L^p(\mathbb{R})} \leq \frac{C\xi^2}{h^3} \int_0^h \left\|\Delta^{\eta,2} u\right\|_{L^p(\mathbb{R})} \, d\eta,$$

which concludes the proof in this case.

**Step 3:** Finally, in the general case $s > 0$, we take $\psi$ to be the function defined in (9). Using the fact that $\int_{\mathbb{R}} \psi_h(x) \, dx = 1$ and the previous lemma, we have

$$\begin{aligned} u_h(x) - u(x) &= \int_{\mathbb{R}} \psi_h(\eta) \, \Delta^\eta u(x) \, d\eta \\ &= \int_{\mathbb{R}} (\omega * \varpi)_h(\eta) \, \Delta^{\eta,m} u(x) \, d\eta. \end{aligned}$$

Hence, by Corollary B.83, the facts that $\operatorname{supp} \omega \subset \left[0, \frac{1}{m+1}\right]$ and $\operatorname{supp} \varpi \subset \left[0, \frac{m}{m+1}\right]$, and a change of variables,

$$\begin{aligned} \|u_h - u\|_{L^p(\mathbb{R})} &\leq \frac{1}{h} \int_{\mathbb{R}} (\omega * \varpi)\left(\frac{\eta}{h}\right) \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} \, d\eta \\ &\leq \frac{C}{h} \int_0^h \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} \, d\eta, \end{aligned}$$

which shows (12).

To prove (11), fix $x \in \mathbb{R}$ and $\xi > 0$. By Exercise 9,

$$\Delta^{\xi,m} u_h(x) = \xi^m \int_0^1 \cdots \int_0^1 u_h^{(m)}(x + (y_1 + \cdots + y_m)\xi) \, dy_1 \cdots dy_m. \qquad (16)$$

By Theorem C.20, the previous lemma, and the fact that $\psi$ is even, we have

$$\begin{aligned} u_h^{(m)}(x) &= \left(u * \psi_h^{(m)}\right)(x) = \int_{\mathbb{R}} \psi_h^{(m)}(-\eta) \, u(x+\eta) \, d\eta \\ &= (-1)^m \int_{\mathbb{R}} (\omega * \varpi)_h^{(m)}(\eta) \, \Delta^{\eta,m} u(x) \, d\eta, \end{aligned}$$

and so, again by Corollary B.83, Remark and the facts that that $\operatorname{supp} \omega \subset \left[-\frac{1}{m+1}, \frac{1}{m+1}\right]$, $\operatorname{supp} \varpi \subset \left[-\frac{m}{m+1}, \frac{m}{m+1}\right]$, and $\left\|(\omega * \varpi)_h^{(m)}\right\|_\infty \leq \frac{C}{h^{m+1}}$,

$$\begin{aligned} &\left\|u_h^{(m)}(\cdot + (y_1 + \cdots + y_m)\xi)\right\|_{L^p(\mathbb{R})} = \left\|u_h^{(m)}\right\|_{L^p(\mathbb{R})} \\ &\leq \xi^m \int_{\mathbb{R}} \left|(\omega * \varpi)_h^{(m)}(\eta)\right| \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} \, d\eta \\ &\leq \frac{C\xi^m}{h^{m+1}} \int_{-h}^h \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} \, d\eta \leq \frac{C\xi^m}{h^{m+1}} \int_0^h \|\Delta^{\eta,m} u\|_{L^p(\mathbb{R})} \, d\eta, \end{aligned}$$

where in the last inequality we have used (2) and a change of variables.Together with (16), this gives

$$\left\|\Delta^{\xi,m}u_h\right\|_{L^p(\mathbb{R})} \leq \frac{C\xi^m}{h^{m+1}} \int_0^h \left\|\Delta^{\eta,m}u\right\|_{L^p(\mathbb{R})} d\eta,$$

which yields (11). ■

**Remark 11** *The previous proof is due to Jonsson and Wallin [2] in the case* $0 < s \leq 1$*. It provides an alternative proof of Lemma 14.18.*

**Proof of Theorem 7.** The proof is the same of Theorem 14.17, with the only difference that we use the previous lemma in place of Lemma 14.18. ■

# 3 Dependence of $B^{s,p,\theta}$ on $s$

In this section we prove that for $0 < t < s$,

$$B^{s,p,\theta_1}\left(\mathbb{R}^N\right) \subset B^{t,p,\theta_2}\left(\mathbb{R}^N\right).$$

**Theorem 12** *Let* $1 \leq p, \theta_1, \theta_2 \leq \infty$ *and* $0 < t < s$*. Then there exists a constant* $C = C\left(p, \theta, t, s\right) > 0$ *such that*

$$|u|_{B^{t,p,\theta_2}(\mathbb{R}^N)} \leq C\,|u|_{B^{s,p,\theta_1}(\mathbb{R}^N)} + C\,\|u\|_{L^p(\mathbb{R}^N)}$$

*for all* $u \in B^{s,p,\theta_1}\left(\mathbb{R}^N\right)$*. In particular,* $B^{s,p,\theta_1}\left(\mathbb{R}^N\right) \subset B^{t,p,\theta_2}\left(\mathbb{R}^N\right)$*.*

**Proof.** Let $u \in B^{s,p,\theta_1}\left(\mathbb{R}^N\right)$. If $\theta_1 < \theta_2$, then by Theorem 7,

$$|u|_{B^{t,p,\theta_2}(\mathbb{R}^N)} \leq C\,|u|_{B^{t,p,\theta_1}(\mathbb{R}^N)}.$$

Thus, it remains to show that

$$|u|_{B^{t,p,\min\{\theta_1,\theta_2\}}(\mathbb{R}^N)} \leq C\,|u|_{B^{s,p,\theta_1}(\mathbb{R}^N)} + C\,\|u\|_{L^p(\mathbb{R}^N)}.$$

Let $\theta = \min\{\theta_1, \theta_2\}$ and $m = \lfloor s \rfloor + 1$. Reasoning as in the proof of Theorem 14.7 (see (14.9)), we have that

$$\begin{aligned}|u|^{(m)}_{B^{t,p,\theta}(\mathbb{R}^N)} \leq & \left(\int_0^1 \left\|\Delta_i^{h,m}u\right\|^{\theta}_{L^p(\mathbb{R}^N)} \frac{dh}{h^{1+t\theta}}\right)^{\frac{1}{\theta}} \\ & + C\,\|u\|_{L^p(\mathbb{R}^N)} \left(\int_1^\infty \frac{1}{h^{1+t\theta}}\,dh\right)^{\frac{1}{\theta}} =: I + II\end{aligned} \tag{17}$$

if $1 \leq \theta < \infty$, while

$$|u|^{(m)}_{B^{t,p,\infty}(\mathbb{R}^N)} \leq \sup_{0<h<1} \frac{1}{h^s}\left\|\Delta_i^{h,m}u\right\|_{L^p(\mathbb{R}^N)} + C\,\|u\|_{L^p(\mathbb{R}^N)}$$

if $\theta = \infty$. It remains to estimate $I$ in (17). If $\theta < \theta_1 < \infty$, by Hölder's inequality

$$\begin{aligned} I &= \left( \int_0^1 \frac{h^{(s-t)\theta}}{h^{(s-t)\theta}} \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}^{\theta} \frac{dh}{h^{1+t\theta}} \right)^{\frac{1}{\theta}} \\ &\le \left( \int_0^1 \left( h^{(s-t)\theta} \right)^{\left(\frac{\theta_1}{\theta}\right)'} \frac{dh}{h} \right)^{\frac{1}{\left(\frac{\theta_1}{\theta}\right)'}} \left( \int_0^1 \left\| \Delta_i^{h,m} u \right\|_{L^p(\mathbb{R}^N)}^{\theta_1} \frac{dh}{h^{1+t\theta_1}} \right)^{\frac{1}{\theta_1}} . \end{aligned}$$

Note that the first integral on the the right-hand side is finite. If $\theta < \theta_1 = \infty$, then

$$I \le \left( \sup_{0<\eta<1} \frac{1}{\eta^s} \left\| \Delta_i^{\eta,m} u \right\|_{L^p(\mathbb{R}^N)} \right) \left( \int_0^1 h^{s\theta} \frac{dh}{h^{1+t\theta}} \right)^{\frac{1}{\theta}} ,$$

which is again finite. Finally, if $\theta = \theta_1 < \infty$, then as in the proof of Theorem 14.7 (see (14.9)) we use the fact that $\frac{1}{h^t} < \frac{1}{h^s}$ for $0 < h < 1$. The result now follows, in view of Proposition 6. This concludes the proof. ■

# 4 A Characterization of $B^{s,p,\theta}$ for $0 < s < 1$

In this section we give a characterization of $B^{s,p,\theta}\left(\mathbb{R}^N\right)$ for $0 < s < 1$. We will show that a function $v \in L^p\left(\mathbb{R}^N\right)$ belongs to $B^{s,p,\theta}\left(\mathbb{R}^N\right)$ if and only if it is the trace of a function $v : \mathbb{R}_+^{N+1} \to \mathbb{R}$ in some weighted Sobolev space. In the special case $\theta = p$ and $s = 1 - \frac{1}{p}$, we will recover Theorem ....

In what follows we set $M := N + 1$, and we write

$$z = (z_1, \ldots, z_{M-1}, z_M) = (z', z_M) \in \mathbb{R}^N \times \mathbb{R}.$$

With a slight abuse of notation, for every $i = 1, \ldots, N$ we also write

$$z' = (z_i'', z_i) \in \mathbb{R}^{M-2} \times \mathbb{R}.$$

Let $\varphi \in C_c^\infty\left(\mathbb{R}^N\right)$ be such that $\operatorname{supp} \varphi \subset \overline{B_N(0,1)}$ and

$$\int_{\mathbb{R}^N} \varphi(z')\, dz' = 1.$$

Given $u \in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$, for $z' \in \mathbb{R}^N$ and $z_M > 0$ define

$$v(z) := \frac{1}{z_M^N} \int_{\mathbb{R}^N} \varphi\left(\frac{z' - y'}{z_M}\right) u(y')\, dy'. \tag{18}$$

Note that $v$ is just the mollification of $u$ with $z_M$ in place of $\varepsilon$.

The proof of the following theorems will make use of the Hardy's inequality

$$\left( \int_a^b \left| \frac{1}{\tau^s} \int_a^\tau f(t)\, dt \right|^p dx \right)^{\frac{1}{p}} \le \frac{1}{s - \frac{1}{p}} \left( \int_a^b \left| \frac{1}{\tau^{s-1}} f(\tau) \right|^p d\tau \right)^{\frac{1}{p}} , \tag{19}$$

where $f : (a, b) \to \mathbb{R}$ is any measurable function, $0 \le a < b \le \infty$, $1 \le p < \infty$ and $s > \frac{1}{p}$. For a proof see, e.g., Theorem 8 in the pdf file on "An extension of the Sobolev–Gagliardo–Nirenberg theorem".

**Theorem 13** *Let $1 \leq p, \theta \leq \infty$, $0 < s < 1$, and $M \geq 2$. Given $u \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$, let $v$ be the function given in (18). Then*

$$\begin{aligned}\frac{s}{N}\left|u\right|_{B^{s,p,\theta}(\mathbb{R}^N)} &\leq \sum_{i=1}^{M}\left(\int_0^{\infty}\left\|\frac{\partial v}{\partial z_i}\left(\cdot, z_M\right)\right\|_{L^p(\mathbb{R}^N)}^{\theta}\frac{dz_M}{z_M^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}} \\ &\leq \frac{C}{1+s}\left|u\right|_{B^{s,p,\theta}(\mathbb{R}^N)}\end{aligned}$$

*if $1 \leq \theta < \infty$, while*

$$\begin{aligned}\frac{s}{N}\left|u\right|_{B^{s,p,\infty}(\mathbb{R}^N)} &\leq \sum_{i=1}^{M}\sup_{z_M>0}\left(z_M^{1-s}\left\|\frac{\partial v}{\partial z_i}\left(\cdot, z_M\right)\right\|_{L^p(\mathbb{R}^N)}\right) \\ &\leq \frac{C}{1+s}\left|u\right|_{B^{s,p,\infty}(\mathbb{R}^N)}\end{aligned}$$

*if $\theta = \infty$, where $C = C\left(M, p\right) > 0$.*

**Proof.** **Step 1:** By Theorem C.20 (where $z_M$ plays the role of $\varepsilon$), for any $i = 1, \ldots, N$ and $z \in \mathbb{R}^M_+$ we have that

$$\begin{aligned}\frac{\partial v}{\partial z_i}\left(z\right) &= \frac{1}{z_M^M}\int_{\mathbb{R}^N}\frac{\partial \varphi}{\partial z_i}\left(\frac{z'-y'}{z_M}\right)u\left(y'\right)\,dy' \\ &= \frac{1}{z_M^M}\int_{\mathbb{R}^N}\frac{\partial \varphi}{\partial z_i}\left(\frac{z'-y'}{z_M}\right)\left[u\left(y'\right)-u\left(y_i'', z_i\right)\right]\,dy' \\ &= \frac{1}{z_M^M}\int_{\mathbb{R}^N}\frac{\partial \varphi}{\partial z_i}\left(\frac{y'}{z_M}\right)\left[u\left(z'-y'\right)-u\left(z_i''-y_i'', z_i\right)\right]\,dy',\end{aligned}$$

where in the second equality we used the fact that

$$\int_{\mathbb{R}}\frac{\partial \varphi}{\partial z_i}\left(\frac{z'-y'}{z_M}\right)u\left(y_i'', z_i\right)\,dy_i = u\left(y_i'', z_i\right)\int_{\mathbb{R}}\frac{\partial \varphi}{\partial z_i}\left(\frac{z'-y'}{z_M}\right)\,dy_i = 0.$$

Since $\operatorname{supp}\varphi \subset \overline{B_N\left(0,1\right)} \subset \overline{B_{M-2}\left(0, z_M\right)} \times \left[-z_M, z_M\right]$, by Tonelli's theorem we have that

$$\left|\frac{\partial v}{\partial z_i}\left(z\right)\right| \leq \frac{C}{z_M^M}\int_{B_{M-2}(0,z_M)}\int_{-z_M}^{z_M}\left|u\left(z_i''-y_i'', z_i-y_i\right)-u\left(z_i''-y_i'', z_i\right)\right|\,dy_i dy_i''.$$

By taking the $L^p\left(\mathbb{R}^N\right)$ norm in $z'$ on both sides, using Corollary B.83, and changing variables, for all $z_M > 0$, we get

$$\begin{aligned}\left\|\frac{\partial v}{\partial z_i}\left(\cdot, z_M\right)\right\|_{L^p(\mathbb{R}^N)} &\leq \frac{C}{z_M^M}\int_{B_{M-2}(0,z_M)}\int_{-z_M}^{z_M}\left\|\Delta_i^{y_i}u\right\|_{L^p(\mathbb{R}^N)}\,dy_i dy_i'' \qquad (20) \\ &\leq \frac{C}{z_M^2}\int_{-z_M}^{z_M}\left\|\Delta_i^{y_i}u\right\|_{L^p(\mathbb{R}^N)}\,dy_i \leq \frac{C}{z_M^2}\int_0^{z_M}\left\|\Delta_i^{y_i}u\right\|_{L^p(\mathbb{R}^N)}\,dy_i,\end{aligned}$$

where in the last inequality we have used the fact that for $y_i < 0$,

$$\Delta_i^{y_i} u(x) = -\Delta_i^{-y_i} u(x + y_i e_i)$$

by (2), together with the change of variables $w = x + y_i e_i$.

If $\theta < \infty$, multiply both sides of (20) by $\frac{1}{z_M^a}$, where

$$a := \frac{1}{\theta} - (1-s),$$

and take the $L^\theta((0,\infty))$ norm in $z_M$ to obtain

$$\left(\int_0^\infty \left\|\frac{\partial v}{\partial z_i}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}} \leq C\left(\int_0^\infty \left(\frac{1}{z_M^{1+\frac{1}{\theta}+s}} \int_0^{z_M} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)}\, dy_i\right)^\theta dz_M\right)^{\frac{1}{\theta}}.$$

By applying Hardy's inequality to the right-hand side, we get

$$\left(\int_0^\infty \left\|\frac{\partial v}{\partial z_i}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}} \leq \frac{C}{1+s}\left(\int_0^\infty \|\Delta_i^{z_M} u\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1+s\theta}}\right)^{\frac{1}{\theta}} \leq \frac{C}{1+s}|u|_{B^{s,p,\theta}(\mathbb{R}^N)}.$$

On the other hand, if $\theta = \infty$, we have that

$$\begin{aligned}\int_0^{z_M} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)}\, dy_i &= \int_0^{z_M} \frac{y_i^s}{y_i^s} \|\Delta_i^{y_i} u\|_{L^p(\mathbb{R}^N)}\, dy_i \\ &\leq \left(\int_0^{z_M} y_i^s\, dy_i\right) \sup_{h>0} \frac{1}{h^s} \left\|\Delta_i^h u\right\|_{L^p(\mathbb{R}^N)} \\ &= \frac{z_M^{1+s}}{1+s} \sup_{h>0} \frac{1}{h^s} \left\|\Delta_i^h u\right\|_{L^p(\mathbb{R}^N)},\end{aligned} \tag{21}$$

and so by (20),

$$z_M^{1-s} \left\|\frac{\partial v}{\partial z_i}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)} \leq \frac{C}{1+s}|u|_{B^{s,p,\infty}(\mathbb{R}^N)},$$

which implies that

$$\sup_{z_M>0}\left(z_M^{1-s} \left\|\frac{\partial v}{\partial z_i}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}\right) \leq \frac{C}{1+s}|u|_{B^{s,p,\infty}(\mathbb{R}^N)}.$$

To estimate $\frac{\partial v}{\partial z_M}$, we write

$$
\begin{aligned}
& u\left(y'\right)-u\left(z'\right) \\
& \quad=\sum_{i=1}^{N}\left[u\left(y_1, \ldots, y_i, z_{i+1}, \ldots, z_N\right)-u\left(y_1, \ldots, y_{i-1}, z_i, \ldots, z_N\right)\right] .
\end{aligned}
$$

Since $\int_{\mathbb{R}^N} \varphi\left(z'\right) d z'=1$, we have that

$$
\begin{aligned}
v\left(z', z_M\right)= & \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{z_M^N} \varphi\left(\frac{z'-y'}{z_M}\right) \\
& \times\left[u\left(y_1, \ldots, y_i, z_{i+1}, \ldots, z_N\right)-u\left(y_1, \ldots, y_{i-1}, z_i, \ldots, z_N\right)\right] d y' \\
& +u\left(z'\right) .
\end{aligned} \tag{22}
$$

By Theorem B.53 we obtain that

$$
\begin{aligned}
\frac{\partial v}{\partial z_M}\left(z', z_M\right)= & \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{\partial}{\partial z_M}\left(\frac{1}{z_M^N} \varphi\left(\frac{z'-y'}{z_M}\right)\right) \\
& \times\left[u\left(y_1, \ldots, y_i, z_{i+1}, \ldots, z_N\right)-u\left(y_1, \ldots, y_{i-1}, z_i, \ldots, z_N\right)\right] d y' .
\end{aligned}
$$

In turn,

$$
\begin{aligned}
& \left|\frac{\partial v}{\partial z_M}\left(z', z_M\right)\right| \\
& \quad \leq \sum_{i=1}^{N} \frac{C}{z_M^M} \int_{B_N\left(0, z_M\right)}\left|u\left(z_1-y_1, \ldots, z_i-y_i, z_{i+1}, \ldots, z_N\right)\right. \\
& \qquad\qquad \left.-u\left(z_1-y_1, \ldots, z_{i-1}-y_{i-1}, z_i, \ldots, z_N\right)\right| d y'
\end{aligned}
$$

We can now continue as before (see (20)) to conclude that

$$
\left(\int_0^{\infty}\left\|\frac{\partial v}{\partial z_M}\left(\cdot, z_M\right)\right\|_{L^p\left(\mathbb{R}^N\right)}^{\theta} \frac{d z_M}{z_M^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}} \leq \frac{C}{1+s}|u|_{B^{s, p, \theta}\left(\mathbb{R}^N\right)}
$$

if $1 \leq \theta<\infty$, while

$$
\sup_{z_M>0}\left(z_M^{1-s}\left\|\frac{\partial v}{\partial z_M}\left(\cdot, z_M\right)\right\|_{L^p\left(\mathbb{R}^N\right)}\right) \leq \frac{C}{1+s}|u|_{B^{s, p, \infty}\left(\mathbb{R}^N\right)}
$$

if $\theta=\infty$.

**Step 2:** To prove the other inequality, assume that $u \in C^1\left(\mathbb{R}^N\right)$. For $z' \in \mathbb{R}^N$, $h>0$ and $i=1, \ldots, N$, we have

$$
\begin{aligned}
\Delta_i^h u\left(z'\right)=u\left(z'+h e_i'\right)-u\left(z'\right)= & {\left[v\left(z'+h e_i', h\right)-v\left(z', h\right)\right] } \\
& +\left[u\left(z'+h e_i'\right)-v\left(z'+h e_i', h\right)\right]+\left[v\left(z', h\right)-u\left(z'\right)\right],
\end{aligned} \tag{23}
$$

where $e_i = (e_i', 0) \in \mathbb{R}^N \times \mathbb{R}$. We now estimate the various terms in (23). By the fundamental theorem of calculus

$$\begin{aligned} v(z' + he_i', h) - v(z', h) &= \int_0^h \frac{\partial v}{\partial z_i}(z_i'', z_i + t, h)\, dt, \\ v(z' + he_i', h) - u(z' + he_i') &= \int_0^h \frac{\partial v}{\partial z_M}(z_i'', z_i + h, t)\, dt, \\ v(z', h) - u(z') &= \int_0^h \frac{\partial v}{\partial z_M}(z', t)\, dt. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\Delta_i^h u(z')\right| \le & \int_0^h \left|\frac{\partial v}{\partial z_i}(z_i'', z_i + t, h)\right| dt \\ & + \int_0^h \left(\left|\frac{\partial v}{\partial z_M}(z_i'', z_i + h, t)\right| + \left|\frac{\partial v}{\partial z_M}(z', t)\right|\right) dt. \end{aligned} \tag{24}$$

If $1 \le p < \infty$, by Corollary B.83, and a change of variables,

$$\begin{aligned} \left\|\Delta_i^h u\right\|_{L^p(\mathbb{R}^N)} \le & \left(\int_{\mathbb{R}^N} \left(\int_0^h \left|\frac{\partial v}{\partial z_i}(z_i'', z_i + t, h)\right| dt\right)^p dz'\right)^{\frac{1}{p}} \\ & + \int_0^h \left\|\frac{\partial v}{\partial z_M}(\cdot, t)\right\|_{L^p(\mathbb{R}^N)} dt =: I + II. \end{aligned}$$

To estimate $I$, note that if $M > 2$, then by Tonelli's theorem and Corollary B.83,

$$\begin{aligned} I &\le \left(\int_{\mathbb{R}} \left(\int_0^h \left\|\frac{\partial v}{\partial z_i}(\cdot, z_i + t, h)\right\|_{L^p(\mathbb{R}^{M-2})} dt\right)^p dz_i\right)^{\frac{1}{p}} \\ &\le \left(\int_{\mathbb{R}} h^{\frac{p}{p'}} \int_0^h \left\|\frac{\partial v}{\partial z_i}(\cdot, z_i + t, h)\right\|_{L^p(\mathbb{R}^{M-2})}^p dt dz_i\right)^{\frac{1}{p}} \\ &= h \left\|\frac{\partial v}{\partial z_i}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}, \end{aligned} \tag{25}$$

where in the second inequality we have used Hölder's inequality and in the last equality, Tonelli's theorem and a change of variables. Hence, we have proved that

$$\left\|\Delta_i^h u\right\|_{L^p(\mathbb{R}^N)} \le h \left\|\frac{\partial v}{\partial z_i}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)} + \int_0^h \left\|\frac{\partial v}{\partial z_M}(\cdot, t)\right\|_{L^p(\mathbb{R}^N)} dt. \tag{26}$$

Note that in view of (24), the same inequality holds if $p = \infty$.

If $1 \leq \theta < \infty$, multiply both sides of the previous inequality by $\frac{1}{h^{\frac{1}{\theta}+s}}$ and then and take the $L^\theta((0,\infty))$ norm in $h$ to obtain,

$$\left(\int_0^\infty \left\|\Delta_i^h u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} \leq \left(\int_0^\infty \left\|\frac{\partial v}{\partial z_i}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}}$$

$$+\left(\int_0^\infty \left(\frac{1}{h^{\frac{1}{\theta}+s}}\int_0^h \left\|\frac{\partial v}{\partial z_M}(\cdot,t)\right\|_{L^p(\mathbb{R}^N)} dt\right)^\theta dh\right)^{\frac{1}{\theta}}$$

$$\leq \left(\int_0^\infty \left\|\frac{\partial v}{\partial z_i}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}}$$

$$+\frac{1}{s}\left(\int_0^\infty \left\|\frac{\partial v}{\partial z_M}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}},$$

where in the last inequality we have used Hardy's inequality. It follows that

$$|u|_{B^{s,p,\theta}(\mathbb{R}^N)} \leq \sum_{i=1}^N \left(\int_0^\infty \left\|\frac{\partial v}{\partial z_i}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}} +$$

$$+\frac{N}{s}\left(\int_0^\infty \left\|\frac{\partial v}{\partial z_M}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(1-s)}}\right)^{\frac{1}{\theta}}.$$

On the other hand, if $\theta = \infty$, multiply both sides of (26) by $\frac{1}{h^s}$ and proceed as in (21) to obtain

$$\sup_{h>0} \frac{1}{h^s}\left\|\Delta_i^h u\right\|_{L^p(\mathbb{R}^N)} \leq \sup_{h>0} h^{1-s}\left\|\frac{\partial v}{\partial z_i}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}$$

$$+\frac{1}{s}\sup_{h>0} h^{1-s}\left\|\frac{\partial v}{\partial z_M}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}.$$

It follows that

$$|u|_{B^{s,p,\infty}(\mathbb{R}^N)} \leq \sum_{i=1}^N \sup_{h>0} h^{1-s}\left\|\frac{\partial v}{\partial z_i}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)} +$$

$$+\frac{N}{s}\sup_{h>0} h^{1-s}\left\|\frac{\partial v}{\partial z_M}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}.$$

To remove the extra assumption that $u \in C^1(\mathbb{R}^N)$, we use a mollification argument. We omit the details. This concludes the proof. ■

## 5 A Characterization of $B^{1,p,\theta}$

In this section we prove the analog of Theorem 13 for $B^{1,p,\theta}$. In what follows, we assume that the function $\varphi$ in the previous section is assumed to be even in each variable $z_i$.

**Theorem 14** *Let $1 \leq p, \theta \leq \infty$ and $M \geq 2$. Given $u \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$, let $v$ be the function given in (18), where $\varphi$ is even in each variable $z_i$, $i = 1, \dots, N$. Then*

$$\frac{1}{C}\left|u\right|_{B^{1,p,\theta}(\mathbb{R}^N)} \leq \sum_{i,j=1}^{M}\left(\int_0^\infty \left\|\frac{\partial^2 v}{\partial z_i \partial z_j}\left(\cdot, z_M\right)\right\|^\theta_{L^p(\mathbb{R}^N)} \frac{dz_M}{z_M^{1-\theta}}\right)^{\frac{1}{\theta}}$$
$$\leq C\left|u\right|_{B^{1,p,\theta}(\mathbb{R}^N)}$$

*if $1 \leq \theta < \infty$, while*

$$\frac{1}{C}\left|u\right|_{B^{1,p,\infty}(\mathbb{R}^N)} \leq \sum_{i,j=1}^{M}\sup_{z_M>0}\left(z_M\left\|\frac{\partial^2 v}{\partial z_i \partial z_j}\left(\cdot, z_M\right)\right\|_{L^p(\mathbb{R}^N)}\right)$$
$$\leq C\left|u\right|_{B^{1,p,\infty}(\mathbb{R}^N)}$$

*if $\theta = \infty$, where $C = C\left(M, p\right) > 0$.*

**Proof. Step 1:** By Theorem C.20, for every $i, j = 1, \dots, N$, we have that

$$\frac{\partial^2 v}{\partial z_i \partial z_j}\left(z\right) = \frac{1}{z_M^{M+1}}\int_{\mathbb{R}^N}\frac{\partial^2 \varphi}{\partial z_i \partial z_j}\left(\frac{y'}{z_M}\right)u\left(z' - y'\right)\,dy'$$
$$= \frac{1}{z_M^{M+1}}\int_{\mathbb{R}^{M-2}}\int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j}\left(\frac{y'}{z_M}\right)\left[u\left(z_i'' - y_i'', z_i - y_i\right) + u\left(z_i'' - y_i'', z_i + y_i\right)\right]\,dy'dy_i'' \tag{27}$$
$$= \frac{1}{z_M^{M+1}}\int_{\mathbb{R}^{M-2}}\int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j}\left(\frac{y'}{z_M}\right)\widetilde{\Delta}_i^{y_i,2}u\left(z_i'' - y_i'', z_i\right)\,dy_i dy_i'',$$

where in the second equality we have used the fact that $\varphi$ is even in the $z_i$ variable and in the last equality the fact that

$$2\int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j}\left(\frac{y'}{z_M}\right)u\left(z_i'' - y_i'', z_i\right)\,dy_i$$
$$= 2u\left(z_i'' - y_i'', z_i\right)\int_0^\infty \frac{\partial^2 \varphi}{\partial z_i \partial z_j}\left(\frac{y_i'', y_i}{z_M}\right)\,dy_i = 0.$$

Since $\varphi \in C_c^\infty\left(B\left(0,1\right)\right)$, we have that

$$\left|\frac{\partial^2 v}{\partial z_i \partial z_j}\left(z\right)\right| \leq \frac{c}{z_M^{M+1}}\int_{B_{M-2}(0,z_M)}\int_0^{z_M}\left|\widetilde{\Delta}_i^{y_i,2}u\left(z_i'' - y_i'', z_i\right)\right|\,dy_i dy_i''. \tag{28}$$

Reasoning as in (20), we get

$$\left\|\frac{\partial^2 v}{\partial z_i \partial z_j}\left(\cdot, z_M\right)\right\|_{L^p(\mathbb{R}^N)} \leq \frac{C}{z_M^3}\int_0^{z_M}\left\|\widetilde{\Delta}_i^{y_i,2}u\right\|_{L^p(\mathbb{R}^N)}\,dy_i \tag{29}$$
$$= \frac{C}{z_M^3}\int_0^{z_M}\left\|\Delta_i^{y_i,2}u\right\|_{L^p(\mathbb{R}^N)}\,dy_i.$$

for all $z_M > 0$. If $\theta < \infty$, multiply both sides of the previous inequality by $\frac{1}{z_M^a}$, where

$$a := \frac{1}{\theta} - 1,$$

and take the $L^\theta((0,\infty))$ norm in $z_M$ to obtain

$$\left(\int_0^\infty \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta}}\right)^{\frac{1}{\theta}}$$
$$\leq C\left(\int_0^\infty \left(\frac{1}{z_M^{2+\frac{1}{\theta}}}\int_0^{z_M} \left\|\Delta_i^{y_i,2} u\right\|_{L^p(\mathbb{R}^N)} dy_i\right)^\theta dz_M\right)^{\frac{1}{\theta}}.$$

By applying Hardy's inequality to the right-hand side, we get

$$\left(\int_0^\infty \left\| \frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta}}\right)^{\frac{1}{\theta}} \leq \frac{C}{2}\left(\int_0^\infty \left\|\Delta_i^{z_M,2} u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1+\theta}}\right)^{\frac{1}{\theta}}$$
$$\leq C\,|u|_{B^{s,p,\theta}(\mathbb{R}^N)}.$$

On the other hand, if $\theta = \infty$, reasoning as in (21), we have that

$$\int_0^{z_M} \left\|\Delta_i^{y_i,2} u\right\|_{L^p(\mathbb{R}^N)} dy_i \leq \frac{z_M^2}{2}\sup_{h>0}\frac{1}{h^s}\left\|\Delta_i^{h,2} u\right\|_{L^p(\mathbb{R}^N)},$$

and so by (29),

$$\sup_{z_M>0}\left(z_M \left\|\frac{\partial^2 v}{\partial z_i \partial z_j}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}\right) \leq |u|_{B^{1,p,\infty}(\mathbb{R}^N)}.$$

To estimate $\frac{\partial^2 v}{\partial z_M^2}$, write

$$\begin{aligned}
&v(z', z_M) \\
&= \frac{1}{z_M^N}\int_{\mathbb{R}^{M-2}}\int_0^\infty \varphi\left(\frac{y'}{z_M}\right)\left[u(z_i'' - y_i'', z_i - y_i) + u(z_i'' - y_i'', z_i + y_i)\right] dy' dy_i'' \\
&= \frac{1}{z_M^N}\int_{\mathbb{R}^N}\int_0^\infty \varphi\left(\frac{y'}{z_M}\right)\widetilde{\Delta}_i^{y_i,2} u(z_i'' - y_i'', z_i)\, dy' dy_i'' + 2u(z'),
\end{aligned}$$

where in the first equality we have used the fact that $\varphi$ is even in the $z_M$ variable and in the second the fact that $\int_{\mathbb{R}^N}\varphi(z')\,dz' = 1$. Hence, by differentiating with respect to $z_M$ and using the fact that $\varphi \in C_c^\infty(B(0,1))$, we obtain

$$\left|\frac{\partial^2 v}{\partial z_M^2}(z', z_M)\right| \leq \frac{c}{z_M^{M+1}}\sum_{i=1}^N \int_{B_{M-2}(0,z_M)}\int_0^{z_M}\left|\widetilde{\Delta}_i^{y_i,2} u(z_i'' - y_i'', z_i)\right| dy_i' dy_i''.$$

We can now proceed as in (28) to prove that

$$\left(\int_0^\infty \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta}}\right)^{\frac{1}{\theta}} \leq C\,|u|_{B^{s,p,\theta}(\mathbb{R}^N)}$$

if $\theta < \infty$, while

$$\sup_{z_M>0}\left(z_M \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}\right) \leq |u|_{B^{1,p,\infty}(\mathbb{R}^N)}\,.$$

Similar estimates hold for $\frac{\partial^2 v}{\partial z_i \partial z_M}$.

**Step 2:** Assume that $u \in C^2\left(\mathbb{R}^N\right)$. For $z' \in \mathbb{R}^N$, $h > 0$ and $i = 1, \ldots, N$, we have

$$\begin{aligned}\Delta_i^{h,2} u\left(z'\right) = &-\Delta_i^{h,2} v\left(z', 2h\right) + 2\Delta_i^{h,2} v\left(z', h\right)\\ &+ \Delta_M^{h,2} v\left(z' + 2he_i, 0\right) - 2\Delta_M^{h,2} v\left(z' + he_i, 0\right) + \Delta_M^{h,2} v\left(z', 0\right).\end{aligned}$$

We now estimate the various terms on the right-hand side. By Exercise 9,

$$\begin{aligned}\left|\Delta_i^{h,2} u\left(z'\right)\right| \leq &\int_0^{2h} \tau \left(\left|\frac{\partial^2 v}{\partial z_i^2}\left(z_i'', z_i + \tau, 2h\right)\right| + 2\left|\frac{\partial^2 v}{\partial z_i^2}\left(z_i'', z_i + \tau, h\right)\right|\right) d\tau\\ + &\int_0^{2h} \tau \left(\left|\frac{\partial^2 v}{\partial z_M^2}\left(z_i'', z_i + 2h, \tau\right)\right| + 2\left|\frac{\partial^2 v}{\partial z_M^2}\left(z_i'', z_i + h, \tau\right)\right| + \left|\frac{\partial^2 v}{\partial z_M^2}\left(z_i'', z_i, \tau\right)\right|\right) d\tau.\end{aligned}$$

If $1 \leq p < \infty$, by Corollary B.83 and a change of variables,

$$\begin{aligned}\left\|\Delta_i^{h,2} u\right\|_{L^p(\mathbb{R}^N)} \leq &\left(\int_{\mathbb{R}^N}\left(\int_0^{2h} \tau \left|\frac{\partial^2 v}{\partial z_i^2}\left(z_i'', z_i + \tau, 2h\right)\right| d\tau\right)^p dz'\right)^{\frac{1}{p}}\\ &+ 2\left(\int_{\mathbb{R}^N}\left(\int_0^{2h} \tau \left|\frac{\partial^2 v}{\partial z_i^2}\left(z_i'', z_i + \tau, h\right)\right| d\tau\right)^p dz'\right)^{\frac{1}{p}}\\ &+ 4\int_0^{2h} \tau \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)} d\tau dt =: I + II + III.\end{aligned}$$

To estimate $I$ and $II$, we reason as in (25) to obtain

$$I + II \leq 2h^2 \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, 2h)\right\|_{L^p(\mathbb{R}^N)} + 4h^2 \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}.$$

Hence,

$$\begin{aligned}\left\|\Delta_i^{h,2} u\right\|_{L^p(\mathbb{R}^N)} \leq &\, 2h^2 \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, 2h)\right\|_{L^p(\mathbb{R}^N)} + 4h^2 \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}\\ &+ 4\int_0^{2h} \tau \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)} d\tau.\end{aligned} \tag{30}$$

The same inequality holds if $p = \infty$.

If $1 \le \theta < \infty$, multiply both sides of the previous inequality by $\frac{1}{h^{\frac{1}{\theta}+1}}$, take the $L^{\theta}((0,\infty))$ norm in $h$, and use the change of variables $h' = 2h$, to obtain,

$$\left(\int_0^{\infty} \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)}^{\theta} \frac{dh}{h^{1+\theta}}\right)^{\frac{1}{\theta}} \le 5 \left(\int_0^{\infty} \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^{\theta} \frac{dh}{h^{1-\theta}}\right)^{\frac{1}{\theta}}$$
$$+ 4 \left(\int_0^{\infty} \left(\frac{1}{h^{\frac{1}{\theta}+1}} \int_0^{2h} \tau \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)} d\tau\right)^{\theta} dh\right)^{\frac{1}{\theta}} =: A + B.$$

To estimate $B$, we apply Hardy's inequality and a change of variables, to obtain

$$B \le C \left(\int_0^{\infty} \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^{\theta} \frac{dh}{h^{1-\theta}}\right)^{\frac{1}{\theta}}.$$

It follows that

$$|u|_{B^{1,p,\theta}(\mathbb{R}^N)} \le 5 \sum_{i=1}^{N} \left(\int_0^{\infty} \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^{\theta} \frac{dh}{h^{1-\theta}}\right)^{\frac{1}{\theta}} +$$
$$+ NC \left(\int_0^{\infty} \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^{\theta} \frac{dh}{h^{1-\theta}}\right)^{\frac{1}{\theta}}.$$

if $\theta < \infty$, and

$$|u|_{B^{1,p,\infty}(\mathbb{R}^M)} := \sum_{i=1}^{M} \sup_{h>0} \frac{1}{h} \left\|\Delta_i^{h,2} u\right\|_{L^p(\mathbb{R}^M)}$$

On the other hand, if $\theta = \infty$, multiply both sides of (30) by $\frac{1}{h}$ to obtain

$$\frac{1}{h} \left\|\Delta_i^{h,2} u\right\|_{L^p(\mathbb{R}^N)} \le 2h \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, 2h)\right\|_{L^p(\mathbb{R}^N)} + 4h \left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}$$
$$+ \frac{4}{h} \int_0^{2h} \tau \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)} d\tau dt =: E + F + G.$$

Since

$$G \le \sup_{\tau>0} \left(\tau \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)}\right) \left(\frac{4}{h} \int_0^{2h} d\tau\right)$$
$$= 8 \sup_{\tau>0} \left(\tau \left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)}\right),$$

we obtain

$$\sup_{h>0}\frac{1}{h}\left\|\Delta_i^{h,2}u\right\|_{L^p(\mathbb{R}^N)}\leq C\sup_{h>0}\left(h\left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}\right)$$
$$+C\sup_{h>0}\left(h\left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}\right).$$

It follows that

$$|u|_{B^{1,p,\infty}(\mathbb{R}^N)}\leq C\sum_{i=1}^N\sup_{h>0}\left(h\left\|\frac{\partial^2 v}{\partial z_i^2}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}\right)+$$
$$+NC\sup_{h>0}\left(h\left\|\frac{\partial^2 v}{\partial z_M^2}(\cdot,h)\right\|_{L^p(\mathbb{R}^N)}\right).$$

To remove the extra assumption that $u\in C^2\left(\mathbb{R}^N\right)$, we use a mollification argument. We omit the details. This concludes the proof. ■

# 6 A Characterization of $B^{s,p,\theta}$ for $s>0$

In this section we extend Theorems 13 and 14 to $B^{s,p,\theta}$ for all $s>0$. Note that although Theorem 15 includes Theorems 13 and 14 as special cases, for didactial purposes we have kept $0<s<1$ and $s=1$ separate,since the proof is more transparent in those cases. The function $\varphi$ used in the definition (18) should now be chosen more carefully.

Define

$$\varphi(z'):=\prod_{i=1}^N\psi(z_i),\quad z'\in\mathbb{R}^N, \tag{31}$$

where $\omega$ is the function defined in (9).

**Theorem 15** *Let $1\leq p,\theta\leq\infty$, $s>0$, $m\in\mathbb{N}$ with $m\geq\lfloor s\rfloor+1$, and $M\geq 2$. Given $u\in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$, let $v$ be the function given in (18), where $\varphi$ is given in (31). Then*

$$\frac{1}{C}|u|_{B^{s,p,\theta}(\mathbb{R}^N)}\leq\sum_{|\alpha|=m}\left(\int_0^\infty\left\|\frac{\partial^\alpha v}{\partial z^\alpha}(\cdot,z_M)\right\|^\theta_{L^p(\mathbb{R}^N)}\frac{dz_M}{z_M^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}} \tag{32}$$
$$\leq C|u|_{B^{s,p,\theta}(\mathbb{R}^N)}$$

*if $1\leq\theta<\infty$, while*

$$\frac{1}{C}|u|_{B^{s,p,\infty}(\mathbb{R}^N)}\leq\sum_{|\alpha|=m}\sup_{z_M>0}\left(z_M^{m-s}\left\|\frac{\partial^\alpha v}{\partial z^\alpha}(\cdot,z_M)\right\|_{L^p(\mathbb{R}^N)}\right) \tag{33}$$
$$\leq C|u|_{B^{s,p,\infty}(\mathbb{R}^N)}$$

*if $\theta=\infty$, where $C=C(M,p,s)>0$.*

We now turn to the proof of Theorem 15.

**Proof of Theorem 15.** In view of Proposition 6, it is enough to show that the inequalities (32) and (33) hold with $|u|_{B^{s,p,\theta}(\mathbb{R}^N)}$ replaced by the seminorm $|u|^{(m)}_{B^{s,p,\theta}(\mathbb{R}^N)}$ defined in (4).

**Step 1:** Let $\alpha \in \mathbb{N}_0^M$ be a multi-index of the form $\alpha = (\alpha_1, \ldots, \alpha_N, 0)$, with $|\alpha| = m$, and fix $i = 1, \ldots, N$ such that $\alpha_i > 0$. By Theorem C.20, (31), and the fact that $\psi$ is an even function of one variable, we have that

$$\frac{\partial^\alpha v}{\partial z^\alpha}(z) = \frac{1}{z_M^{M+m-1}} \int_{\mathbb{R}^N} \frac{\partial^\alpha \varphi}{\partial z^\alpha}\left(\frac{z'-y'}{z_M}\right) u(y')\, dy'$$

$$= \frac{1}{z_M^{M+m-1}} \int_{\mathbb{R}^{M-2}} \prod_{j\neq i} \psi^{(\alpha_j)}\left(\frac{z_j-y_j}{z_M}\right) \int_{\mathbb{R}} \psi^{(\alpha_i)}\left(\frac{z_i-y_i}{z_M}\right) u(y_i'', y_i)\, dy_i dy_i'' \tag{34}$$

$$= \frac{(-1)^{|\alpha_i|}}{z_M^{M+m-1}} \int_{\mathbb{R}^{M-2}} \prod_{j\neq i} \psi^{(\alpha_j)}\left(\frac{z_j-y_j}{z_M}\right) \int_{\mathbb{R}} \psi^{(\alpha_i)}\left(\frac{\eta_i}{z_N}\right) u(y_i'', z_i+\eta_i)\, d\eta_i dy_i''.$$

Using the notation (3), we have that

$$\frac{d^{(\alpha_i)}(\psi_{z_N})}{dt^{(\alpha_i)}}(t) = \frac{1}{z_M^{|\alpha_i|+1}} \frac{d^{(\alpha_i)}\psi}{dt^{(\alpha_i)}}\left(\frac{t}{z_N}\right), \quad t \in \mathbb{R},$$

and so, by Lemma 8,

$$\int_{\mathbb{R}} \psi^{(\alpha_i)}\left(\frac{\eta_i}{z_N}\right) u(y_i'', z_i+\eta_i)\, d\eta_i = \int_{\mathbb{R}} (\omega * \varpi)^{(\alpha_i)}\left(\frac{\eta_i}{z_N}\right) \Delta_i^{\eta_i,m} u(y_i'', z_i+\eta_i)\, d\eta_i.$$

Since $\varpi, \psi, \omega \in C_c^\infty(B(0,1))$, by (34) we have that

$$\left|\frac{\partial^\alpha v}{\partial z^\alpha}(z)\right| \leq \frac{C}{z_M^{M+m-1}} \int_{B_{M-2}(0,z_M)} \int_{-z_M}^{z_M} |\Delta_i^{\eta_i,m} u(y_i'', z_i+\eta_i)|\, d\eta_i dy_i''. \tag{35}$$

Reasoning as in (20) (see also 2), we get

$$\left\|\frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)} \leq \frac{C}{z_M^{m+1}} \int_0^{z_M} \|\Delta_i^{\eta_i,m} u\|_{L^p(\mathbb{R}^N)}\, d\eta_i \tag{36}$$

for all $z_M > 0$. If $\theta < \infty$, multiply both sides of the previous inequality by $\frac{1}{z_M^a}$, where

$$a := \frac{1}{\theta} - (m-s),$$

and take the $L^\theta((0,\infty))$ norm in $z_M$ to obtain

$$\left(\int_0^\infty \left\|\frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}}$$

$$\leq C\left(\int_0^\infty \left(\frac{1}{z_M^{1+\frac{1}{\theta}+s}} \int_0^{z_M} \|\Delta_i^{\eta_i,m} u\|_{L^p(\mathbb{R}^N)}\, d\eta_i\right)^\theta dz_M\right)^{\frac{1}{\theta}}.$$

By applying Hardy's inequality to the right-hand side, we get

$$\begin{aligned}
&\left(\int_0^\infty \left\|\frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}} \\
&\le \frac{C}{1+s}\left(\int_0^\infty \left\|\Delta_i^{z_M,m} u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1+s\theta}}\right)^{\frac{1}{\theta}} \le \frac{C}{1+s}\,|u|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(m)}.
\end{aligned}$$

On the other hand, if $\theta = \infty$, reasoning as in (21), from (36), we get

$$\sup_{z_M>0}\left(z_M^{m-s}\left\|\frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}\right) \le \frac{C}{1+s}\,|u|_{B^{s,p,\infty}(\mathbb{R}^N)}^{(m)}.$$

To estimate $\frac{\partial^m v}{\partial z_M^m}$, we use (22) and (31) to write

$$\begin{aligned}
v(z', z_M) &= \sum_{i=1}^N \int_{\mathbb{R}^{M-2}} \frac{1}{z_M^{N-1}} \prod_{j\ne i} \psi\left(\frac{y_j}{z_M}\right) \\
&\times \int_{\mathbb{R}} \frac{1}{z_M}\psi\left(\frac{y_i}{z_M}\right) \Delta_i^{-y_i,1} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\, dy_i dy_i'' \\
&+ u(z').
\end{aligned}$$

Since $\psi$ is even, after a change of variables, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{z_M}\psi\left(\frac{y_i}{z_M}\right) \Delta_i^{-y_i,1} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\, dy_i \\
&= \int_{\mathbb{R}} \frac{1}{z_M}\psi\left(\frac{\eta_i}{z_M}\right) \Delta_i^{\eta_i,1} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\, d\eta_i \\
&= \int_{\mathbb{R}} \frac{1}{z_M}(\omega * \varpi)\left(\frac{\eta_i}{z_M}\right) \Delta_i^{\eta_i,m} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\, d\eta_i,
\end{aligned}$$

where in the last equality we have used Lemma 8. Hence, after renaming $\eta_i$ and using Fubini's theorem,

$$\begin{aligned}
v(z', z_M) &= u(z') \\
&+ \sum_{i=1}^N \int_{\mathbb{R}^{M-1}} \frac{1}{z_M^N} K_i\left(\frac{y'}{z_M}\right) \Delta_i^{y_i,m} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\, dy',
\end{aligned}$$

where

$$K_i(z') := (\omega * \varpi)(z_i) \prod_{j\ne i} \psi(z_j), \quad z' \in \mathbb{R}^N.$$

We now differentiate the previous identity with respect to $z_M$ and use Leibnitz formula for the derivative of a product, to get

$$\begin{aligned}\frac{\partial^m v}{\partial z_M^m}(z', z_M) &= \sum_{i=1}^{N} \int_{\mathbb{R}^{M-1}} \frac{\partial^m}{\partial z_M^m}\left(\frac{1}{z_M^N} K_i\left(\frac{y'}{z_M}\right)\right) \\ &\quad \times \Delta_i^{y_i, m} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\, dy' \\ &= \sum_{i=1}^{N} \sum_{n=0}^{m} \frac{C(M, m, n)}{z_M^{N+m-n}} \int_{\mathbb{R}^{M-1}} \frac{\partial^n}{\partial z_M^n}\left(K_i\left(\frac{y'}{z_M}\right)\right) \\ &\quad \times \Delta_i^{y_i, m} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\, dy'.\end{aligned}$$

For $n = 0, \ldots, m$, $\frac{\partial^n}{\partial z_M^n}\left(K_i\left(\frac{y'}{z_M}\right)\right)$ is given by sums of the form

$$\frac{1}{z_M^n}\left(\frac{y'}{z_M}\right)^{\alpha} \frac{\partial^\alpha K_i}{\partial z^\alpha}\left(\frac{y'}{z_M}\right),$$

where $\alpha = (\alpha_1, \ldots, \alpha_N, 0)$, with $0 \le |\alpha| \le n$. It follows that

$$\begin{aligned}\left|\frac{\partial^m v}{\partial z_M^m}(z', z_M)\right| &\le \sum_{i=1}^{N} \frac{C}{z_M^{M+m-1}} \\ &\times \int_{B_{M-2}(0, z_M)} \int_{-z_M}^{z_M} \left|\Delta_i^{y_i, m} u(z_1 - y_1, \ldots, z_{i-1} - y_{i-1}, z_i, \ldots, z_N)\right| d\eta_i dy_i''.\end{aligned}$$

Reasoning as in (35), we have that

$$\left(\int_0^\infty \left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}^{\theta} \frac{dz_M}{z_M^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}} \le \frac{C}{1+s} |u|_{B^{s,p,\theta}(\mathbb{R}^N)}^{(m)}$$

if $\theta < \infty$, while

$$\sup_{z_M > 0}\left(z_M^{m-s}\left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, z_M)\right\|_{L^p(\mathbb{R}^N)}\right) \le \frac{C}{1+s} |u|_{B^{s,p,\infty}(\mathbb{R}^N)}^{(m)}.$$

Similar estimates hold for $\frac{\partial^\alpha v}{\partial z^\alpha}$, when $\alpha = (\alpha_1, \ldots, \alpha_N, \alpha_N)$, with $|\alpha| = m$, and $0 < \alpha_M < m$. We omit the details.

**Step 2:** Assume that $u \in C^m(\mathbb{R}^N)$. For $z' \in \mathbb{R}^N$, $h > 0$ and $i = 1, \ldots, N$, we have

$$\begin{aligned}\Delta_i^{h,m} u(z') = \Delta_i^{h,m} v(z', 0) &= -\sum_{j=1}^{m} (-1)^j \binom{m}{j} \Delta_i^{h,m} v(z', jh) \\ &\quad + \sum_{j=0}^{m} (-1)^j \binom{m}{j} \Delta_M^{h,m} v(z' + jhe_i, 0).\end{aligned} \tag{37}$$

To see this, we use the following identity for polynomials

$$
\begin{aligned}
(x-1)^m &= (x-1)^m\left(1-(-1)^m(y-1)^m\right)+(-1)^m(x-1)^m(y-1)^m\\
&= -\sum_{j=1}^m(-1)^j\binom{m}{j}y^j(x-1)^m+\sum_{j=0}^m(-1)^j\binom{m}{j}x^j(y-1)^m
\end{aligned}
$$

replace $x$ and $y$ with $T_i^h$ and $T_M^h$, respectively, and use the fact that $\Delta_i^{h,m} = \left(T_i^h - I\right)^m$.

We now estimate the various terms on the right-hand side of (37). By Exercise 9,

$$
\begin{aligned}
\left|\Delta_i^{h,m}u\left(z'\right)\right| &\le \sum_{j=1}^m\binom{m}{j}\int_0^{mh}\tau^{m-1}\left|\frac{\partial^m v}{\partial z_i^m}\left(z_i'',z_i+\tau,jh\right)\right|d\tau\\
&\quad+\sum_{j=0}^m\binom{m}{j}\int_0^{mh}\tau^{m-1}\left|\frac{\partial^m v}{\partial z_M^m}\left(z_i'',z_i+jh,\tau\right)\right|d\tau.
\end{aligned}
$$

If $1\le p<\infty$, by Corollary B.83 and a change of variables,

$$
\begin{aligned}
&\left\|\Delta_i^{h,m}u\right\|_{L^p(\mathbb{R}^N)}\\
&\le C(m,p)\sum_{j=1}^m\left(\int_{\mathbb{R}^N}\left(\int_0^{mh}\tau^{m-1}\left|\frac{\partial^m v}{\partial z_i^m}\left(z_i'',z_i+\tau,jh\right)\right|d\tau\right)^p dz'\right)^{\frac{1}{p}}\\
&\quad+C(m,p)\sum_{j=0}^m\int_0^{mh}\tau^{m-1}\left\|\frac{\partial^m v}{\partial z_M^m}(\cdot,\tau)\right\|_{L^p(\mathbb{R}^N)}d\tau\\
&=: C(m,p)\sum_{j=1}^m I_j+C(m,p)\sum_{j=0}^m II_j.
\end{aligned}
$$

To estimate $I_j$ we reason as in (25) to obtain

$$
I_j\le C(m,p)\,h^m\left\|\frac{\partial^m v}{\partial z_i^m}(\cdot,jh)\right\|_{L^p(\mathbb{R}^N)}.
$$

Hence,

$$
\begin{aligned}
\left\|\Delta_i^{h,m}u\right\|_{L^p(\mathbb{R}^N)} &\le C(m,p)\sum_{j=1}^m h^m\left\|\frac{\partial^m v}{\partial z_i^m}(\cdot,jh)\right\|_{L^p(\mathbb{R}^N)}\\
&\quad+C(m,p)\sum_{j=0}^m\int_0^{mh}\tau^{m-1}\left\|\frac{\partial^m v}{\partial z_M^m}(\cdot,\tau)\right\|_{L^p(\mathbb{R}^N)}d\tau.
\end{aligned}
\tag{38}
$$

The same inequality holds if $p=\infty$.

If $1 \leq \theta < \infty$, multiply both sides of the previous inequality by $\frac{1}{h^{\frac{1}{\theta}+s}}$ and then and take the $L^{\theta}((0,\infty))$ norm in $h$ to obtain,

$$
\begin{aligned}
&\left(\int_0^\infty \left\|\Delta_i^{h,m} u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}}\right)^{\frac{1}{\theta}} \\
&\leq C(m,p) \sum_{j=1}^m \left(\int_0^\infty \left\|\frac{\partial^m v}{\partial z_i^m}(\cdot, jh)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}} \\
&\quad + C(m,p) \sum_{j=0}^m \left(\int_0^\infty \left(\frac{1}{h^{\frac{1}{\theta}+s}} \int_0^{mh} \tau^{m-1} \left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)} d\tau\right)^\theta dh\right)^{\frac{1}{\theta}} \\
&\leq C(m,p) \left(\int_0^\infty \left\|\frac{\partial^m v}{\partial z_i^m}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}} \\
&\quad + \frac{C(m,p)}{s} \left(\int_0^\infty \left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}},
\end{aligned}
$$

where in the last inequality we have used Hardy's inequality and a change of variables. It follows that

$$
\begin{aligned}
|u|_{B^{s,p,\theta}(\mathbb{R}^N)} \leq C(m,p) \sum_{i=1}^N \left(\int_0^\infty \left\|\frac{\partial^m v}{\partial z_i^m}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}} + \\
+ N\frac{C(m,p)}{s} \left(\int_0^\infty \left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1-\theta(m-s)}}\right)^{\frac{1}{\theta}}.
\end{aligned}
$$

On the other hand, if $\theta = \infty$, multiply both sides of (38) by $\frac{1}{h^s}$ and proceed as in (21) to obtain

$$
\frac{1}{h^s} \int_0^{mh} \tau^{m-1} \left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, \tau)\right\|_{L^p(\mathbb{R}^N)} d\tau \leq \frac{m^s}{s} \sup_{h>0} h^{m-s} \left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)},
$$

It follows that

$$
\begin{aligned}
|u|_{B^{s,p,\infty}(\mathbb{R}^N)} \leq C(m,p) \sum_{i=1}^N \sup_{h>0} h^{m-s} \left\|\frac{\partial^m v}{\partial z_i^m}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)} + \\
+ N\frac{C(m,p)\, m^s}{s} \sup_{h>0} h^{m-s} \left\|\frac{\partial^m v}{\partial z_M^m}(\cdot, h)\right\|_{L^p(\mathbb{R}^N)}.
\end{aligned}
$$

To remove the extra assumption that $u \in C^m(\mathbb{R}^N)$, we use a mollification argument. We omit the details. This concludes the proof. ■

**Corollary 16** *Let $1 \leq p, \theta \leq \infty$, $s > 1$, and*

$$k := \begin{cases} \lfloor s \rfloor & \text{if } s \text{ is not an integer or } \theta = 1, \\ s - 1 & \text{if } s \text{ is an integer and } \theta > 1. \end{cases}$$

*Then*

$$B^{s,p,\theta}\left(\mathbb{R}^N\right) \subset W^{k,p}\left(\mathbb{R}^N\right).$$

**Proof.** Here we take $m = \lfloor s \rfloor + 1$.
**Step 1:** Assume first that $s$ is not an integer or $\theta = 1$. Since the function $v$ given in (18) belongs to $C^\infty\left(\mathbb{R}^M_+\right)$, by the fundamental theorem of calculus, for any multi-index $\beta \in \mathbb{N}_0^M$ of the form $\beta = (\beta_1, \ldots, \beta_N, 0)$, with $|\beta| = k$, and for any $i = 1, \ldots, N$ such that $\beta_i > 0$, we have

$$\frac{\partial^\beta v}{\partial z^\beta}(z', h) - \frac{\partial^\beta v}{\partial z^\beta}(z', \varepsilon) = \int_\varepsilon^h \frac{\partial^\alpha v}{\partial z^\alpha}(z', t)\, dt$$

for all $z' \in \mathbb{R}^N$ and $0 < \varepsilon < h < 1$, where $\alpha := (\beta_1, \ldots, \beta_N, 1)$. If $1 \leq p < \infty$, by Corollary B.83,

$$\left\| \frac{\partial^\beta v}{\partial z^\beta}(\cdot, h) - \frac{\partial^\beta v}{\partial z^\beta}(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^N)} \leq \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt.$$

The same inequality holds for $p = \infty$.

If $1 \leq \theta < \infty$, we consider two cases. If $\theta > 1$ and $s$ is not an integer, then $m < s + 1$, and so by Hölder's inequality,

$$\begin{aligned} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt &= \int_\varepsilon^h \frac{t^{\frac{1}{\theta} - (m-s)}}{t^{\frac{1}{\theta} - (m-s)}} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt \\ &\leq \left( \int_\varepsilon^h t^{\frac{\theta'}{\theta} + \theta'(s-m)}\, dt \right)^{\frac{1}{\theta'}} \left( \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dt}{t^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\ &\leq \frac{h^{1-(m-s)}}{\left[\theta'\left(1 - (m-s)\right)\right]^{\frac{1}{\theta'}}} \left( \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dt}{t^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}}, \end{aligned}$$

where we have used Hölder's inequality. If $\theta = 1$ and $s$ is possibly an integer, then

$$\begin{aligned} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt &= \int_\varepsilon^h \frac{t^{1-(m-s)}}{t^{1-(m-s)}} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt \\ &\leq h^{1-(m-s)} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} \frac{dt}{t^{1-(m-s)}}, \end{aligned}$$

where we have used the fact that $1 - (m - s) \geq 0$, since $m = \lfloor s \rfloor + 1 \leq s + 1$.

In both cases, we obtain

$$\left\| \frac{\partial^\beta v}{\partial z^\beta}(\cdot, h) - \frac{\partial^\beta v}{\partial z^\beta}(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^N)} \leq C h^{1-(m-s)} \left( \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dt}{t^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}}. \tag{39}$$

Since the right-hand side goes to zero as $h \to 0^+$, we have that $\left\{ \frac{\partial^\beta v}{\partial z^\beta}(\cdot, h) \right\}_h$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$, and hence it converges in $L^p(\mathbb{R}^N)$ to some function $g_\beta$. Since $v(\cdot, h) = u * \varphi_h$, where $\varphi_h$ is a standard mollifier, it follows by standard interpolation results and (39) that the sequence $\{u * \varphi_h\}$ is a cauchy sequence in $W^{k,p}(\mathbb{R}^N)$. Hence, $u$ belongs to $W^{k,p}(\mathbb{R}^N)$.

If $\theta = \infty$ and $s$ is not an integer, then

$$\begin{aligned} \int_\varepsilon^h \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt &= \int_\varepsilon^h \frac{t^{m-s}}{t^{m-s}} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} dt \\ &\leq \int_\varepsilon^h \frac{1}{t^{m-s}}\, dt \sup_{\varepsilon < t < h} \left( t^{m-s} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} \right) \\ &\leq \frac{h^{1-(m-s)}}{(1-(m-s))} \sup_{\varepsilon < t < h} \left( t^{m-s} \left\| \frac{\partial^\alpha v}{\partial z^\alpha}(\cdot, t) \right\|_{L^p(\mathbb{R}^N)} \right), \end{aligned}$$

We can now continue as before to conclude that $u$ belongs to $W^{k,p}(\mathbb{R}^N)$.

**Step 2:** Next we consider the case in which $s$ is an integer and $\theta > 1$, so that $k = s - 1$. Let $u \in B^{s,p,\theta}(\mathbb{R}^N)$. In this case, by Theorem 12, we have that $B^{k+1,p,\theta} \subset B^{k+s,p,\theta}$ for every $0 < s < 1$. Hence, by the previos part, applied to $B^{k+s,p,\theta}$, we have that $u$ belongs to $W^{k,p}(\mathbb{R}^N)$. ■

**Remark 17** *Reasoning as in Step 1 of the previous proof, we can show that if $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ is such that $|u|_{B^{s,p,\theta}(\mathbb{R}^N)} < \infty$ for some $1 \leq p, \theta \leq \infty$, $s > 1$, with $s$ is not an integer or $\theta = 1$, then all the distributional derivatives of $u$ of order $k$ belong to $L^p(\mathbb{R}^N)$.*

# 7 A Characterization of $B^{s,p,\theta}$ for $s > 1$

The next theorem allows to extend most of the results of Chapter 14 to higher order Besov spaces.

**Theorem 18** *Let $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, let $s > 1$, and let $n \in \mathbb{N}$ with $n < s$. Then a function $u \in L^p(\mathbb{R}^N)$ belongs to $B^{s,p,\theta}(\mathbb{R}^N)$ if and only if for every multi-index $\alpha$ with $|\alpha| = n$, the distributional derivative $\frac{\partial^\alpha u}{\partial x_\alpha}$ belongs to $B^{s-n,p,\theta}(\mathbb{R}^N)$. Moreover, the expression*

$$\|u\|_{L^p(\mathbb{R}^N)} + \sum_{|\alpha|=n} \left| \frac{\partial^\alpha u}{\partial x_\alpha} \right|_{B^{s-n,p,\theta}(\mathbb{R}^N)}$$

*is an equivalent norm in $B^{s,p,\theta}(\mathbb{R}^N)$.*

**Proof.** By induction, it suffices to prove the theorem in the case $n = 1$.
**Step 1:** Assume that $\frac{\partial u}{\partial x_i} \in B^{s-1,p,\theta}\left(\mathbb{R}^N\right)$ for all $i = 1, \ldots, N$. In particular, $u \in W^{1,p}\left(\mathbb{R}^N\right)$, and so by (14.10),

$$\frac{1}{h}\left\|\Delta_i^h u\right\|_{L^p(\mathbb{R}^N)} \leq \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)}$$

for every $h > 0$. Since $u \in W^{1,p}\left(\mathbb{R}^N\right)$, for every fixed $h > 0$ the function $v := \Delta_i^{h,m-1} u$ belongs to $W^{1,p}\left(\mathbb{R}^N\right)$ and

$$\frac{\partial}{\partial x_i}\left(\Delta_i^{h,m-1} u\right) = \Delta_i^{h,m-1}\left(\frac{\partial u}{\partial x_i}\right). \tag{40}$$

Thus, applying the previous inequality with $v$ in place of $u$, we get

$$\frac{1}{h}\left\|\Delta_i^{h,m} u\right\|_{L^p(\mathbb{R}^N)} \leq \left\|\Delta_i^{h,m-1}\left(\frac{\partial u}{\partial x_i}\right)\right\|_{L^p(\mathbb{R}^N)} \tag{41}$$

for every $h > 0$. If $\theta < \infty$, raising both sides to the power $\theta$, multiplying by $\frac{1}{h^{1+(s-1)\theta}}$ and integrating over $h$ gives

$$\int_0^\infty \left\|\Delta_i^{h,m} u\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+s\theta}} \leq \int_0^\infty \left\|\Delta_i^{h,m-1}\left(\frac{\partial u}{\partial x_i}\right)\right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dh}{h^{1+(s-1)\theta}},$$

which implies that $u$ belongs to $B^{s,p,\theta}\left(\mathbb{R}^N\right)$. If $\theta = \infty$ the same conclusion follows by multiplying both sides of (41) by $\frac{1}{h^{s-1}}$ and taking the supremum over $h > 0$.

**Step 2:** Assume that $u$ belongs to $B^{s,p,\theta}\left(\mathbb{R}^N\right)$. In view of Corollary 16, we have that $B^{s,p,\theta}\left(\mathbb{R}^N\right) \subset W^{1,p}\left(\mathbb{R}^N\right)$ and thus the distributional derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \ldots, N$, of $u$ belong to $L^p\left(\mathbb{R}^N\right)$. We want to show that actually they belong to $B^{s-1,p,\theta}\left(\mathbb{R}^N\right)$. Fix $i = 1, \ldots, N$, let $m \in \mathbb{N}$ with $m \geq \lfloor s \rfloor + 1$, and for $z' \in \mathbb{R}^N$ and $z_M > 0$ define

$$w_i(z) := \frac{1}{z_M^N} \int_{\mathbb{R}^N} \varphi\left(\frac{z' - y'}{z_M}\right) \frac{\partial u}{\partial y_i}(y')\, dy',$$

where $\varphi$ is given in (31). Integrating by parts, we get

$$\begin{aligned} w_i(z) &= \frac{1}{z_M^{N+1}} \int_{\mathbb{R}^N} \frac{\partial \varphi}{\partial y_i}\left(\frac{z' - y'}{z_M}\right) u(y')\, dy' \\ &= \frac{\partial v}{\partial z_i}(z), \end{aligned}$$

where $v$ is the function defined in (18). Hence, for any multi-index $\beta \in \mathbb{N}_0^M$ with $|\beta| = m - 1$, we have that

$$\frac{\partial^\beta w_i}{\partial z_\beta} = \frac{\partial^\alpha v}{\partial z_\alpha},$$

where $\alpha$ is the multi-index of components

$$\alpha_j := \begin{cases} \beta_j & \text{if } j \neq i, \\ \beta_i + 1 & \text{if } j = i, \end{cases}$$

so that $|\alpha| = m$. It follows from Theorem 15 that

$$\begin{aligned} \sum_{|\beta|=m-1} & \left( \int_0^\infty \left\| \frac{\partial^\beta w_i}{\partial z^\beta} (\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-1-(s-1))}} \right)^{\frac{1}{\theta}} \\ & \leq \sum_{|\alpha|=m} \left( \int_0^\infty \left\| \frac{\partial^\alpha v}{\partial z^\alpha} (\cdot, z_M) \right\|_{L^p(\mathbb{R}^N)}^\theta \frac{dz_M}{z_M^{1-\theta(m-s)}} \right)^{\frac{1}{\theta}} \\ & \leq C \, |u|_{B^{s,p,\theta}(\mathbb{R}^N)} < \infty, \end{aligned}$$

which implies that $\frac{\partial u}{\partial x_i}$ belongs to $B^{s-1,p,\theta}\left(\mathbb{R}^N\right)$, again by Theorem 15. ■

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