

# 1 Interpolation Inequalities for Intermediate Derivatives

In this section we prove that if  $\Omega$  is sufficiently regular, then a function  $u$  belongs to  $W^{m,p}(\Omega)$  if and if  $u \in L^p(\Omega)$  and all its  $m$ -th distributional derivatives  $\frac{\partial^\alpha u}{\partial x^\alpha}$ ,  $|\alpha| = m$ , belong to  $L^p(\Omega)$ . We begin with the one-dimensional case  $N = 1$ .

## 1.1 The One-Dimensional Case

**Theorem 1** ( $m = 2$ ) *Let  $I \subset \mathbb{R}$  be an open interval and let  $u \in L^1_{\text{loc}}(I)$ ,  $1 \leq p \leq \infty$ . Then  $u$  belongs to  $W^{2,p}(I)$  if and only if  $u$  belongs to  $L^p(I)$  and its second order distributional derivative  $u''$  belongs to  $L^p(I)$ . In this case, for every  $0 < \ell < \text{length } I$ ,*

$$\|u'\|_{L^p(I)} \leq C(p) \left( \frac{1}{\ell} \|u\|_{L^p(I)} + \ell \|u''\|_{L^p(I)} \right). \quad (1)$$

**Proof. Step 1:** Assume first that  $u \in C^\infty(I)$  and that  $I = (0, b)$  for some  $b > 0$ . Fix  $s \in (0, \frac{1}{3}b)$  and  $t \in (\frac{2}{3}b, b)$ . By the mean value theorem, there exists  $\xi \in (s, t)$  such that

$$u'(\xi) = \frac{u(t) - u(s)}{t - s}.$$

Hence, by the fundamental theorem of calculus, for all  $x \in (0, b)$ ,

$$u'(x) = u'(\xi) + \int_\xi^x u''(y) dy = \frac{u(t) - u(s)}{t - s} + \int_\xi^x u''(y) dy.$$

Since  $t - s \geq \frac{b}{3}$ , It follows that

$$|u'(x)| \leq \frac{3}{b} (|u(t)| + |u(s)|) + \int_0^b |u''(y)| dy$$

for all  $x \in (0, b)$ . If  $p = \infty$ , then

$$|u'(x)| \leq \frac{6}{b} \sup_{t \in (0, b)} |u(t)| + b \sup_{t \in (0, b)} |u''(t)|, \quad (2)$$

while if  $1 \leq p < \infty$ , then by Hölder's inequality

$$|u'(x)| \leq \frac{3}{b} (|u(t)| + |u(s)|) + b^{\frac{p-1}{p}} \left( \int_0^b |u''(y)|^p dy \right)^{\frac{1}{p}}. \quad (3)$$

Using the convexity of the function  $\tau \mapsto |\tau|^p$ , we have that

$$|u'(x)|^p \leq \frac{3^{2p-1}}{b^p} (|u(t)|^p + |u(s)|^p) + 3^{p-1} b^{p-1} \int_0^b |u''(y)|^p dy.$$

By averaging first in  $s$  over  $(0, \frac{1}{3}b)$  and then in  $t$  over  $(\frac{2}{3}b, b)$ , we get

$$\begin{aligned} |u'(x)|^p &\leq \frac{3^{2p-1}}{b^p} \frac{3}{b} \left( \int_{\frac{2}{3}b}^b |u(t)|^p dt + \int_0^{\frac{b}{3}} |u(s)|^p ds \right) + 3^{p-1} b^{p-1} \int_0^b |u''(y)|^p dy \\ &\leq \frac{3^{2p}}{b^{p+1}} \int_0^b |u(y)|^p dy + 3^{p-1} b^{p-1} \int_0^b |u''(y)|^p dy. \end{aligned}$$

Finally, we integrate in  $x$  over  $(0, b)$ , to obtain

$$\int_0^b |u'(x)|^p dx \leq \frac{3^{2p}}{b^p} \int_0^b |u(y)|^p dy + 3^{p-1} b^p \int_0^b |u''(y)|^p dy. \quad (4)$$

**Step 2:** If  $I$  has infinite length, fix  $b > 0$  and subdivide  $I$  in subintervals of length  $b$ . Since (2), respectively, (4), holds in every such subinterval, we get

$$\sup_{x \in I} |u'(x)| \leq \frac{6}{b} \sup_{t \in I} |u(t)| + b \sup_{t \in I} |u''(t)| \quad (5)$$

if  $p = \infty$ , while

$$\int_I |u'(x)|^p dx \leq \frac{3^{2p}}{b^p} \int_I |u(x)|^p dx + 3^{p-1} b^p \int_I |u''(x)|^p dx \quad (6)$$

if  $1 \leq p < \infty$ . By taking  $b$  to be  $\ell$ , we get (1).

On the other hand, if  $I$  has finite length, let  $m \in \mathbb{N}$  and divide  $I$  into  $m$  subintervals of length  $b := \frac{1}{m} \text{length } I$ . Then we get (5) and (6). It suffices to take  $m$  to be the integer part of  $\frac{\text{length } I}{\ell}$ .

**Step 3:** To remove the additional hypothesis that  $u \in C^\infty(I)$ , one can use standard mollifiers (see, e.g. Step 4 of the proof of Theorem 10.55. We omit the details. ■

**Remark 2** When  $p = 1$  and  $u \in L^1(I)$  is such that its second order distributional derivative  $u''$  belongs to  $\mathcal{M}_b(I; \mathbb{R})$ , then, using Theorem 13.9, inequality (1) continues to hold with  $\|u''\|_{L^1(I)}$  replaced by the total variation  $|u''|(I)$ . In turn,  $u \in W^{1,1}(I)$  and  $u' \in BV(I)$ .

**Remark 3** Note that in Steps 1 and 2 we have not used the fact that  $u$  and  $u''$  belong to  $L^p(I)$ . Thus, for  $u \in C^\infty(I)$  (or  $u \in C^2(I)$ ) inequality (1) always hold, with the right-hand side possibly infinite.

Next we consider the case  $m \geq 2$ .

**Theorem 4** ( $m \geq 2$ ) Let  $I \subset \mathbb{R}$  be an open interval, let  $u \in L^1_{\text{loc}}(I)$ , let  $1 \leq p \leq \infty$ , and let  $m \in \mathbb{N}$ , with  $m \geq 2$ . Then  $u$  belongs to  $W^{m,p}(I)$  if and only if  $u$  belongs to  $L^p(I)$  and its  $m$ -th distributional derivative  $u^{(m)}$  belongs to  $L^p(I)$ . In this case, for every  $0 < \ell < \text{length } I$  and  $j \in \mathbb{N}$  with  $1 \leq j < m$ ,

$$\|u^{(j)}\|_{L^p(I)} \leq C(p, j, m) \left( \ell^{-j} \|u\|_{L^p(I)} + \ell^{m-j} \|u^{(m)}\|_{L^p(I)} \right). \quad (7)$$

**Proof.** Assume that  $u \in C^\infty(I)$ . In what follows the constant  $C = C(p, j, m)$  may change from line to line.

**Step 1:** We begin by proving that

$$\left\| u^{(j)} \right\|_{L^p(I)} \leq C \left( \ell^{-j} \|u\|_{L^p(I)} + \ell \left\| u^{(j+1)} \right\|_{L^p(I)} \right) \quad (8)$$

for every  $0 < \ell < \text{length } I$  and for all  $j \in \mathbb{N}$  with  $1 \leq j < m$ . The proof is by induction on  $j$ . For  $j = 1$  the result follows from Theorem 1 and Remark 3. Thus assume that (8) holds for every  $0 < \ell < \text{length } I$  and for some  $j \in \mathbb{N}$  with  $1 \leq j < m - 1$  and let's prove that

$$\left\| u^{(j+1)} \right\|_{L^p(I)} \leq C \left( \ell^{-j-1} \|u\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right)$$

for every  $0 < \ell < \text{length } I$ . Let  $\theta \in (0, 1)$ . Applying Theorem 1 and Remark 3 to the function  $v = u^{(j)}$  and we obtain

$$\begin{aligned} \left\| u^{(j+1)} \right\|_{L^p(I)} &\leq C(p) \left( \ell^{-1} \left\| u^{(j)} \right\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right) \\ &\leq C(p) \left( \ell^{-1} C \left( (\ell\theta)^{-j} \|u\|_{L^p(I)} + \ell\theta \left\| u^{(j+1)} \right\|_{L^p(I)} \right) + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right) \\ &= C(p) \left( \left( C\theta^{-j} \ell^{-j-1} \|u\|_{L^p(I)} + C\theta \left\| u^{(j+1)} \right\|_{L^p(I)} \right) + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right), \end{aligned}$$

where in the second inequality we have used (8) for  $j$  (which holds by the induction hypothesis) and with  $\ell$  replaced by  $\ell\theta$ , which is less than  $\text{length } I$ , since  $\theta \in (0, 1)$ . Taking  $\theta$  so small that  $C(p)C\theta < \frac{1}{2}$ , we obtain

$$\frac{1}{2} \left\| u^{(j+1)} \right\|_{L^p(I)} \leq C(p) (1 + C\theta^{-j}) \left( \ell^{-j-1} \|u\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right),$$

which gives the desired inequality for  $j + 1$ .

**Step 2:** Next we prove that inequality (7) holds for every  $0 < \ell < \text{length } I$  and  $j \in \mathbb{N}$  with  $1 \leq j < m$ . The proof is by induction on  $i = m - j$ . For  $i = 1$ , we have that  $j = m - 1$ , and so the result follows from the previous step. Thus assume that (7) holds for every  $0 < \ell < \text{length } I$  and for some  $j \in \mathbb{N}$  with  $1 < j < m$  and let's prove that

$$\left\| u^{(j-1)} \right\|_{L^p(I)} \leq C \left( \ell^{-j+1} \|u\|_{L^p(I)} + \ell^{m-j+1} \left\| u^{(m)} \right\|_{L^p(I)} \right)$$

for every  $0 < \ell < \text{length } I$ . Note that if  $i = m - j$ , then  $i + 1 = m - (j - 1)$ .

Let  $\theta \in (0, 1)$ . Applying (8) for  $j - 1$  and then (7) for  $j$  (which holds by the induction hypothesis), we obtain

$$\begin{aligned} \left\| u^{(j-1)} \right\|_{L^p(I)} &\leq C \left( \ell^{-j+1} \|u\|_{L^p(I)} + \ell \left\| u^{(j)} \right\|_{L^p(I)} \right) \\ &\leq C \left( \ell^{-j+1} \|u\|_{L^p(I)} + \ell C \left( \ell^{-j} \|u\|_{L^p(I)} + \ell^{m-j} \left\| u^{(m)} \right\|_{L^p(I)} \right) \right) \\ &\leq C \left( \ell^{-j+1} \|u\|_{L^p(I)} + \ell^{m-j+1} \left\| u^{(j+2)} \right\|_{L^p(I)} \right), \end{aligned}$$

which gives the desired inequality for  $j - 1$ .

**Step 3:** To remove the additional hypothesis that  $u \in C^\infty(I)$ , one can use standard mollifiers (see, e.g. Step 4 of the proof of Theorem 10.55. We omit the details. ■

**Corollary 5** *Let  $I \subset \mathbb{R}$  be an open interval, let  $1 \leq p \leq \infty$ , let  $m \in \mathbb{N}$ , with  $m \geq 2$ , and let  $u \in W^{2,p}(I)$ . Then for every  $j \in \mathbb{N}$  with  $1 \leq j < m$ ,*

$$\left\| u^{(j)} \right\|_{L^p(I)} \leq C(p, j, m) \|u\|_{L^p(I)}^{(m-j)/m} \left\| u^{(m)} \right\|_{L^p(I)}^{j/m} \quad (9)$$

if either  $\text{length } I = \infty$  or  $\text{length } I < \infty$  and

$$j \|u\|_{L^p(I)} \leq (m - j) (\text{length } I)^m \left\| u^{(m)} \right\|_{L^p(I)},$$

while

$$\left\| u^{(j)} \right\|_{L^p(I)} \leq mC(p, j, m) (\text{length } I)^{-j} \|u\|_{L^p(I)} \quad (10)$$

if  $\text{length } I < \infty$  and  $(m - j) (\text{length } I)^m \left\| u^{(m)} \right\|_{L^p(I)} < j \|u\|_{L^p(I)}$ .

**Proof.** If  $\left\| u^{(m)} \right\|_{L^p(I)} = 0$ , then  $u$  is a polynomial of degree  $m - 1$ . If  $\text{length } I = \infty$ , then  $u = 0$ , while if  $\text{length } I < \infty$ , the result follows from direct calculations. Thus in what follow we assume that  $\left\| u^{(m)} \right\|_{L^p(I)} > 0$  and  $\|u\|_{L^p(I)} > 0$ . Consider the function

$$g(t) = \frac{1}{t^j} A + t^{m-j} B,$$

where  $A, B > 0$  and  $0 < t \leq t_0$ , with  $t_0 \in (0, \infty]$ . Then

$$\inf_{0 < t \leq t_0} g(t) = (jA)^{\frac{m-j}{m}} ((m-j)B)^{\frac{j}{m}} \quad (11)$$

if  $\sqrt[m]{\frac{j}{m-j} \frac{A}{B}} \leq t_0$ , while

$$\inf_{0 < t \leq t_0} g(t) = \frac{1}{t_0^j} A + t_0^{m-j} B \quad (12)$$

if  $\sqrt[m]{\frac{j}{m-j} \frac{A}{B}} > t_0$ . In particular, if  $t_0 = \infty$ , then we are always in the first case.

Taking  $A = \|u\|_{L^p(I)}$ ,  $B = \|u^{(m)}\|_{L^p(I)}$  and  $t_0 = \text{length } I$ , the inequality (9) follows from (11) and Theorem 4.

To obtain (10), note that if  $(m-j)(\text{length } I)^m \|u^{(m)}\|_{L^p(I)} < j \|u\|_{L^p(I)}$ , then by Theorem 4 and (12),

$$\begin{aligned} \|u^{(j)}\|_{L^p(I)} &\leq C(p, j, m) \left( (\text{length } I)^{-j} \|u\|_{L^p(I)} + (\text{length } I)^{m-j} \|u^{(m)}\|_{L^p(I)} \right) \\ &\leq mC(p, j, m) (\text{length } I)^{-j} \|u\|_{L^p(I)}, \end{aligned}$$

which concludes the proof. ■

In what follows, given  $T > 0$ , we denote by  $W_{\#}^{m,p}(0, T)$  the space of all functions in  $W_{\text{loc}}^{m,p}(\mathbb{R})$  that are  $T$ -periodic, endowed with the norm in  $W^{m,p}(0, T)$ .

**Corollary 6** *Let  $I = (0, T)$ , let  $1 \leq p \leq \infty$ , let  $m \in \mathbb{N}$ , with  $m \geq 2$ , and let  $u \in W_{\#}^{m,p}(I)$ . Then for every  $j \in \mathbb{N}$  with  $1 \leq j < m$ ,*

$$\|u^{(j)}\|_{L^p(I)} \leq C(p, j, m) \|u\|_{L^p(I)}^{(m-j)/m} \|u^{(m)}\|_{L^p(I)}^{j/m}. \quad (13)$$

**Proof.** If  $\|u^{(m)}\|_{L^p(I)} = 0$ , then  $u$  is a polynomial of degree  $m-1$ . But since  $u$  is periodic, it follows that  $u$  must be constant. Hence,  $u^{(j)} = 0$ , and so (13) is actually an equality.

Thus in what follow, assume that  $\|u^{(m)}\|_{L^p(I)} > 0$ . Choose an integer  $n \in \mathbb{N}$  so large that

$$n^m (m-j) (\text{length } I)^m \|u^{(m)}\|_{L^p(I)} \geq j \|u\|_{L^p(I)} \quad (14)$$

and consider the interval  $J = (0, nT)$ . If  $1 \leq p < \infty$ , then by the periodicity of  $u$  and  $u^{(m)}$ ,

$$\begin{aligned} &(m-j) (\text{length } J)^{mp} \int_J |u^{(m)}(x)|^p dx \\ &= (m-j) (\text{length } J)^{mp} \sum_{i=1}^n \int_{(i-1)T}^{iT} |u^{(m)}(x)|^p dx \\ &= n^{mp} (m-j) (\text{length } I)^{mp} \sum_{i=1}^n \int_{(i-1)T}^{iT} |u^{(m)}(x)|^p dx \\ &= n^{mp+1} (\text{length } I)^{2p} \int_0^T |u^{(m)}(x)|^p dx \\ &\geq j^p n \int_0^T |u(x)|^p dx = j^p \int_J |u^{(m)}(x)|^p dx. \end{aligned}$$

Hence, by the previous corollary applied to  $J$ , and the periodicity of  $u$ ,  $u^{(j)}$ , and  $u^{(m)}$ ,

$$\begin{aligned} n^{1/p} \left\| u^{(j)} \right\|_{L^p(I)} &= \left\| u^{(j)} \right\|_{L^p(J)} \leq C(p, j, m) \|u\|_{L^p(J)}^{(m-j)/m} \left\| u^{(m)} \right\|_{L^p(J)}^{j/m} \\ &= C(p, j, m) n^{(m-j)/(mp)} \|u\|_{L^p(I)}^{(m-j)/m} n^{j/(mp)} \left\| u^{(m)} \right\|_{L^p(I)}^{j/m}, \end{aligned}$$

which gives the desired inequality.

If  $p = \infty$ , then again by (14), and the facts that, by the periodicity of  $u$  and  $u^{(m)}$ ,  $\|u\|_{L^\infty(J)} = \|u\|_{L^\infty(I)}$  and  $\|u^{(m)}\|_{L^\infty(J)} = \|u^{(m)}\|_{L^\infty(I)}$ , we have that  $(\text{length } J)^2 \|u''\|_{L^\infty(J)} \geq \|u\|_{L^\infty(J)}$ , and so by the previous corollary applied to  $J$ , and the periodicity of  $u$ ,  $u^{(j)}$ , and  $u^{(m)}$ ,

$$\begin{aligned} \left\| u^{(j)} \right\|_{L^\infty(I)} &= \left\| u^{(j)} \right\|_{L^\infty(J)} \leq C(p) \|u\|_{L^\infty(J)}^{(m-j)/m} \left\| u^{(m)} \right\|_{L^\infty(J)}^{j/m} \\ &= C(p) \|u\|_{L^\infty(I)}^{(m-j)/m} \left\| u^{(m)} \right\|_{L^\infty(I)}^{j/m}, \end{aligned}$$

which concludes the proof. ■

## 1.2 The $N$ -th Dimensional Case

In this section, using a slicing argument, we obtain the  $N$ -dimensional version of Theorem 1. We begin with the simple case of a rectangle.

**Theorem 7** *Let  $R := I_1 \times \cdots \times I_N \subset \mathbb{R}^N$ , where  $I_i \subset \mathbb{R}$  is an open interval, let  $u \in L^1_{\text{loc}}(R)$ , and let  $1 \leq p \leq \infty$ . Then  $u$  belongs to  $W^{2,p}(R)$  if and only if  $u$  belongs to  $L^p(R)$  and all its second order distributional derivatives  $\frac{\partial^2 u}{\partial x_i \partial x_j}$ ,  $i, j = 1, \dots, N$ , belong to  $L^p(R)$ . In this case, for every  $0 < \ell < \min_i \text{length } I_i$ ,*

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(R)} \leq C(p, N) \left( \frac{1}{\ell} \|u\|_{L^p(R)} + \ell \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{L^p(R)} \right)$$

for all  $i = 1, \dots, N$ .

**Proof.** Assume that  $u \in C^\infty(R)$  with  $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(R)$  for all  $i, j = 1, \dots, N$ . Fix  $i = 1, \dots, N$  and let  $R'_i := (a_1, b_1) \times \cdots \times (a_{i-1}, b_{i-1}) \times (a_{i+1}, b_{i+1}) \times \cdots \times (a_N, b_N)$ . Since  $b_i - a_i > \ell$ , for all  $x'_i \in R_i$ , we are in a position to apply Theorem 1 to obtain that

$$\int_{a_i}^{b_i} \left| \frac{\partial u}{\partial x_i}(x'_i, x_i) \right|^p dx_i \leq \frac{3^{2p}}{\ell^p} \int_{a_i}^{b_i} |u(x'_i, x_i)|^p dx_i + 3^{p-1} \ell^p \int_{a_i}^{b_i} \left| \frac{\partial^2 u}{\partial x_i^2}(x'_i, x_i) \right|^p dx_i,$$

where we have used the notation (E2). Integrating the previous inequality over  $R'_i$  and using Tonelli's theorem, we obtain

$$\int_R \left| \frac{\partial u}{\partial x_i} \right|^p dx \leq \frac{3^{2p}}{\ell^p} \int_R |u|^p dx + 3^{p-1} \ell^p \int_R \left| \frac{\partial^2 u}{\partial x_i^2} \right|^p dx,$$

which gives the desired inequality. The general case is treated in the next exercise. ■

**Exercise 8** Let  $\Omega \subset \mathbb{R}^N$  be an open set and let  $u \in L^1_{\text{loc}}(\Omega)$  be such that all second order distributional derivatives  $\frac{\partial^2 u}{\partial x_i \partial x_j}$ ,  $i, j = 1, \dots, N$ , belong to  $L^p(\Omega)$ .

(i) Prove that all the first order distributional derivatives  $\frac{\partial u}{\partial x_i}$ ,  $i = 1, \dots, N$ , belong to  $L^1_{\text{loc}}(\Omega)$ . Hint: Given  $\Omega' \subset\subset \Omega$ , cover  $\Omega'$  with rectangles compactly contained in  $\Omega$  and use mollification together with the previous theorem.

(ii) Prove that if  $\Omega = R$  and  $u$  and all the second order distributional derivatives  $\frac{\partial^2 u}{\partial x_i \partial x_j}$ ,  $i, j = 1, \dots, N$ , belong to  $L^p(\Omega)$ , then  $u$  belongs to  $W^{1,p}(\Omega)$ .

Next we consider uniformly Lipschitz domains (see Definition 12.10).

**Exercise 9** Given  $L > 0$ , consider the sector

$$\Xi_L := S^{N-1} \cap \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : L|x'| < x_N\}.$$

Prove that there exists a constant  $C_{N,p,L} > 0$  such that

$$\int_{\Xi_L} |z \cdot \nu|^p d\mathcal{H}^{N-1}(\nu) \geq C_{N,p,L} |z|^p$$

for every  $z \in \mathbb{R}^N$ .

**Theorem 10** ( $m = 2$ ) Let  $\Omega \subset \mathbb{R}^N$  be an open set with uniformly Lipschitz boundary (with parameters  $\varepsilon, L, M$ ), let  $u \in L^1_{\text{loc}}(\Omega)$ , and let  $1 \leq p \leq \infty$ . Then  $u$  belongs to  $W^{2,p}(\Omega)$  if and only if  $u$  belongs to  $L^p(\Omega)$  and all its second order distributional derivatives  $\frac{\partial^2 u}{\partial x_i \partial x_j}$ ,  $i, j = 1, \dots, N$ , belong to  $L^p(\Omega)$ . In this case, for every  $0 < \ell < \frac{\varepsilon}{4(1+L)}$ ,

$$\|\nabla u\|_{L^p(\Omega; \mathbb{R}^N)} \leq C(p, N, L) \left( \frac{1}{\ell} \|u\|_{L^p(\Omega)} + \ell \|\nabla^2 u\|_{L^p(\Omega; \mathbb{R}^{N \times N})} \right)$$

for all  $i = 1, \dots, N$ .

**Proof. Step 1:** Assume that  $1 \leq p < \infty$  and that  $u \in C^\infty(\Omega)$  with  $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$  for all  $i, j = 1, \dots, N$ . Given  $\ell > 0$  and  $\nu \in S^{N-1}$ , we define  $\Omega(\nu, \ell)$  as the set of all points  $x \in \Omega$  belonging to a segment, with length greater than  $\ell$ , parallel to  $\nu$  and contained in  $\Omega$ . Let

$$\Pi_\nu := \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$$

and for  $y \in \Pi_\nu$  define the slice  $\Omega(\nu, \ell, y)$  by

$$\Omega(\nu, \ell, y) := \{t \in \mathbb{R} : y + t\nu \in \Omega(\nu, \ell)\}.$$

Note that if  $y \in \Pi_\nu$  is such that  $\Omega(\nu, \ell, y)$  is nonempty, then  $\Omega(\nu, \ell, y)$  is the union of a family of pairwise disjoint open intervals with length greater than  $\ell$  and we may define the function  $u_{y,\nu} : \Omega(\nu, \ell, y) \rightarrow \mathbb{R}$  by

$$u_{y,\nu}(t) := u(y + t\nu).$$

Applying Theorem 1 to the function  $u_{y,\nu}$  in each connected component of  $\Omega(\nu, \ell, y)$ , we obtain

$$\begin{aligned} \int_{\Omega(\nu, \ell, y)} |\nabla u(y + t\nu) \cdot \nu|^p dt &\leq \frac{3^{2p}}{\ell^p} \int_{\Omega(\nu, \ell, y)} |u(y + t\nu)|^p dt \\ &\quad + 3^{p-1} \ell^p \int_{\Omega(\nu, \ell, y)} |\nabla^2 u(y + t\nu) \nu \cdot \nu|^p dt. \end{aligned}$$

Integrating both sides of the previous inequality over all  $y \in \Pi_\nu$  for which  $\Omega(\nu, \ell, y)$  is nonempty, by Tonelli's Theorem, we get

$$\int_{\Omega(\nu, \ell)} |\nabla u(x) \cdot \nu|^p dx \leq \frac{3^{2p}}{\ell^p} \int_{\Omega} |u(x)|^p dx + 3^{p-1} \ell^p \int_{\Omega} |\nabla^2 u(x)|^p dx.$$

We now integrate both sides of the previous inequality in the variable  $\nu$  over  $S^{N-1}$  and use again Tonelli's theorem, to obtain

$$\begin{aligned} \int_{\Omega} \int_{G(x, \ell)} |\nabla u(x) \cdot \nu|^p d\mathcal{H}^{N-1}(\nu) dx &= \int_{S^{N-1}} \int_{\Omega(\nu, \ell)} |\nabla u(x) \cdot \nu|^p dx d\mathcal{H}^{N-1}(\nu) \\ &\leq \frac{3^{2p} \beta_N}{\ell^p} \int_{\Omega} |u(x)|^p dx + 3^{p-1} \beta_N \ell^p \int_{\Omega} |\nabla^2 u(x)|^p dx, \end{aligned} \tag{15}$$

where  $\beta_N = \mathcal{H}^{N-1}(S^{N-1})$  and  $G(x, \ell) := \{\nu \in S^{N-1} : x \in \Omega(\nu, \ell)\}$ .

Now fix  $0 < \ell < \frac{\varepsilon}{4(1+L)}$  and  $x \in \Omega$ . There are two cases: If  $\text{dist}(x, \partial\Omega) \geq \ell$ , then  $\overline{B(x, \ell)} \subset \Omega$ , and so  $G(x, \ell) = S^{N-1}$ . In this case,

$$\int_{G(x, \ell)} |\nabla u(x) \cdot \nu|^p d\mathcal{H}^{N-1}(\nu) = C_{N,p} |\nabla u(x)|^p, \tag{16}$$

where we used the fact that for every  $z \in \mathbb{R}^N \setminus \{0\}$ , by a rotation,

$$\begin{aligned} \int_{S^{N-1}} |z \cdot \nu|^p d\mathcal{H}^{N-1}(\nu) &= |z|^p \int_{S^{N-1}} \left| \frac{z}{|z|} \cdot \nu \right|^p d\mathcal{H}^{N-1}(\nu) \\ &= |z|^p \int_{S^{N-1}} |e_1 \cdot \nu|^p d\mathcal{H}^{N-1}(\nu) =: |z|^p C_{N,p}. \end{aligned}$$

On the other hand, if  $\text{dist}(x, \partial\Omega) < \ell$ , then there exists  $x_0 \in \partial\Omega$  with  $|x - x_0| < \ell$ . By parts (i) and (iii) of Definition 12.10, there exist  $n \in \mathbb{N}$ , local coordinates  $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ , and a Lipschitz function  $f : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  (both depending on  $n$ ), with  $\text{Lip } f \leq L$ , such that  $B(x_0, \varepsilon) \subset \Omega_n$  and

$$\Omega_n \cap \Omega = \Omega_n \cap \{(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : y_N > f(y')\}.$$

In particular,

$$B(x_0, \varepsilon) \cap \Omega = B(x_0, \varepsilon) \cap \{(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : y_N > f(y')\}.$$

Since  $\ell < \frac{\varepsilon}{4(1+L)}$ , we have that  $x \in B\left(x_0, \frac{\varepsilon}{4(1+L)}\right)$ . Using local coordinates, we can write  $x = (\bar{y}', \bar{y}_N)$ . Consider the point  $(\bar{y}', f(\bar{y}'))$ . Since  $f$  is Lipschitz, with  $\text{Lip } f \leq L$ , for  $y' \in \mathbb{R}^{N-1}$ , we have that

$$f(y') \leq f(\bar{y}') + L|y' - \bar{y}'| < \bar{y}_N + L|y' - \bar{y}'|.$$

Hence, the cone

$$K_x = \left\{ (y', y_N) \in B_{N-1}\left(\bar{y}', \frac{\varepsilon}{4(1+L)}\right) \times \mathbb{R} : \bar{y}_N + L|y' - \bar{y}'| < y_N < \bar{y}_N + \frac{\varepsilon}{4} \right\}$$

is contained in  $B(x_0, \varepsilon) \cap \Omega$ . Using local coordinates, consider the sector

$$\Xi_x := \{\nu = (\nu', \nu_N) \in S^{N-1} : L|\nu'| < \nu_N\}.$$

Since  $\ell < \frac{\varepsilon}{4(1+L)}$ , if  $t \in (0, \ell)$  and  $\nu \in \Xi_x$ , then the point  $x + t\nu$  belongs to  $K_x \subset \Omega$ . Hence,  $G(x, \ell) \supset \Xi_x$ , and so

$$\int_{G(x, \ell)} |\nabla u(x) \cdot \nu|^p d\mathcal{H}^{N-1}(\nu) \geq \int_{\Xi_x} |\nabla u(x) \cdot \nu|^p d\mathcal{H}^{N-1}(\nu) \geq C_{N,p,L} |\nabla u(x)|^p,$$

where we have used the previous exercise. Together with (15)-(16), this shows that

$$C_{N,p,L} \int_{\Omega} |\nabla u|^p dx \leq \frac{3^{2p}\beta_N}{\ell^p} \int_{\Omega} |u|^p dx + 3^{p-1}\beta_N \ell^p \int_{\Omega} |\nabla^2 u|^p dx,$$

which is what we wanted to prove.

The additional hypothesis that  $u \in C^\infty(\Omega)$  can be removed as in Exercise 8. We omit the details.

**Step 2:** The case  $p = \infty$  is simpler and is left as an exercise. ■

**Remark 11** Note that in the previous proof we only used a uniform cone property.

**Remark 12** When  $p = 1$  and  $u \in L^1(\Omega)$  is such that its second order distributional derivative belongs to  $\mathcal{M}_b(\Omega; \mathbb{R}^{N \times N})$ , then, using Theorem 13.9, inequality (1) continues to hold with  $\|\nabla^2 u\|_{L^1(\Omega; \mathbb{R}^{N \times N})}$  replaced by the total variation  $|D(\nabla u)|(\Omega)$  of  $\nabla u$ . In turn,  $u \in W^{1,1}(\Omega)$  and  $\nabla u \in BV(\Omega; \mathbb{R}^N)$ .

Next we consider the case  $m \geq 2$ .

**Theorem 13** ( $m \geq 2$ ) Let  $\Omega \subset \mathbb{R}^N$  be an open set with uniformly Lipschitz boundary (with parameters  $\varepsilon, L, M$ ), let  $u \in L^1_{\text{loc}}(\Omega)$ , let  $1 \leq p \leq \infty$ , and let  $m \in \mathbb{N}$ , with  $m \geq 2$ . Then  $u$  belongs to  $W^{m,p}(\Omega)$  if and only if  $u$  belongs to  $L^p(\Omega)$  and all its distributional derivatives  $\frac{\partial^\alpha u}{\partial x^\alpha}$ ,  $|\alpha| = m$ , belong to  $L^p(\Omega)$ . In this case, for every  $0 < \ell < \frac{\varepsilon}{4(1+L)}$ , for every  $j \in \mathbb{N}$  with  $1 \leq j < m$ ,

$$\|\nabla^j u\|_{L^p} \leq C(p, j, m, N, L) (\ell^{-j} \|u\|_{L^p} + \ell^{m-j} \|\nabla^m u\|_{L^p}).$$

**Proof.** The proof is very similar to the one of Theorem 4 and thus we omit it. ■

**Corollary 14** *Let  $\Omega \subset \mathbb{R}^N$  be an open set with uniformly Lipschitz boundary (with parameters  $\varepsilon, L, M$ ), let  $1 \leq p \leq \infty$ , let  $m \in \mathbb{N}$ , with  $m \geq 2$ , and let  $u \in W^{m,p}(\Omega)$ . Then for every  $j \in \mathbb{N}$  with  $1 \leq j < m$ ,*

$$\|\nabla^j u\|_{L^p} \leq C(p, j, m, N, L) \|u\|_{L^p}^{(m-j)/m} \|\nabla^m u\|_{L^p}^{j/m} \quad (17)$$

*if either  $\varepsilon = \infty$  or  $\varepsilon < \infty$  and  $j \|u\|_{L^p} \leq (m-j) R^m \|\nabla^m u\|_{L^p}$ , while*

$$\|\nabla^j u\|_{L^p} \leq mC(p, j, m, N, L) R^{-j} \|u\|_{L^p}$$

*if  $\varepsilon < \infty$  and  $(m-j) R^m \|\nabla^m u\|_{L^p} < j \|u\|_{L^p}$ , where  $R := \frac{\varepsilon}{4(1+L)}$ .*

**Proof.** The proof is the same of the one of Corollary 5. ■

**Remark 15** *If  $\Omega = \mathbb{R}^N$  or  $\Omega = \mathbb{R}_+^N$  or*

$$\Omega := \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > f(x')\},$$

*where  $f : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  is a Lipschitz function, then  $\varepsilon = \infty$  and inequality (17) holds.*