1 Interpolation Inequalities for Intermediate Derivatives

In this section we prove that if $\Omega$ is sufficiently regular, then a function $u$ belongs to $W^{m,p}(\Omega)$ if and if $u \in L^p(\Omega)$ and all its $m$-th distributional derivatives $\frac{\partial^m u}{\partial x^m}$, $|\alpha| = m$, belong to $L^p(\Omega)$. We begin with the one-dimensional case $N = 1$.

1.1 The One-Dimensional Case

**Theorem 1** ($m = 2$) Let $I \subset \mathbb{R}$ be an open interval and let $u \in L^1_{\text{loc}}(I)$, $1 \leq p \leq \infty$. Then $u$ belongs to $W^{2,p}(I)$ if and only if $u \in L^p(I)$ and its second order distributional derivative $u''$ belongs to $L^p(I)$. In this case, for every $0 < \ell < \text{length } I$,

$$\|u''\|_{L^p(I)} \leq C(p) \left( \frac{1}{\ell} \|u\|_{L^p(I)} + \ell \|u''\|_{L^p(I)} \right).$$

**Proof.** Step 1: Assume first that $u \in C^\infty(I)$ and that $I = (0,b)$ for some $b > 0$. Fix $s \in (0, \frac{1}{3}b)$ and $t \in (\frac{2}{3}b, b)$. By the mean value theorem, there exists $\xi \in (s,t)$ such that

$$u'(\xi) = \frac{u(t) - u(s)}{t - s}.$$ 

Hence, by the fundamental theorem of calculus, for all $x \in (0, b)$,

$$u'(x) = u'(\xi) + \int_\xi^x u''(y) \, dy = \frac{u(t) - u(s)}{t - s} + \int_\xi^x u''(y) \, dy.$$ 

Since $t - s \geq \frac{1}{3}b$, it follows that

$$|u'(x)| \leq \frac{3}{b} (|u(t)| + |u(s)|) + \int_0^b |u''(y)| \, dy$$

for all $x \in (0, b)$. If $p = \infty$, then

$$|u'(x)| \leq \frac{6}{b} \sup_{t \in (0,b)} |u(t)| + b \sup_{t \in (0,b)} |u''(t)|,$$  \hspace{1cm} (2)

while if $1 \leq p < \infty$, then by Hölder’s inequality

$$|u'(x)| \leq \frac{3}{b} (|u(t)| + |u(s)|) + b^{\frac{p-1}{p}} \left( \int_0^b |u''(y)|^p \, dy \right)^{\frac{1}{p}}.$$  \hspace{1cm} (3)

Using the convexity of the function $\tau \mapsto \tau^p$, we have that

$$|u'(x)|^p \leq \frac{3^{p-1}}{b^p} (|u(t)|^p + |u(s)|^p) + 3^{p-1} b^{p-1} \int_0^b |u''(y)|^p \, dy.$$
By averaging first in \( s \) over \((0, \frac{1}{3}b)\) and then in \( t \) over \((\frac{2}{3}b, b)\), we get

\[
|u'(x)|^p \leq \frac{3^{2p-1} 3}{b^{p-1}} \left( \int_{\frac{b}{3}}^b |u(t)|^p \, dt + \int_{0}^{\frac{b}{3}} |u(s)|^p \, ds \right) + 3^{p-1} b^{p-1} \int_{0}^{b} |u''(y)|^p \, dy
\]

\[
\leq \frac{3^p}{b^{p+1}} \int_{0}^{b} |u(y)|^p \, dy + 3^{p-1} b^{p-1} \int_{0}^{b} |u''(y)|^p \, dy.
\]

Finally, we integrate in \( x \) over \((0, b)\), to obtain

\[
\int_{0}^{b} |u'(x)|^p \, dx \leq \frac{3^{2p}}{b^{p}} \int_{0}^{b} |u(y)|^p \, dy + 3^{p-1} b^{p} \int_{0}^{b} |u''(y)|^p \, dy.
\]  

**Step 2:** If \( I \) has infinite length, fix \( b > 0 \) and subdivide \( I \) in subintervals of length \( b \). Since (2), respectively, (4), holds in every such subinterval, we get

\[\sup_{x \in I} |u'(x)| \leq \frac{6}{b} \sup_{t \in I} |u(t)| + b \sup_{t \in I} |u''(t)|\]  

(5)

if \( p = \infty \), while

\[
\int_{I} |u'(x)|^p \, dx \leq \frac{3^{2p}}{b^{p}} \int_{I} |u(x)|^p \, dx + 3^{p-1} b^{p} \int_{I} |u''(x)|^p \, dx
\]  

(6)

if \( 1 \leq p < \infty \). By taking \( b \) to be \( \ell \), we get (1).

On the other hand, if \( I \) has finite length, let \( m \in \mathbb{N} \) and divide \( I \) into \( m \) subintervals of length \( b := \frac{1}{m} \) length \( I \). Then we get (5) and (6). It suffices to take \( m \) to be the integer part of \( \frac{\text{length } I}{\ell} \).

**Step 3:** To remove the additional hypothesis that \( u \in C^\infty(I) \), one can use standard mollifiers (see, e.g. Step 4 of the proof of Theorem 10.55. We omit the details. \[\blacksquare\]

**Remark 2** When \( p = 1 \) and \( u \in L^1(I) \) is such that its second order distributional derivative \( u'' \) belongs to \( M_b ; (I; \mathbb{R}) \), then, using Theorem 13.9, inequality (1) continues to hold with \( \|u''\|_{L^1(I)} \) replaced by the total variation \( |u''| (I) \). In turn, \( u \in W^{1,1}(I) \) and \( u' \in BV(I) \).

**Remark 3** Note that in Steps 1 and 2 we have not used the fact that \( u \) and \( u'' \) belong to \( L^p(I) \). Thus, for \( u \in C^\infty(I) \) (or \( u \in C^2(I) \)) inequality (1) always hold, with the right-hand side possibly infinite.

Next we consider the case \( m \geq 2 \).

**Theorem 4** \((m \geq 2)\). Let \( I \subset \mathbb{R} \) be an open interval, let \( u \in L^1_{\text{loc}}(I) \), let \( 1 \leq p \leq \infty \), and let \( m \in \mathbb{N} \), with \( m \geq 2 \). Then \( u \) belongs to \( W^{m,p}(I) \) if and only if \( u \) belongs to \( L^p(I) \) and its \( m \)-th distributional derivative \( u^{(m)} \) belongs to \( L^p(I) \).

In this case, for every \( 0 < \ell < \text{length } I \) and \( j \in \mathbb{N} \) with \( 1 \leq j < m \),

\[
\left\| u^{(j)} \right\|_{L^p(I)} \leq C(p, j, m) \left( \ell^{-j} \|u\|_{L^p(I)} + \ell^{m-j} \left\| u^{(m)} \right\|_{L^p(I)} \right),
\]  

(7)
Proof. Assume that \( u \in C^\infty (I) \). In what follows the constant \( C = C (p, j, m) \) may change from line to line.

Step 1: We begin by proving that
\[
\left\| u^{(j)} \right\|_{L^p(I)} \leq C \left( \ell^{-j} \left\| u \right\|_{L^p(I)} + \ell \left\| u^{(j+1)} \right\|_{L^p(I)} \right)
\]  
(8)
for every \( 0 < \ell < \text{length } I \) and for all \( j \in \mathbb{N} \) with \( 1 \leq j < m \). The proof is by induction on \( j \). For \( j = 1 \) the result follows from Theorem 1 and Remark 3.

Thus assume that (8) holds for every \( 0 < \ell < \text{length } I \) and for some \( j \in \mathbb{N} \) with \( 1 \leq j < m - 1 \) and let’s prove that
\[
\left\| u^{(j+1)} \right\|_{L^p(I)} \leq C \left( \ell^{-j-1} \left\| u \right\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right)
\]
for every \( 0 < \ell < \text{length } I \). Let \( \theta \in (0,1) \). Applying Theorem 1 and Remark 3 to the function \( v = u^{(j)} \) and we obtain
\[
\left\| u^{(j+1)} \right\|_{L^p(I)} \leq C (p) \left( \ell^{-1} \left\| u^{(j)} \right\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right)
\]
\[
\leq C (p) \left( \ell^{-1} C \left( (\ell \theta)^{-j} \left\| u \right\|_{L^p(I)} + \ell \theta \left\| u^{(j+1)} \right\|_{L^p(I)} \right) + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right)
\]
\[
= C (p) \left( \left( C \theta^{-j} \ell^{-j-1} \left\| u \right\|_{L^p(I)} + C \theta \left\| u^{(j+1)} \right\|_{L^p(I)} \right) + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right),
\]
where in the second inequality we have used (8) for \( j \) (which holds by the induction hypothesis) and with \( \ell \) replaced by \( \ell \theta \), which is less than \( \text{length } I \), since \( \theta \in (0,1) \). Taking \( \theta \) so small that \( C (p) C \theta < \frac{1}{2} \), we obtain
\[
\frac{1}{2} \left\| u^{(j+1)} \right\|_{L^p(I)} \leq C (p) \left( 1 + C \theta^{-j} \right) \left( \ell^{-j-1} \left\| u \right\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right),
\]
which gives the desired inequality for \( j + 1 \).

Step 2: Next we prove that inequality (7) holds for every \( 0 < \ell < \text{length } I \) and \( j \in \mathbb{N} \) with \( 1 \leq j < m \). The proof is by induction on \( i = m - j \). For \( i = 1 \), we have that \( j = m - 1 \), and so the result follows from the previous step. Thus assume that (7) holds for every \( 0 < \ell < \text{length } I \) and for some \( j \in \mathbb{N} \) with \( 1 < j < m \) and let’s prove that
\[
\left\| u^{(j-1)} \right\|_{L^p(I)} \leq C \left( \ell^{-j+1} \left\| u \right\|_{L^p(I)} + \ell^{m-j+1} \left\| u^{(m)} \right\|_{L^p(I)} \right)
\]
for every \( 0 < \ell < \text{length } I \). Note that if \( i = m - j \), then \( i + 1 = m - (j - 1) \).
Let $\theta \in (0,1)$. Applying (8) for $j - 1$ and then (7) for $j$ (which holds by the induction hypothesis), we obtain

$$
\left\| u^{(j-1)} \right\|_{L^p(I)} \leq C \left( \ell^{-j+1} \left\| u \right\|_{L^p(I)} + \ell \left\| u^{(j)} \right\|_{L^p(I)} \right)
$$

$$
\leq C \left( \ell^{-j+1} \left\| u \right\|_{L^p(I)} + \ell C \left( \ell^{-j} \left\| u \right\|_{L^p(I)} + \ell^{m-j} \left\| u^{(m)} \right\|_{L^p(I)} \right) \right)
$$

$$
\leq C \left( \ell^{-j+1} \left\| u \right\|_{L^p(I)} + \ell^{m-j+1} \left\| u^{(j+2)} \right\|_{L^p(I)} \right),
$$

which gives the desired inequality for $j - 1$.

**Step 3:** To remove the additional hypothesis that $u \in C^\infty(I)$, one can use standard mollifiers (see, e.g. Step 4 of the proof of Theorem 10.55. We omit the details. ■

**Corollary 5** Let $I \subset \mathbb{R}$ be an open interval, let $1 \leq p \leq \infty$, let $m \in \mathbb{N}$, with $m \geq 2$, and let $u \in W^{2,p}(I)$. Then for every $j \in \mathbb{N}$ with $1 \leq j < m$,

$$
\left\| u^{(j)} \right\|_{L^p(I)} \leq C (p, j, m) \left\| u \right\|_{L^p(I)}^{(m-j)/m} \left\| u^{(m)} \right\|_{L^p(I)}^{j/m}
$$

(9)

if either length $I = \infty$ or length $I < \infty$ and

$$
\frac{m}{m-j} \left\| u^{(m)} \right\|_{L^p(I)} \leq \frac{m-j}{m} \left\| u \right\|_{L^p(I)}
$$

while

$$
\left\| u^{(j)} \right\|_{L^p(I)} \leq mC (p, j, m) (\text{length } I)^{-j} \left\| u \right\|_{L^p(I)}
$$

(10)

if length $I < \infty$ and $(m-j) \left\| u^{(m)} \right\|_{L^p(I)} \leq \frac{m-j}{m} \left\| u \right\|_{L^p(I)}$.

**Proof.** If $\left\| u^{(m)} \right\|_{L^p(I)} = 0$, then $u$ is a polynomial of degree $m - 1$. If length $I = \infty$, then $u = 0$, while if length $I < \infty$, the result follows from direct calculations. Thus in what follow we assume that $\left\| u^{(m)} \right\|_{L^p(I)} > 0$ and $\left\| u \right\|_{L^p(I)} > 0$. Consider the function

$$
g(t) = \frac{1}{t} A + t^{m-j}B,
$$

where $A, B > 0$ and $0 < t \leq t_0$, with $t_0 \in (0, \infty]$. Then

$$
\inf_{0 < t \leq t_0} g(t) = (j A)^{\frac{m-j}{m}} ((m-j) B)^{\frac{1}{m}}
$$

(11)

if $\sqrt{\frac{j}{m-j} \frac{A}{B}} \leq t_0$, while

$$
\inf_{0 < t \leq t_0} g(t) = \frac{1}{t_0^j} A + t_0^{m-j}B
$$

(12)

if $\sqrt{\frac{j}{m-j} \frac{A}{B}} > t_0$. In particular, if $t_0 = \infty$, then we are always in the first case.
Taking $A = \|u\|_{L^p(I)}$, $B = \|u^{(m)}\|_{L^p(I)}$ and $t_0 = \text{length } I$, the inequality (9) follows from (11) and Theorem 4.

To obtain (10), note that if $(m - j) (\text{length } I)^m \|u^{(m)}\|_{L^p(I)} < j \|u\|_{L^p(I)}$, then by Theorem 4 and (12),

$$\|u^{(j)}\|_{L^p(I)} \leq C (p, j, m) \left((\text{length } I)^{-j} \|u\|_{L^p(I)} + (\text{length } I)^{m-j} \|u^{(m)}\|_{L^p(I)}\right)$$

$$\leq mC (p, j, m) (\text{length } I)^{-j} \|u\|_{L^p(I)},$$

which concludes the proof.

In what follows, given $T > 0$, we denote by $W^{m,p}_\# (0,T)$ the space of all functions in $W^{m,p}_{\text{loc}} (\mathbb{R})$ that are $T$-periodic, endowed with the norm in $W^{m,p}_\# (0,T)$.

**Corollary 6** Let $I = (0,T)$, let $1 \leq p \leq \infty$, let $m \in \mathbb{N}$, with $m \geq 2$, and let $u \in W^{m,p}_\# (I)$. Then for every $j \in \mathbb{N}$ with $1 \leq j < m$,

$$\|u^{(j)}\|_{L^p(I)} \leq C (p, j, m) \|u^{(m-j)/m}\|_{L^p(I)} \|u^{(m)}\|_{L^p(I)}^{j/m}. \quad (13)$$

**Proof.** If $\|u^{(m)}\|_{L^p(I)} = 0$, then $u$ is a polynomial of degree $m - 1$. But since $u$ is periodic, it follows that $u$ must be constant. Hence, $u^{(j)} = 0$, and so (13) is actually an equality.

Thus in what follow, assume that $\|u^{(m)}\|_{L^p(I)} > 0$. Choose an integer $n \in \mathbb{N}$ so large that

$$n^m (m - j) (\text{length } I)^m \|u^{(m)}\|_{L^p(I)} \geq j \|u\|_{L^p(I)} \quad (14)$$

and consider the interval $J = (0,nT)$. If $1 \leq p < \infty$, then by the periodicity of $u$ and $u^{(m)}$,

$$(m - j) (\text{length } J)^{mp} \int_J |u^{(m)} (x)|^p \, dx$$

$$= (m - j) (\text{length } J)^{mp} \sum_{i=1}^n \int_{(i-1)T}^{iT} |u^{(m)} (x)|^p \, dx$$

$$= n^{mp} (m - j) (\text{length } I)^{mp} \sum_{i=1}^n \int_{(i-1)T}^{iT} |u^{(m)} (x)|^p \, dx$$

$$= n^{mp+1} (\text{length } I)^{2p} \int_0^T |u^{(m)} (x)|^p \, dx$$

$$\geq j^p n \int_0^T |u (x)|^p \, dx = j^p \int_J |u^{(m)} (x)|^p \, dx.$$
Hence, by the previous corollary applied to \( J \), and the periodicity of \( u, u^{(i)} \), and \( u^{(m)} \),
\[
\begin{align*}
n^{1/p} \left\| u^{(j)} \right\|_{L^p(I)} &= \left\| u^{(j)} \right\|_{L^p(J)} \leq C(p, j, m) \left\| u \right\|_{L^p(I)}^{(m-j)/m} \left\| u^{(m)} \right\|^{j/m}_{L^p(I)} \\
&= C(p, j, m) n^{(m-j)/(mp)} \left\| u \right\|_{L^p(I)}^{(m-j)/m} n^{j/(mp)} \left\| u^{(m)} \right\|^{j/m}_{L^p(I)},
\end{align*}
\]
which gives the desired inequality.

If \( p = \infty \), then again by (14), and the facts that, by the periodicity of \( u \) and \( u^{(m)} \), \( \left\| u \right\|_{L^\infty(I)} = \left\| u \right\|_{L^\infty(I)} \) and \( \left\| u^{(m)} \right\|_{L^\infty(I)} = \left\| u^{(m)} \right\|_{L^\infty(I)} \), we have that
\[
\left\| u^{(j)} \right\|_{L^\infty(I)} = \left\| u^{(j)} \right\|_{L^\infty(J)} \leq C(p) \left\| u \right\|_{L^\infty(I)}^{(m-j)/m} \left\| u^{(m)} \right\|^{j/m}_{L^\infty(I)},
\]
which concludes the proof. ■

1.2 The \( N \)-th Dimensional Case

In this section, using a slicing argument, we obtain the \( N \)-dimensional version of Theorem 1. We begin with the simple case of a rectangle.

**Theorem 7** Let \( R := I_1 \times \cdots \times I_N \subset \mathbb{R}^N \), where \( I_i \subset \mathbb{R} \) is an open interval, let \( u \in L^1_{\text{loc}}(R) \), and let \( 1 \leq p \leq \infty \). Then \( u \) belongs to \( W^{2,p}(R) \) if and only if \( u \) belongs to \( L^p(R) \) and all its second order distributional derivatives \( \frac{\partial^2 u}{\partial x_i \partial x_j} \), \( i, j = 1, \ldots, N \), belong to \( L^p(R) \). In this case, for every \( 0 < \ell < \min_i \text{length} I_i \),
\[
\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(R)} \leq C(p, N) \left( \frac{1}{\ell} \left\| u \right\|_{L^p(R)} + \ell \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{L^p(R)} \right)
\]
for all \( i = 1, \ldots, N \).

**Proof.** Assume that \( u \in C^\infty(R) \) with \( \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(R) \) for all \( i, j = 1, \ldots, N \). Fix \( i = 1, \ldots, N \) and let \( R_i' := (a_1, b_1) \times \cdots \times (a_{i-1}, b_{i-1}) \times (a_{i+1}, b_{i+1}) \times \cdots \times (a_N, b_N) \). Since \( b_i - a_i > \ell \), for all \( x_i' \in R_i \), we are in a position to apply Theorem 1 to obtain that
\[
\int_{a_i}^{b_i} \left| \frac{\partial u}{\partial x_i}(x_i', x_i) \right|^p \, dx_i \leq \frac{32^p}{\ell^p} \int_{a_i}^{b_i} |u(x_i', x_i)|^p \, dx_i + 3^{p-1} \ell^p \int_{a_i}^{b_i} \left| \frac{\partial^2 u}{\partial x_i^2}(x_i', x_i) \right|^p \, dx_i,
\]
where we have used the notation (E2). Integrating the previous inequality over \( R_i' \) and using Tonelli’s theorem, we obtain
\[
\int_{R} \left| \frac{\partial u}{\partial x_i} \right|^p \, dx \leq \frac{32^p}{\ell^p} \int_{R} |u|^p \, dx + 3^{p-1} \ell^p \int_{R} \left| \frac{\partial^2 u}{\partial x_i^2} \right|^p \, dx,
\]
which gives the desired inequality. The general case is treated in the next exercise.

**Exercise 8** Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ be such that all second order distributional derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, N$, belong to $L^p(\Omega)$.

(i) Prove that all the first order distributional derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \ldots, N$, belong to $L^1_{\text{loc}}(\Omega)$. Hint: Given $\Omega' \subset \subset \Omega$, cover $\Omega'$ with rectangles compactly contained in $\Omega$ and use mollification together with the previous theorem.

(ii) Prove that if $\Omega = \mathbb{R}$ and $u$ and all the second order distributional derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, N$, belong to $L^p(\Omega)$, then $u$ belongs to $W^{1,p}(\Omega)$.

Next we consider uniformly Lipschitz domains (see Definition 12.10).

**Exercise 9** Given $L > 0$, consider the sector

$$
\Xi_L := S^{N-1} \cap \{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < x_N \}.
$$

Prove that there exists a constant $C_{N,p,L} > 0$ such that

$$
\int_{\Xi_L} |z \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) \geq C_{N,p,L} |z|^p
$$

for every $z \in \mathbb{R}^N$.

**Theorem 10** ($m = 2$) Let $\Omega \subset \mathbb{R}^N$ be an open set with uniformly Lipschitz boundary (with parameters $\varepsilon, L, M$), let $u \in L^1_{\text{loc}}(\Omega)$, and let $1 \leq p \leq \infty$. Then $u$ belongs to $W^{2,p}(\Omega)$ if and only if $u$ belongs to $L^p(\Omega)$ and all its second order distributional derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, N$, belong to $L^p(\Omega)$. In this case, for every $0 < \ell < \frac{1}{M+1}$,

$$
\| \nabla u \|_{L^p(\Omega; \mathbb{R}^N)} \leq C(p, N, L) \left( \frac{1}{\ell} \| u \|_{L^p(\Omega)} + \ell \| \nabla^2 u \|_{L^p(\Omega; \mathbb{R}^{N \times N})} \right)
$$

for all $i = 1, \ldots, N$.

**Proof. Step 1:** Assume that $1 \leq p < \infty$ and that $u \in C^\infty(\Omega)$ with $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$ for all $i, j = 1, \ldots, N$. Given $\ell > 0$ and $\nu \in S^{N-1}$, we define $\Omega(\nu, \ell)$ as the set of all points $x \in \Omega$ belonging to a segment, with length greater than $\ell$, parallel to $\nu$ and contained in $\Omega$. Let

$$
\Pi_\nu := \{ x \in \mathbb{R}^N : x \cdot \nu = 0 \}
$$

and for $y \in \Pi_\nu$ define the slice $\Omega(\nu, \ell, y)$ by

$$
\Omega(\nu, \ell, y) := \{ t \in \mathbb{R} : y + t\nu \in \Omega(\nu, \ell) \}.
$$
Note that if \( y \in \Pi_\nu \) is such that \( \Omega(\nu, \ell, y) \) is nonempty, then \( \Omega(\nu, \ell, y) \) is the union of a family of pairwise disjoint open intervals with length greater than \( \ell \) and we may define the function \( u_{y,\nu} : \Omega(\nu, \ell, y) \to \mathbb{R} \) by

\[
u_{y,\nu}(t) := u(y + t\nu).
\]

Applying Theorem 1 to the function \( u_{y,\nu} \) in each connected component of \( \Omega(\nu, \ell, y) \), we obtain

\[
\int_{\Omega(\nu, \ell, y)} |\nabla u(y + t\nu) \cdot \nu|^p \, dt \leq \frac{2^p}{\ell^p} \int_{\Omega(\nu, \ell, y)} |u(y + t\nu)|^p \, dt + 3^{p-1}\ell^p \int_{\Omega(\nu, \ell, y)} |\nabla^2 u(y + t\nu)\nu \cdot \nu|^p \, dt.
\]

Integrating both sides of the previous inequality over all \( y \in \Pi_\nu \) for which \( \Omega(\nu, \ell, y) \) is nonempty, by Tonelli’s Theorem, we get

\[
\int_{\Omega(\nu, \ell)} |\nabla u(x) \cdot \nu|^p \, dx \leq \frac{2^p}{\ell^p} \int_{\Omega} |u(x)|^p \, dx + 3^{p-1}\ell^p \int_{\Omega} |\nabla^2 u(x)|^p \, dx.
\]

We now integrate both sides of the previous inequality in the variable \( \nu \) over \( S^{n-1} \) and use again Tonelli’s theorem, to obtain

\[
\int \int_{G(\nu, \ell)} |\nabla u(x) \cdot \nu|^p \, d\mathcal{H}^{n-1}(\nu) \, dx = \int_{S^{n-1}} \int_{\Omega(\nu, \ell)} |\nabla u(x) \cdot \nu|^p \, dx \, d\mathcal{H}^{n-1}(\nu)
\]

\[
\leq \frac{3^p}{\ell^p} \int_{\Omega} |u(x)|^p \, dx + 3^{p-1}\ell^p \int_{\Omega} |\nabla^2 u(x)|^p \, dx,
\]

where \( \beta_N = \mathcal{H}^{n-1}(\Sigma^{n-1}) \) and \( G(\nu, \ell) := \{ \nu \in S^{n-1} : x \in \Omega(\nu, \ell) \} \).

Now fix \( 0 < \ell < \frac{\varepsilon}{2(1+\ell)} \) and \( x \in \Omega \). There are two cases: If \( \text{dist}(x, \partial \Omega) \geq \ell \), then \( B(x, \ell) \subseteq \Omega \), and so \( G(\nu, \ell) = S^{n-1} \). In this case,

\[
\int_{G(\nu, \ell)} \nabla u(x) \cdot \nu|^p \, d\mathcal{H}^{n-1}(\nu) = C_{N,p} |\nabla u(x)|^p,
\]

(15)

where we used the fact that for every \( z \in \mathbb{R}^n \setminus \{0\} \), by a rotation,

\[
\int_{S^{n-1}} |z \cdot \nu|^p \, d\mathcal{H}^{n-1}(\nu) = |z|^p \int_{S^{n-1}} \left| \frac{z}{|z|} \right| \cdot \nu|^p \, d\mathcal{H}^{n-1}(\nu)
\]

\[
= |z|^p \int_{S^{n-1}} |e_1 \cdot \nu|^p \, d\mathcal{H}^{n-1}(\nu) = |z|^p C_{N,p}.
\]

On the other hand, if \( \text{dist}(x, \partial \Omega) < \ell \), then there exists \( x_0 \in \partial \Omega \) with \( |x - x_0| < \ell \).

By parts (i) and (iii) of Definition 12.10, there exist \( n \in \mathbb{N} \), local coordinates \( y = (y', y_N) \in \mathbb{R}^{n-1} \times \mathbb{R} \), and a Lipschitz function \( f : \mathbb{R}^{n-1} \to \mathbb{R} \) (both depending on \( n \)), with \( \text{Lip} f \leq L \), such that \( B(x_0, \varepsilon) \subseteq \Omega_n \) and

\[
\Omega_n \cap \Omega = \Omega_n \cap \{(y', y_N) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_N > f(y') \}.
\]
In particular,
\[ B(x_0, \varepsilon) \cap \Omega = B(x_0, \varepsilon) \cap \{(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : y_N > f(y')\}. \]

Since \( \ell < \frac{\varepsilon}{4(1+L^4)} \), we have that \( x \in B(x_0, \frac{\varepsilon}{4(1+L^4)}) \). Using local coordinates, we can write \( x = (\tilde{y}', \tilde{y}_N) \). Consider the point \((\tilde{y}', f(\tilde{y}'))\). Since \( f \) is Lipschitz, with \( \text{Lip} f \leq L \), for \( y' \in \mathbb{R}^{N-1} \), we have that
\[ f(y') \leq f(\tilde{y}') + L|y' - \tilde{y}'| < \tilde{y}_N + L|y' - \tilde{y}'|. \]

Hence, the cone
\[ K_x = \left\{(y', y_N) \in B_{N-1} \left(\tilde{y}', \frac{\varepsilon}{4(1+L^4)}\right) \times \mathbb{R} : \tilde{y}_N + L|y' - \tilde{y}'| < y_N < \tilde{y}_N + \frac{\varepsilon}{4}\right\}. \]
is contained in \( B(x_0, \varepsilon) \cap \Omega \). Using local coordinates, consider the sector
\[ \Xi_x := \left\{ \nu = (\nu', \nu_N) \in S^{N-1} : L|\nu'| < \nu_N \right\}. \]
Since \( \ell < \frac{\varepsilon}{4(1+L^4)} \), if \( t \in (0, \ell) \) and \( \nu \in \Xi_x \), then the point \( x + t\nu \) belongs to \( K_x \cap \Omega \). Hence, \( G(x, \ell) \supset \Xi_x \), and so
\[ \int_{G(x, \ell)} |\nabla u(x) \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) \geq \int_{\Xi_x} |\nabla u(x) \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) \geq C_{N,p,L} |\nabla u(x)|^p, \]
where we have used the previous exercise. Together with (15)-(16), this shows that
\[ C_{N,p,L} \int_{\Omega} |\nabla u|^p \, dx \leq \frac{2^{2p^*} \beta_N}{\ell^p} \int_{\Omega} |u|^p \, dx + 3^{p-1} \beta_N \ell^p \int_{\Omega} |\nabla^2 u|^p \, dx, \]
which is what we wanted to prove.

The additional hypothesis that \( u \in C^\infty(\Omega) \) can be removed as in Exercise 8. We omit the details.

**Step 2:** The case \( p = \infty \) is simpler and is left as an exercise. ■

**Remark 11** Note that in the previous proof we only used a uniform cone property.

**Remark 12** When \( p = 1 \) and \( u \in L^1(\Omega) \) is such that its second order distributional derivative belongs to \( \mathcal{M}_b(\Omega; \mathbb{R}^{N \times N}) \), then, using Theorem 13.9, inequality (1) continues to hold with \( \|\nabla^2 u\|_{L^1(\Omega; \mathbb{R}^{N \times N})} \) replaced by the total variation \( |\text{D} (\nabla u)|(\Omega) \) of \( \nabla u \). In turn, \( u \in W^{1,1}(\Omega) \) and \( \nabla u \in BV(\Omega; \mathbb{R}^N) \).

Next we consider the case \( m \geq 2 \).

**Theorem 13** \((m \geq 2)\) Let \( \Omega \subset \mathbb{R}^N \) be an open set with uniformly Lipschitz boundary (with parameters \( \varepsilon, L, M \)), let \( u \in L^1_{\text{loc}}(\Omega) \), let \( 1 \leq p \leq \infty \), and let \( m \in \mathbb{N} \), with \( m \geq 2 \). Then \( u \) belongs to \( W^{m,p}(\Omega) \) if and only if \( u \) belongs to \( L^p(\Omega) \) and all its distributional derivatives \( \frac{\partial^\alpha u}{\partial x^\alpha} \), \( |\alpha| = m \), belong to \( L^p(\Omega) \). In this case, for every \( 0 < \ell < \frac{\varepsilon}{4(1+L^4)} \), for every \( j \in \mathbb{N} \) with \( 1 \leq j \leq m \),
\[ \|\nabla^j u\|_{L^p} \leq C(p, j, m, N, L) (\ell^{-j} \|u\|_{L^p} + \ell^{m-j} \|\nabla^m u\|_{L^p}). \]
**Proof.** The proof is very similar to the one of Theorem 4 and thus we omit it.

**Corollary 14** Let $\Omega \subset \mathbb{R}^N$ be an open set with uniformly Lipschitz boundary (with parameters $\varepsilon, L, M$), let $1 \leq p \leq \infty$, let $m \in \mathbb{N}$, with $m \geq 2$, and let $u \in W^{m,p}(\Omega)$. Then for every $j \in \mathbb{N}$ with $1 \leq j < m$,

$$\|\nabla^j u\|_{L^p} \leq C(p, j, m, N, L) \|u\|_{L^p}^{(m-j)/m} \|\nabla^m u\|_{L^p}^{j/m} \quad (17)$$

if either $\varepsilon = \infty$ or $\varepsilon < \infty$ and $j \|u\|_{L^p} \leq (m - j) R^m \|\nabla^m u\|_{L^p}$, while

$$\|\nabla^j u\|_{L^p} \leq mC(p, j, m, N, L) R^{-j} \|u\|_{L^p}$$

if $\varepsilon < \infty$ and $(m - j) R^m \|\nabla^m u\|_{L^p} < j \|u\|_{L^p}$, where $R := \frac{\varepsilon}{4(1+L)}$.

**Proof.** The proof is the same of the one of Corollary 5.

**Remark 15** If $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+$ or

$$\Omega := \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > f(x')\},$$

where $f : \mathbb{R}^{N-1} \to \mathbb{R}$ is a Lipschitz function, then $\varepsilon = \infty$ and inequality (17) holds.