1 Interpolation Inequalities for Intermediate Derivatives

In this section we prove that if Ω is sufficiently regular, then a function u belongs to $W^{m,p}(\Omega)$ if and if $u \in L^p(\Omega)$ and all its m-th distributional derivatives $\frac{\partial^\alpha u}{\partial x^\alpha}$, $|\alpha| = m$, belong to $L^p(\Omega)$. We begin with the one-dimensional case $N = 1$.

1.1 The One-Dimensional Case

Theorem 1 ($m = 2$) *Let $I \subset \mathbb{R}$ be an open interval and let $u \in L^1_{\text{loc}}(I)$, $1 \le p \le \infty$. Then u belongs to $W^{2,p}(I)$ if and only if u belongs to $L^p(I)$ and its second order distributional derivative u'' belongs to $L^p(I)$. In this case, for every $0 < \ell < \operatorname{length} I$,*

$$\|u'\|_{L^p(I)} \le C(p)\left(\frac{1}{\ell}\|u\|_{L^p(I)} + \ell\|u''\|_{L^p(I)}\right). \tag{1}$$

Proof. Step 1: Assume first that $u \in C^\infty(I)$ and that $I = (0, b)$ for some $b > 0$. Fix $s \in \left(0, \frac{1}{3}b\right)$ and $t \in \left(\frac{2}{3}b, b\right)$. By the mean value theorem, there exists $\xi \in (s, t)$ such that

$$u'(\xi) = \frac{u(t) - u(s)}{t - s}.$$

Hence, by the fundamental theorem of calculus, for all $x \in (0, b)$,

$$u'(x) = u'(\xi) + \int_\xi^x u''(y)\, dy = \frac{u(t) - u(s)}{t - s} + \int_\xi^x u''(y)\, dy.$$

Since $t - s \ge \frac{b}{3}$, It follows that

$$|u'(x)| \le \frac{3}{b}(|u(t)| + |u(s)|) + \int_0^b |u''(y)|\, dy$$

for all $x \in (0, b)$. If $p = \infty$, then

$$|u'(x)| \le \frac{6}{b}\sup_{t\in(0,b)}|u(t)| + b\sup_{t\in(0,b)}|u''(t)|, \tag{2}$$

while if $1 \le p < \infty$, then by Hölder's inequality

$$|u'(x)| \le \frac{3}{b}(|u(t)| + |u(s)|) + b^{\frac{p-1}{p}}\left(\int_0^b |u''(y)|^p\, dy\right)^{\frac{1}{p}}. \tag{3}$$

Using the convexity of the function $\tau \mapsto |\tau|^p$, we have that

$$|u'(x)|^p \le \frac{3^{2p-1}}{b^p}(|u(t)|^p + |u(s)|^p) + 3^{p-1}b^{p-1}\int_0^b |u''(y)|^p\, dy.$$

By averaging first in s over $\left(0, \frac{1}{3}b\right)$ and then in t over $\left(\frac{2}{3}b, b\right)$, we get

$$
\begin{aligned}
|u'(x)|^p &\le \frac{3^{2p-1}}{b^p}\frac{3}{b}\left(\int_{\frac{2}{3}b}^{\frac{b}{3}} |u(t)|^p \, dt + \int_0^{\frac{b}{3}} |u(s)|^p \, ds\right) + 3^{p-1}b^{p-1}\int_0^b |u''(y)|^p \, dy \\
&\le \frac{3^{2p}}{b^{p+1}}\int_0^b |u(y)|^p \, dy + 3^{p-1}b^{p-1}\int_0^b |u''(y)|^p \, dy.
\end{aligned}
$$

Finally, we integrate in x over $(0, b)$, to obtain

$$
\int_0^b |u'(x)|^p \, dx \le \frac{3^{2p}}{b^p}\int_0^b |u(y)|^p \, dy + 3^{p-1}b^p\int_0^b |u''(y)|^p \, dy. \tag{4}
$$

Step 2: If I has infinite length, fix $b > 0$ and subdivide I in subintervals of length b. Since (2), respectively, (4), holds in every such subinterval, we get

$$
\sup_{x\in I} |u'(x)| \le \frac{6}{b}\sup_{t\in I}|u(t)| + b\sup_{t\in I}|u''(t)| \tag{5}
$$

if $p = \infty$, while

$$
\int_I |u'(x)|^p \, dx \le \frac{3^{2p}}{b^p}\int_I |u(x)|^p \, dx + 3^{p-1}b^p\int_I |u''(x)|^p \, dx \tag{6}
$$

if $1 \le p < \infty$. By taking b to be ℓ, we get (1).

On the other hand, if I has finite length, let $m \in \mathbb{N}$ and divide I into m subintervals of length $b := \frac{1}{m}\operatorname{length} I$. Then we get (5) and (6). It suffices to take m to be the integer part of $\frac{\operatorname{length} I}{\ell}$.

Step 3: To remove the additional hypothesis that $u \in C^\infty(I)$, one can use standard mollifiers (see, e.g. Step 4 of the proof of Theorem 10.55. We omit the details. ■

Remark 2 *When $p = 1$ and $u \in L^1(I)$ is such that its second order distributional derivative u'' belongs to $\mathcal{M}_{\mathrm{b}}(I;\mathbb{R})$, then, using Theorem 13.9, inequality (1) continues to hold with $\|u''\|_{L^1(I)}$ replaced by the total ariation $|u''|(I)$. In turn, $u \in W^{1,1}(I)$ and $u' \in BV(I)$.*

Remark 3 *Note that in Steps 1 and 2 we have not used the fact that u and u'' belong to $L^p(I)$. Thus, for $u \in C^\infty(I)$ (or $u \in C^2(I)$) inequality (1) always hold, with the right-hand side possibly infinite.*

Next we consider the case $m \ge 2$.

Theorem 4 ($m \ge 2$) *Let $I \subset \mathbb{R}$ be an open interval, let $u \in L^1_{\mathrm{loc}}(I)$, let $1 \le p \le \infty$, and let $m \in \mathbb{N}$, with $m \ge 2$. Then u belongs to $W^{m,p}(I)$ if and only if u belongs to $L^p(I)$ and its m-th distributional derivative $u^{(m)}$ belongs to $L^p(I)$. In this case, for every $0 < \ell < \operatorname{length} I$ and $j \in \mathbb{N}$ with $1 \le j < m$,*

$$
\left\|u^{(j)}\right\|_{L^p(I)} \le C(p,j,m)\left(\ell^{-j}\|u\|_{L^p(I)} + \ell^{m-j}\left\|u^{(m)}\right\|_{L^p(I)}\right). \tag{7}
$$

Proof. Assume that $u \in C^{\infty}(I)$. In what follows the constant $C = C(p, j, m)$ may change from line to line.
Step 1: We begin by proving that

$$\left\| u^{(j)} \right\|_{L^p(I)} \leq C \left(\ell^{-j} \left\| u \right\|_{L^p(I)} + \ell \left\| u^{(j+1)} \right\|_{L^p(I)} \right) \tag{8}$$

for every $0 < \ell < \operatorname{length} I$ and for all $j \in \mathbb{N}$ with $1 \leq j < m$. The proof is by induction on j. For $j = 1$ the result follows from Theorem 1 and Remark 3. Thus assume that (8) holds for every $0 < \ell < \operatorname{length} I$ and for some $j \in \mathbb{N}$ with $1 \leq j < m - 1$ and let's prove that

$$\left\| u^{(j+1)} \right\|_{L^p(I)} \leq C \left(\ell^{-j-1} \left\| u \right\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right)$$

for every $0 < \ell < \operatorname{length} I$. Let $\theta \in (0, 1)$. Applying Theorem 1 and Remark 3 to the function $v = u^{(j)}$ and we obtain

$$\begin{aligned}
&\left\| u^{(j+1)} \right\|_{L^p(I)} \leq C(p) \left(\ell^{-1} \left\| u^{(j)} \right\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right) \\
&\leq C(p) \left(\ell^{-1} C \left((\ell\theta)^{-j} \left\| u \right\|_{L^p(I)} + \ell\theta \left\| u^{(j+1)} \right\|_{L^p(I)} \right) + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right) \\
&= C(p) \left(\left(C\theta^{-j} \ell^{-j-1} \left\| u \right\|_{L^p(I)} + C\theta \left\| u^{(j+1)} \right\|_{L^p(I)} \right) + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right),
\end{aligned}$$

where in the second inequality we have used (8) for j (which holds by the induction hypothesis) and with ℓ replaced by $\ell\theta$, which is less than $\operatorname{length} I$, since $\theta \in (0, 1)$. Taking θ so small that $C(p) C\theta < \frac{1}{2}$, we obtain

$$\frac{1}{2} \left\| u^{(j+1)} \right\|_{L^p(I)} \leq C(p) \left(1 + C\theta^{-j} \right) \left(\ell^{-j-1} \left\| u \right\|_{L^p(I)} + \ell \left\| u^{(j+2)} \right\|_{L^p(I)} \right),$$

which gives the desired inequality for $j + 1$.
Step 2: Next we prove that inequality (7) holds for every $0 < \ell < \operatorname{length} I$ and $j \in \mathbb{N}$ with $1 \leq j < m$. The proof is by induction on $i = m - j$. For $i = 1$, we have that $j = m - 1$, and so the result follows from the previous step. Thus assume that (7) holds for every $0 < \ell < \operatorname{length} I$ and for some $j \in \mathbb{N}$ with $1 < j < m$ and let's prove that

$$\left\| u^{(j-1)} \right\|_{L^p(I)} \leq C \left(\ell^{-j+1} \left\| u \right\|_{L^p(I)} + \ell^{m-j+1} \left\| u^{(m)} \right\|_{L^p(I)} \right)$$

for every $0 < \ell < \operatorname{length} I$. Note that if $i = m - j$, then $i + 1 = m - (j - 1)$.

Let $\theta \in (0,1)$. Applying (8) for $j-1$ and then (7) for j (which holds by the induction hypothesis), we obtain

$$\begin{aligned}\left\|u^{(j-1)}\right\|_{L^p(I)} &\le C\left(\ell^{-j+1}\|u\|_{L^p(I)}+\ell\left\|u^{(j)}\right\|_{L^p(I)}\right)\\ &\le C\left(\ell^{-j+1}\|u\|_{L^p(I)}+\ell C\left(\ell^{-j}\|u\|_{L^p(I)}+\ell^{m-j}\left\|u^{(m)}\right\|_{L^p(I)}\right)\right)\\ &\le C\left(\ell^{-j+1}\|u\|_{L^p(I)}+\ell^{m-j+1}\left\|u^{(j+2)}\right\|_{L^p(I)}\right),\end{aligned}$$

which gives the desired inequality for $j-1$.
Step 3: To remove the additional hypothesis that $u \in C^{\infty}(I)$, one can use standard mollifiers (see, e.g. Step 4 of the proof of Theorem 10.55. We omit the details. ■

Corollary 5 *Let $I \subset \mathbb{R}$ be an open interval, let $1 \le p \le \infty$, let $m \in \mathbb{N}$, with $m \ge 2$, and let $u \in W^{2,p}(I)$. Then for every $j \in \mathbb{N}$ with $1 \le j < m$,*

$$\left\|u^{(j)}\right\|_{L^p(I)} \le C(p,j,m)\|u\|_{L^p(I)}^{(m-j)/m}\left\|u^{(m)}\right\|_{L^p(I)}^{j/m} \tag{9}$$

if either length $I=\infty$ *or* length $I<\infty$ *and*

$$j\|u\|_{L^p(I)} \le (m-j)(\operatorname{length} I)^m\left\|u^{(m)}\right\|_{L^p(I)},$$

while

$$\left\|u^{(j)}\right\|_{L^p(I)} \le mC(p,j,m)(\operatorname{length} I)^{-j}\|u\|_{L^p(I)} \tag{10}$$

if length $I<\infty$ *and* $(m-j)(\operatorname{length} I)^m\left\|u^{(m)}\right\|_{L^p(I)} < j\|u\|_{L^p(I)}$.

Proof. If $\left\|u^{(m)}\right\|_{L^p(I)}=0$, then u is a polynomial of degree $m-1$. If length $I=\infty$, then $u=0$, while if length $I<\infty$, the result follows from direct calculations. Thus in what follow we assume that $\left\|u^{(m)}\right\|_{L^p(I)}>0$ and $\|u\|_{L^p(I)}>0$. Consider the function

$$g(t)=\frac{1}{t^j}A+t^{m-j}B,$$

where $A,B>0$ and $0<t\le t_0$, with $t_0\in(0,\infty]$. Then

$$\inf_{0<t\le t_0} g(t)=(jA)^{\frac{m-j}{m}}((m-j)B)^{\frac{j}{m}} \tag{11}$$

if $\sqrt[m]{\frac{j}{m-j}\frac{A}{B}}\le t_0$, while

$$\inf_{0<t\le t_0} g(t)=\frac{1}{t_0^j}A+t_0^{m-j}B \tag{12}$$

if $\sqrt[m]{\frac{j}{m-j}\frac{A}{B}}>t_0$. In particular, if $t_0=\infty$, then we are always in the first case.

Taking $A = \|u\|_{L^p(I)}$, $B = \left\|u^{(m)}\right\|_{L^p(I)}$ and $t_0 = \text{length}\, I$, the inequality (9) follows from (11) and Theorem 4.

To obtain (10), note that if $(m-j)(\text{length}\, I)^m \left\|u^{(m)}\right\|_{L^p(I)} < j\|u\|_{L^p(I)}$, then by Theorem 4 and (12),

$$\begin{aligned}\left\|u^{(j)}\right\|_{L^p(I)} &\le C(p,j,m)\left((\text{length}\, I)^{-j}\|u\|_{L^p(I)} + (\text{length}\, I)^{m-j}\left\|u^{(m)}\right\|_{L^p(I)}\right)\\ &\le mC(p,j,m)(\text{length}\, I)^{-j}\|u\|_{L^p(I)},\end{aligned}$$

which concludes the proof. ■

In what follows, given $T > 0$, we denote by $W^{m,p}_{\#}(0,T)$ the space of all functions in $W^{m,p}_{\text{loc}}(\mathbb{R})$ that are T-periodic, endowed with the norm in $W^{m,p}(0,T)$.

Corollary 6 *Let $I = (0,T)$, let $1 \le p \le \infty$, let $m \in \mathbb{N}$, with $m \ge 2$, and let $u \in W^{m,p}_{\#}(I)$. Then for every $j \in \mathbb{N}$ with $1 \le j < m$,*

$$\left\|u^{(j)}\right\|_{L^p(I)} \le C(p,j,m)\|u\|^{(m-j)/m}_{L^p(I)}\left\|u^{(m)}\right\|^{j/m}_{L^p(I)}. \tag{13}$$

Proof. If $\left\|u^{(m)}\right\|_{L^p(I)} = 0$, then u is a polynomial of degree $m-1$. But since u is periodic, it follows that u must be constant. Hence, $u^{(j)} = 0$, and so (13) is actually an equality.

Thus in what follow, assume that $\left\|u^{(m)}\right\|_{L^p(I)} > 0$. Choose an integer $n \in \mathbb{N}$ so large that

$$n^m(m-j)(\text{length}\, I)^m\left\|u^{(m)}\right\|_{L^p(I)} \ge j\|u\|_{L^p(I)} \tag{14}$$

and consider the interval $J = (0, nT)$. If $1 \le p < \infty$, then by the periodicity of u and $u^{(m)}$,

$$\begin{aligned}&(m-j)(\text{length}\, J)^{mp}\int_J\left|u^{(m)}(x)\right|^p dx\\ &\qquad = (m-j)(\text{length}\, J)^{mp}\sum_{i=1}^{n}\int_{(i-1)T}^{iT}\left|u^{(m)}(x)\right|^p dx\\ &\qquad = n^{mp}(m-j)(\text{length}\, I)^{mp}\sum_{i=1}^{n}\int_{(i-1)T}^{iT}\left|u^{(m)}(x)\right|^p dx\\ &\qquad = n^{mp+1}(\text{length}\, I)^{2p}\int_0^T\left|u^{(m)}(x)\right|^p dx\\ &\qquad \ge j^p n\int_0^T|u(x)|^p\, dx = j^p\int_J\left|u^{(m)}(x)\right|^p dx.\end{aligned}$$

Hence, by the previous corollary applied to J, and the periodicity of u, $u^{(j)}$, and $u^{(m)}$,

$$\begin{aligned} n^{1/p}\left\|u^{(j)}\right\|_{L^p(I)} &= \left\|u^{(j)}\right\|_{L^p(J)} \leq C(p,j,m)\|u\|_{L^p(J)}^{(m-j)/m}\left\|u^{(m)}\right\|_{L^p(J)}^{j/m} \\ &= C(p,j,m)\, n^{(m-j)/(mp)}\|u\|_{L^p(I)}^{(m-j)/m} n^{j/(mp)}\left\|u^{(m)}\right\|_{L^p(I)}^{j/m}, \end{aligned}$$

which gives the desired inequality.

If $p=\infty$, then again by (14), and the facts that, by the periodicity of u and $u^{(m)}$, $\|u\|_{L^\infty(J)}=\|u\|_{L^\infty(I)}$ and $\left\|u^{(m)}\right\|_{L^\infty(J)}=\left\|u^{(m)}\right\|_{L^\infty(I)}$, we have that $(\operatorname{length} J)^2\|u''\|_{L^\infty(J)} \geq \|u\|_{L^\infty(J)}$, and so by the previous corollary applied to J, and the periodicity of u, $u^{(j)}$, and $u^{(m)}$,

$$\begin{aligned} \left\|u^{(j)}\right\|_{L^\infty(I)} &= \left\|u^{(j)}\right\|_{L^\infty(J)} \leq C(p)\|u\|_{L^\infty(J)}^{(m-j)/m}\left\|u^{(m)}\right\|_{L^\infty(J)}^{j/m} \\ &= C(p)\|u\|_{L^\infty(I)}^{(m-j)/m}\left\|u^{(m)}\right\|_{L^\infty(I)}^{j/m}, \end{aligned}$$

which concludes the proof. ■

1.2 The N-th Dimensional Case

In this section, using a slicing argument, we obtain the N-dimensional version of Theorem 1. We begin with the simple case of a rectangle.

Theorem 7 *Let $R:=I_1\times\cdots\times I_N\subset\mathbb{R}^N$, where $I_i\subset\mathbb{R}$ is an open interval, let $u\in L^1_{\mathrm{loc}}(R)$, and let $1\leq p\leq\infty$. Then u belongs to $W^{2,p}(R)$ if and only if u belongs to $L^p(R)$ and all its second order distributional derivatives $\frac{\partial^2 u}{\partial x_i\partial x_j}$, $i,j=1,\ldots,N$, belong to $L^p(R)$. In this case, for every $0<\ell<\min_i \operatorname{length} I_i$,*

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(R)} \leq C(p,N)\left(\frac{1}{\ell}\|u\|_{L^p(R)}+\ell\left\|\frac{\partial^2 u}{\partial x_i^2}\right\|_{L^p(R)}\right)$$

for all $i=1,\ldots,N$.

Proof. Assume that $u\in C^\infty(R)$ with $\frac{\partial^2 u}{\partial x_i\partial x_j}\in L^p(R)$ for all $i,j=1,\ldots,N$. Fix $i=1,\ldots,N$ and let $R_i':=(a_1,b_1)\times\cdots\times(a_{i-1},b_{i-1})\times(a_{i+1},b_{i+1})\times\cdots\times(a_N,b_N)$. Since $b_i-a_i>\ell$, for all $x_i'\in R_i$, we are in a position to apply Theorem 1 to obtain that

$$\int_{a_i}^{b_i}\left|\frac{\partial u}{\partial x_i}(x_i',x_i)\right|^p dx_i \leq \frac{3^{2p}}{\ell^p}\int_{a_i}^{b_i}|u(x_i',x_i)|^p\,dx_i+3^{p-1}\ell^p\int_{a_i}^{b_i}\left|\frac{\partial^2 u}{\partial x_i^2}(x_i',x_i)\right|^p dx_i,$$

where we have used the notation (E2). Integrating the previous inequality over R_i' and using Tonelli's theorem, we obtain

$$\int_R\left|\frac{\partial u}{\partial x_i}\right|^p dx \leq \frac{3^{2p}}{\ell^p}\int_R|u|^p\,dx+3^{p-1}\ell^p\int_R\left|\frac{\partial^2 u}{\partial x_i^2}\right|^p dx,$$

which gives the desired inequality. The general case is treated in the next exercise. ■

Exercise 8 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ be such that all second order distributional derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, N$, belong to $L^p(\Omega)$.*

(i) Prove that all the first order distributional derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \ldots, N$, belong to $L^1_{\text{loc}}(\Omega)$. Hint: Given $\Omega' \subset\subset \Omega$, cover Ω' with rectangles compactly contained in Ω and use mollification together with the previous theorem.

(ii) Prove that if $\Omega = R$ and u and all the second order distributional derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, N$, belong to $L^p(\Omega)$, then u belongs to $W^{1,p}(\Omega)$.

Next we consider uniformly Lipschitz domains (see Definition 12.10).

Exercise 9 *Given $L > 0$, consider the sector*

$$\Xi_L := S^{N-1} \cap \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : L|x'| < x_N\right\}.$$

Prove that there exists a constant $C_{N,p,L} > 0$ such that

$$\int_{\Xi_L} |z \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) \geq C_{N,p,L} |z|^p$$

for every $z \in \mathbb{R}^N$.

Theorem 10 **($m = 2$)** *Let $\Omega \subset \mathbb{R}^N$ be an open set with uniformly Lipschitz boundary (with parameters ε, L, M), let $u \in L^1_{\text{loc}}(\Omega)$, and let $1 \leq p \leq \infty$. Then u belongs to $W^{2,p}(\Omega)$ if and only if u belongs to $L^p(\Omega)$ and all its second order distributional derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, N$, belong to $L^p(\Omega)$. In this case, for every $0 < \ell < \frac{\varepsilon}{4(1+L)}$,*

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} \leq C(p, N, L) \left(\frac{1}{\ell} \|u\|_{L^p(\Omega)} + \ell \left\|\nabla^2 u\right\|_{L^p(\Omega;\mathbb{R}^{N\times N})} \right)$$

for all $i = 1, \ldots, N$.

Proof. Step 1: Assume that $1 \leq p < \infty$ and that $u \in C^\infty(\Omega)$ with $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$ for all $i, j = 1, \ldots, N$. Given $\ell > 0$ and $\nu \in S^{N-1}$, we define $\Omega(\nu, \ell)$ as the set of all points $x \in \Omega$ belonging to a segment, with length greater than ℓ, parallel to ν and contained in Ω. Let

$$\Pi_\nu := \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$$

and for $y \in \Pi_\nu$ define the slice $\Omega(\nu, \ell, y)$ by

$$\Omega(\nu, \ell, y) := \{t \in \mathbb{R} : y + t\nu \in \Omega(\nu, \ell)\}.$$

Note that if $y \in \Pi_\nu$ is such that $\Omega(\nu, \ell, y)$ is nonempty, then $\Omega(\nu, \ell, y)$ is the union of a family of pairwise disjoint open intervals with length greater than ℓ and we may define the function $u_{y,\nu} \colon \Omega(\nu, \ell, y) \to \mathbb{R}$ by

$$u_{y,\nu}(t) := u(y + t\nu).$$

Applying Theorem 1 to the function $u_{y,\nu}$ in each connected component of $\Omega(\nu, \ell, y)$, we obtain

$$\begin{aligned}\int_{\Omega(\nu,\ell,y)} |\nabla u(y + t\nu) \cdot \nu|^p \, dt \leq \frac{3^{2p}}{\ell^p} \int_{\Omega(\nu,\ell,y)} |u(y + t\nu)|^p \, dt \\ + 3^{p-1}\ell^p \int_{\Omega(\nu,\ell,y)} \left|\nabla^2 u(y + t\nu)\nu \cdot \nu\right|^p \, dt.\end{aligned}$$

Integrating both sides of the previous inequality over all $y \in \Pi_\nu$ for which $\Omega(\nu, \ell, y)$ is nonempty, by Tonelli's Theorem, we get

$$\int_{\Omega(\nu,\ell)} |\nabla u(x) \cdot \nu|^p \, dx \leq \frac{3^{2p}}{\ell^p} \int_\Omega |u(x)|^p \, dx + 3^{p-1}\ell^p \int_\Omega \left|\nabla^2 u(x)\right|^p \, dx.$$

We now integrate both sides of the previous inequality in the variable ν over S^{N-1} and use again Tonelli's theorem, to obtain

$$\begin{aligned}\int_\Omega \int_{G(x,\ell)} |\nabla u(x) \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) \, dx = \int_{S^{N-1}} \int_{\Omega(\nu,\ell)} |\nabla u(x) \cdot \nu|^p \, dx d\mathcal{H}^{N-1}(\nu) \\ \leq \frac{3^{2p}\beta_N}{\ell^p} \int_\Omega |u(x)|^p \, dx + 3^{p-1}\beta_N \ell^p \int_\Omega \left|\nabla^2 u(x)\right|^p \, dx,\end{aligned} \tag{15}$$

where $\beta_N = \mathcal{H}^{N-1}(S^{N-1})$ and $G(x, \ell) := \{\nu \in S^{N-1} : x \in \Omega(\nu, \ell)\}$.

Now fix $0 < \ell < \frac{\varepsilon}{4(1+L)}$ and $x \in \Omega$. There are two cases: If $\operatorname{dist}(x, \partial\Omega) \geq \ell$, then $\overline{B(x, \ell)} \subset \Omega$, and so $G(x, \ell) = S^{N-1}$. In this case,

$$\int_{G(x,\ell)} |\nabla u(x) \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) = C_{N,p} |\nabla u(x)|^p, \tag{16}$$

where we used the fact that for every $z \in \mathbb{R}^N \setminus \{0\}$, by a rotation,

$$\begin{aligned}\int_{S^{N-1}} |z \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) &= |z|^p \int_{S^{N-1}} \left|\frac{z}{|z|} \cdot \nu\right|^p \, d\mathcal{H}^{N-1}(\nu) \\ &= |z|^p \int_{S^{N-1}} |e_1 \cdot \nu|^p \, d\mathcal{H}^{N-1}(\nu) =: |z|^p C_{N,p}.\end{aligned}$$

On the other hand, if $\operatorname{dist}(x, \partial\Omega) < \ell$, then there exists $x_0 \in \partial\Omega$ with $|x - x_0| < \ell$. By parts (i) and (iii) of Definition 12.10, there exist $n \in \mathbb{N}$, local coordinates $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, and a Lipschitz function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ (both depending on n), with $\operatorname{Lip} f \leq L$, such that $B(x_0, \varepsilon) \subset \Omega_n$ and

$$\Omega_n \cap \Omega = \Omega_n \cap \left\{(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : y_N > f(y')\right\}.$$

In particular,

$$B(x_0,\varepsilon)\cap\Omega = B(x_0,\varepsilon)\cap\left\{(y',y_N)\in\mathbb{R}^{N-1}\times\mathbb{R}:\ y_N>f(y')\right\}.$$

Since $\ell<\frac{\varepsilon}{4(1+L)}$, we have that $x\in B\left(x_0,\frac{\varepsilon}{4(1+L)}\right)$. Using local coordinates, we can write $x=(\bar{y}',\bar{y}_N)$. Consider the point $(\bar{y}',f(\bar{y}'))$. Since f is Lipschitz, with $\operatorname{Lip} f\le L$, for $y'\in\mathbb{R}^{N-1}$, we have that

$$f(y')\le f(\bar{y}')+L|y'-\bar{y}'|<\bar{y}_N+L|y'-\bar{y}'|.$$

Hence, the cone

$$K_x=\left\{(y',y_N)\in B_{N-1}\left(\bar{y}',\frac{\varepsilon}{4(1+L)}\right)\times\mathbb{R}:\ \bar{y}_N+L|y'-\bar{y}'|<y_N<\bar{y}_N+\frac{\varepsilon}{4}\right\}$$

is contained in $B(x_0,\varepsilon)\cap\Omega$. Using local coordinates, consider the sector

$$\Xi_x:=\left\{\nu=(\nu',\nu_N)\in S^{N-1}:\ L|\nu'|<\nu_N\right\}.$$

Since $\ell<\frac{\varepsilon}{4(1+L)}$, if $t\in(0,\ell)$ and $\nu\in\Xi_x$, then the point $x+t\nu$ belongs to $K_x\subset\Omega$. Hence, $G(x,\ell)\supset\Xi_x$, and so

$$\int_{G(x,\ell)}|\nabla u(x)\cdot\nu|^p\,d\mathcal{H}^{N-1}(\nu)\ge\int_{\Xi_x}|\nabla u(x)\cdot\nu|^p\,d\mathcal{H}^{N-1}(\nu)\ge C_{N,p,L}|\nabla u(x)|^p,$$

where we have used the previous exercise. Together with (15)-(16), this shows that

$$C_{N,p,L}\int_\Omega|\nabla u|^p\,dx\le\frac{3^{2p}\beta_N}{\ell^p}\int_\Omega|u|^p\,dx+3^{p-1}\beta_N\ell^p\int_\Omega\left|\nabla^2u\right|^p\,dx,$$

which is what we wanted to prove.

The additional hypothesis that $u\in C^\infty(\Omega)$ can be removed as in Exercise 8. We omit the details.

Step 2: The case $p=\infty$ is simpler and is left as an exercise. ■

Remark 11 *Note that in the previous proof we only used a uniform cone property.*

Remark 12 *When $p=1$ and $u\in L^1(\Omega)$ is such that its second order distributional derivative belongs to $\mathcal{M}_{\mathrm{b}}\left(\Omega;\mathbb{R}^{N\times N}\right)$, then, using Theorem 13.9, inequality (1) continues to hold with $\left\|\nabla^2u\right\|_{L^1(\Omega;\mathbb{R}^{N\times N})}$ replaced by the total variation $|D(\nabla u)|(\Omega)$ of ∇u. In turn, $u\in W^{1,1}(\Omega)$ and $\nabla u\in BV\left(\Omega;\mathbb{R}^N\right)$.*

Next we consider the case $m\ge2$.

Theorem 13 ($m\ge2$) *Let $\Omega\subset\mathbb{R}^N$ be an open set with uniformly Lipschitz boundary (with parameters ε,L,M), let $u\in L^1_{\mathrm{loc}}(\Omega)$, let $1\le p\le\infty$, and let $m\in\mathbb{N}$, with $m\ge2$. Then u belongs to $W^{m,p}(\Omega)$ if and only if u belongs to $L^p(\Omega)$ and all its distributional derivatives $\frac{\partial^\alpha u}{\partial x^\alpha}$, $|\alpha|=m$, belong to $L^p(\Omega)$. In this case, for every $0<\ell<\frac{\varepsilon}{4(1+L)}$, for every $j\in\mathbb{N}$ with $1\le j<m$,*

$$\left\|\nabla^j u\right\|_{L^p}\le C(p,j,m,N,L)\left(\ell^{-j}\|u\|_{L^p}+\ell^{m-j}\|\nabla^m u\|_{L^p}\right).$$

Proof. The proof is very similar to the one of Theorem 4 and thus we omit it. ■

Corollary 14 *Let $\Omega \subset \mathbb{R}^N$ be an open set with uniformly Lipschitz boundary (with parameters ε, L, M), let $1 \leq p \leq \infty$, let $m \in \mathbb{N}$, with $m \geq 2$, and let $u \in W^{m,p}(\Omega)$. Then for every $j \in \mathbb{N}$ with $1 \leq j < m$,*

$$\left\|\nabla^j u\right\|_{L^p} \leq C(p,j,m,N,L)\,\|u\|_{L^p}^{(m-j)/m}\,\|\nabla^m u\|_{L^p}^{j/m} \tag{17}$$

if either $\varepsilon = \infty$ or $\varepsilon < \infty$ and $j\,\|u\|_{L^p} \leq (m-j)\,R^m\,\|\nabla^m u\|_{L^p}$, while

$$\left\|\nabla^j u\right\|_{L^p} \leq mC(p,j,m,N,L)\,R^{-j}\,\|u\|_{L^p}$$

if $\varepsilon < \infty$ and $(m-j)\,R^m\,\|\nabla^m u\|_{L^p} < j\,\|u\|_{L^p}$, where $R := \frac{\varepsilon}{4(1+L)}$.

Proof. The proof is the same of the one of Corollary 5. ■

Remark 15 *If $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}_+^N$ or*

$$\Omega := \left\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \, x_N > f(x')\right\},$$

where $f : \mathbb{R}^{N-1} \to \mathbb{R}$ is a Lipschitz function, then $\varepsilon = \infty$ and inequality (17) holds.