# 1 An Alternative Proof of Proposition 16.21

We present a different proof of Proposition 16.21 that does not use the Hardy-Littlewood inequality. It is due to Malý and Pick [1].

**Proposition 1 (Proposition 16.21)** *Let* $u \in W^{1,1}\left(\mathbb{R}^N\right)$, $N \geq 2$. *Then*

$$\sup_{s>0} s\left[\mathcal{L}^N\left(\left\{x \in \mathbb{R}^N : |u(x)| \geq s\right\}\right)\right]^{\frac{N-1}{N}} \leq \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla u(x)|\, dx.$$

**Proof.** Using spherical coordinates, for every $r > 0$ and $x \in \mathbb{R}^N$, we have that

$$\int_{B(x,r)} \frac{1}{|x-y|^{N-1}}\, dy = \beta_N r = N\alpha_N r. \tag{1}$$

We claim that for every $x \in \mathbb{R}^N$ and every Lebesgue measurable set $E \subset \mathbb{R}^N$ of finite measure,

$$\int_E \frac{1}{|x-y|^{N-1}}\, dy \leq N\alpha_N^{\frac{N-1}{N}}\left(\mathcal{L}^N(E)\right)^{\frac{1}{N}}. \tag{2}$$

Indeed, let $r > 0$ be such that $\mathcal{L}^N(B(x,r)) = \mathcal{L}^N(E)$. Then

$$\begin{aligned}\mathcal{L}^N(B(x,r)\setminus E) &= \mathcal{L}^N(B(x,r)) - \mathcal{L}^N(B(x,r)\cap E)\\ &= \mathcal{L}^N(E) - \mathcal{L}^N(B(x,r)\cap E) = \mathcal{L}^N(E\setminus B(x,r)),\end{aligned}$$

and so

$$\begin{aligned}\int_{E\setminus B(x,r)} \frac{1}{|x-y|^{N-1}}\, dy &\leq \int_{E\setminus B(x,r)} \frac{1}{r^{N-1}}\, dy\\ &= \int_{B(x,r)\setminus E} \frac{1}{r^{N-1}}\, dy \leq \int_{B(x,r)\setminus E} \frac{1}{|x-y|^{N-1}}\, dy,\end{aligned}$$

where we used the fact that $\frac{1}{|x-y|^{N-1}} \leq \frac{1}{r^{N-1}}$ for all $y \in E \setminus B(x,r)$, while the opposite inequality holds for $y \in B(x,r)\setminus E$. Adding $\int_{E\cap B(x,r)} \frac{1}{|x-y|^{N-1}}\, dy$ to both sides of the previous inequality and using (1) gives

$$\begin{aligned}\int_E \frac{1}{|x-y|^{N-1}}\, dy &\leq \int_{B(x,r)} \frac{1}{|x-y|^{N-1}}\, dy = N\alpha_N r\\ &= N\alpha_N^{\frac{N-1}{N}}\left(\mathcal{L}^N(B(x,r))\right)^{\frac{1}{N}} = N\alpha_N^{\frac{N-1}{N}}\left(\mathcal{L}^N(E)\right)^{\frac{1}{N}},\end{aligned}$$

which proves (2).

Now fix $u \in W^{1,1}\left(\mathbb{R}^N\right)$. By Exercise 16.20, and a change of variables, for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$,

$$|u(x)| \leq \frac{1}{N\alpha_N} \int_{\mathbb{R}^N} |\nabla u(y)| \frac{1}{|x-y|^{N-1}}\, dy.$$

Let $s > 0$ and fix a compact $K \subset E_s := \left\{x \in \mathbb{R}^N : |u(x)| \geq s\right\}$. Integrating the previous inequality over $K$ and using Tonelli's theorem and (2) yields

$$
\begin{aligned}
s\mathcal{L}^N(K) &\leq \int_K |u(x)|\, dx \leq \frac{1}{N\alpha_N}\int_{\mathbb{R}^N} |\nabla u(y)| \int_K \frac{1}{|x-y|^{N-1}}\, dxdy \\
&\leq \frac{1}{\alpha_N^{1/N}}\left(\mathcal{L}^N(K)\right)^{1/N}\int_{\mathbb{R}^N} |\nabla u(y)|\, dy.
\end{aligned}
$$

Hence,

$$s\mathcal{L}^N(K)^{1-\frac{1}{N}} \leq \frac{1}{\alpha_N^{1/N}}\int_{\mathbb{R}^N} |\nabla u(y)|\, dy$$

and letting $K \nearrow E_s$, it follows that

$$s\mathcal{L}^N(E_s)^{1-\frac{1}{N}} \leq \frac{1}{\alpha_N^{1/N}}\int_{\mathbb{R}^N} |\nabla u(y)|\, dy.$$

It now suffices to take the supremum over all $s > 0$. ■

Since in Theorem 16.19, the value of the precise constant in Proposition 16.21 does not play any role, there is a simpler proof, which was suggested by Georgios Psaradakis.

**Second proof.** Let $s > 0$ and define $E_s := \left\{x \in \mathbb{R}^N : |u(x)| \geq s\right\}$. Then

$$s\mathcal{L}^N(E_s)^{1-\frac{1}{N}} \leq \left(\int_{E_s} |u(y)|^{\frac{N}{N-1}}\, dy\right)^{1-\frac{1}{N}} \leq \left(\int_{\mathbb{R}^N} |u(y)|^{\frac{N}{N-1}}\, dy\right)^{1-\frac{1}{N}}.$$

Applying the Sobolev–Gagliardo–Nirenberg embedding theorem (see Theorem 11.2) we get

$$s\mathcal{L}^N(E_s)^{1-\frac{1}{N}} \leq C(N)\int_{\mathbb{R}^N} |\nabla u(y)|\, dy.$$

It now suffices to take the supremum over all $s > 0$. ■

**Remark 2** *If one settles for the constant $C(N)$ found in the proof of Step 1 of Theorem 11.2, which is $1$, then the constant $\frac{1}{\alpha_N^{1/N}}$ should be replaced by $1$ in the statement of Proposition 16.21. This does not affect the proof of Theorem 16.19.*

## References

[1] J. Malý and L. Pick, *An elementary proof of sharp Sobolev embeddings.* Proc. Amer. Math. Soc. **130** (2002), no. 2, 555–563 (electronic).