

1 An Alternative Proof of Proposition 16.21

We present a different proof of Proposition 16.21 that does not use the Hardy-Littlewood inequality. It is due to Malý and Pick [1].

Proposition 1 (Proposition 16.21) *Let $u \in W^{1,1}(\mathbb{R}^N)$, $N \geq 2$. Then*

$$\sup_{s>0} s [\mathcal{L}^N(\{x \in \mathbb{R}^N : |u(x)| \geq s\})]^{\frac{N-1}{N}} \leq \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla u(x)| dx.$$

Proof. Using spherical coordinates, for every $r > 0$ and $x \in \mathbb{R}^N$, we have that

$$\int_{B(x,r)} \frac{1}{|x-y|^{N-1}} dy = \beta_N r = N\alpha_N r. \quad (1)$$

We claim that for every $x \in \mathbb{R}^N$ and every Lebesgue measurable set $E \subset \mathbb{R}^N$ of finite measure,

$$\int_E \frac{1}{|x-y|^{N-1}} dy \leq N\alpha_N^{\frac{N-1}{N}} (\mathcal{L}^N(E))^{\frac{1}{N}}. \quad (2)$$

Indeed, let $r > 0$ be such that $\mathcal{L}^N(B(x,r)) = \mathcal{L}^N(E)$. Then

$$\begin{aligned} \mathcal{L}^N(B(x,r) \setminus E) &= \mathcal{L}^N(B(x,r)) - \mathcal{L}^N(B(x,r) \cap E) \\ &= \mathcal{L}^N(E) - \mathcal{L}^N(B(x,r) \cap E) = \mathcal{L}^N(E \setminus B(x,r)), \end{aligned}$$

and so

$$\begin{aligned} \int_{E \setminus B(x,r)} \frac{1}{|x-y|^{N-1}} dy &\leq \int_{E \setminus B(x,r)} \frac{1}{r^{N-1}} dy \\ &= \int_{B(x,r) \setminus E} \frac{1}{r^{N-1}} dy \leq \int_{B(x,r) \setminus E} \frac{1}{|x-y|^{N-1}} dy, \end{aligned}$$

where we used the fact that $\frac{1}{|x-y|^{N-1}} \leq \frac{1}{r^{N-1}}$ for all $y \in E \setminus B(x,r)$, while the opposite inequality holds for $y \in B(x,r) \setminus E$. Adding $\int_{E \cap B(x,r)} \frac{1}{|x-y|^{N-1}} dy$ to both sides of the previous inequality and using (1) gives

$$\begin{aligned} \int_E \frac{1}{|x-y|^{N-1}} dy &\leq \int_{B(x,r)} \frac{1}{|x-y|^{N-1}} dy = N\alpha_N r \\ &= N\alpha_N^{\frac{N-1}{N}} (\mathcal{L}^N(B(x,r)))^{\frac{1}{N}} = N\alpha_N^{\frac{N-1}{N}} (\mathcal{L}^N(E))^{\frac{1}{N}}, \end{aligned}$$

which proves (2).

Now fix $u \in W^{1,1}(\mathbb{R}^N)$. By Exercise 16.20, and a change of variables, for \mathcal{L}^N -a.e. $x \in \mathbb{R}^N$,

$$|u(x)| \leq \frac{1}{N\alpha_N} \int_{\mathbb{R}^N} |\nabla u(y)| \frac{1}{|x-y|^{N-1}} dy.$$

Let $s > 0$ and fix a compact $K \subset E_s := \{x \in \mathbb{R}^N : |u(x)| \geq s\}$. Integrating the previous inequality over K and using Tonelli's theorem and (2) yields

$$\begin{aligned} s\mathcal{L}^N(K) &\leq \int_K |u(x)| \, dx \leq \frac{1}{N\alpha_N} \int_{\mathbb{R}^N} |\nabla u(y)| \int_K \frac{1}{|x-y|^{N-1}} \, dx dy \\ &\leq \frac{1}{\alpha_N^{1/N}} (\mathcal{L}^N(K))^{1/N} \int_{\mathbb{R}^N} |\nabla u(y)| \, dy. \end{aligned}$$

Hence,

$$s\mathcal{L}^N(K)^{1-\frac{1}{N}} \leq \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla u(y)| \, dy$$

and letting $K \nearrow E_s$, it follows that

$$s\mathcal{L}^N(E_s)^{1-\frac{1}{N}} \leq \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla u(y)| \, dy.$$

It now suffices to take the supremum over all $s > 0$. ■

Since in Theorem 16.19, the value of the precise constant in Proposition 16.21 does not play any role, there is a simpler proof, which was suggested by Georgios Psaradakis.

Second proof. Let $s > 0$ and define $E_s := \{x \in \mathbb{R}^N : |u(x)| \geq s\}$. Then

$$s\mathcal{L}^N(E_s)^{1-\frac{1}{N}} \leq \left(\int_{E_s} |u(y)|^{\frac{N}{N-1}} \, dy \right)^{1-\frac{1}{N}} \leq \left(\int_{\mathbb{R}^N} |u(y)|^{\frac{N}{N-1}} \, dy \right)^{1-\frac{1}{N}}.$$

Applying the Sobolev–Gagliardo–Nirenberg embedding theorem (see Theorem 11.2) we get

$$s\mathcal{L}^N(E_s)^{1-\frac{1}{N}} \leq C(N) \int_{\mathbb{R}^N} |\nabla u(y)| \, dy.$$

It now suffices to take the supremum over all $s > 0$. ■

Remark 2 *If one settles for the constant $C(N)$ found in the proof of Step 1 of Theorem 11.2, which is 1, then the constant $\frac{1}{\alpha_N^{1/N}}$ should be replaced by 1 in the statement of Proposition 16.21. This does not affect the proof of Theorem 16.19.*

References

- [1] J. Malý and L. Pick, *An elementary proof of sharp Sobolev embeddings*. Proc. Amer. Math. Soc. **130** (2002), no. 2, 555–563 (electronic).