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★A first course in Sobolev spaces.
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This book is intended as a graduate textbook and contains a somewhat unusual approach to Sobolev spaces. The author explains this motivation in the preface: when he faced the situation of teaching Sobolev spaces to students with very little knowledge of functional analysis and measure theory, he decided to find a new and rather direct way to introduce the subject to them. This also explains the structure, the detail, and the whole program followed in this comprehensive textbook of more than 600 pages. The book is divided into 16 chapters (subsumed in two parts), each of similar length, complemented by three substantial appendices on 'Functional analysis', 'Measures', and 'Lebesgue and Hausdorff measures', as well as the usual indices (which could have been longer), some notes, and the bibliography.

The first part, Chapters 1–7, deals with functions of one variable and contains results which may be in part well known. The advantage of this presentation is, however, an essentially self-contained and unified approach that not only includes important facts and theorems with their proofs, but also provides many examples, some instructive figures, and exercises throughout the text. This makes the subject accessible for students in advanced undergraduate courses, but the book is interesting to read for researchers, too.

Chapter 1, 'Monotone functions', deals with continuity and differentiability properties of such functions, culminating in Lebesgue's differentiation theorem with two different proofs. This is followed in Chapter 2, 'Functions of bounded pointwise variation', by the discussion of the smallest vector space that contains all monotone functions, the space of functions of bounded pointwise variation in the interval I, $\mathrm{BPV}(I)$ (what is often called $\mathrm{BV}(I)$). What may be of particular interest, for advanced readers also, is the last part in this chapter, devoted to a characterisation of continuous functions in BPV due to Banach (and connected with the area formula). Chapter 3, 'Absolutely continuous functions', starts with the observation that for monotone functions the fundamental theorem of calculus fails in general for Lebesgue integration. Thus one considers the subclass of absolutely continuous functions $\mathrm{AC}(I)$ of $\mathrm{BPV}(I)$ to recover this feature. The rest of the chapter is devoted to the comparison of AC with BPV, including also the decomposition of BPV functions into AC and singular functions, and the analogues of classical concepts within this context. Chapter 4, 'Curves', is aimed at extending the notion of pointwise variation to functions with values in metric spaces, starting with (curves in) $\mathbb{R}^d$. Naturally, this is closely related to the notion of rectifiability. It is refined by the concept of Fréchet curves in order to allow non-simple continuous curves to be parametrised by arclength. Another tool to characterise the length of a curve $L(\gamma)$ is by its Hausdorff measure $\mathcal{H}^1(\gamma)$. The chapter ends with Jordan's curve theorem. As a preparation for Chapter 7 (where functions from BPV are discovered as those whose derivatives are finite measures), Chapter 5, 'Lebesgue-Stieltjes measures', is devoted to the correspondence between BPV functions and Radon measures, starting with the interplay between increasing functions and measures. Then it is extended to BPV functions and (Lebesgue-Stieltjes) signed measures. As a counterpart for the decomposition of BPV functions, the decomposition of measures is studied before further essentials for Lebesgue-Stieltjes integrals are proved. Chapter 6, 'Decreasing rearrangement', deals with the decreasing (often called 'non-increasing') rearrangement of functions. (One of the standard monographs in this area, that of C. Bennett and R. C. Sharpley [*Interpolation of operators*, Academic Press, Boston, MA, 1988; MR0928802 (89e:46001)], should have been referred to here.) The connection to the preceding chapters is visible when proving the absolute continuity of the function u^* and studying its derivative. In Chapter 7, 'Functions of bounded variation and Sobolev functions', the tools prepared lead to the characterisation of functions $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}$, $1 \le p < \infty$, by means of an absolutely continuous representative and its derivative. The Sobolev space $W^{1,p}(\Omega)$, $1 \le p \le \infty$, is introduced here as the subspace of $L^p(\Omega)$ such that for all test functions $\varphi \in C_c^1(\Omega)$, it holds that

$$\int_\Omega u\varphi' \mathrm{d}x = -\int_\Omega v\varphi \mathrm{d}x$$

for some suitable $v \in L^p(\Omega)$. The space $W^{1,1}(\Omega)$ is identified as a subspace of $\mathrm{BV}(\Omega)$, where the latter denotes the space of functions of bounded variation (not to be mixed up with BPV).

In the second part of the book, Chapters 8–16, the author turns to functions of several variables. This means, in particular, that some concepts developed in the first part are now extended to higher dimensions. Chapter 8, 'Absolutely continuous functions and change of variables', can thus be seen as the counterpart of Chapter 3 in dealing with absolute continuity. Chapter 9, 'Distributions', contains material that one would have expected at the very beginning of a 'usual' textbook on Sobolev spaces: concepts of test functions, regular and singular distributions, their derivatives, supports, convolutions, and certain characterisations needed in what follows. Naturally this chapter contains more exercises than some earlier parts. Parallel to Chapter 7 (in the case of one variable), Sobolev spaces on sets $\Omega \subset \mathbb{R}^N$ are introduced in Chapter 10, 'Sobolev spaces', again via the above identification, now with respect to all (distributional) partial derivatives. Basic properties of the space $W^{1,p}(\Omega)$, $1 \le p \le \infty$ (with the special case $p = 2$), are proved, e.g. density of smooth functions (if $p < \infty$), comparison with the spaces defined by restriction (if the boundary is sufficiently smooth), inner descriptions, characterisation by differences, and duality. Special attention is paid to the description of $W^{1,p}(\Omega)$-functions, $1 \le p < \infty$, $\Omega \subset \mathbb{R}^N$, in terms of absolutely continuous representatives (on lines). Together with the preparatory chapters on absolutely continuous functions, this leads immediately to nice properties for functions in $W^{1,p}(\Omega)$. Sobolev's famous embedding theorem is the central goal of Chapter 11, 'Sobolev spaces: embeddings'. One studies the interplay between the loss of smoothness and the gain of integrability, i.e., the continuity and compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L_q(\Omega)$, $\Omega \subset \mathbb{R}^N$, for $1 \le q \le p^* = \frac{Np}{N-p}$, $1 \le p < N$. The limiting case $p = N$ deserves special attention; in order to find the suitable target space for $W^{1,N}(\Omega)$, Orlicz spaces are introduced. In the super-limiting case $p > N$, the natural choice is Hölder spaces $C^{0,\alpha}$ with $\alpha = 1 - \frac{N}{p}$. Apart from the somehow exceptional case $p = \infty$, these famous theorems are indispensable key results in every book on Sobolev spaces. Motivated by applications in partial differential equations (the original reason to study and introduce these spaces), other central features are extensions and traces. The first topic can be found in Chapter 12, 'Sobolev spaces: further properties', whereas the second one is postponed to Chapter 15, 'Sobolev spaces: traces' (since Besov spaces have to be introduced first). Poincaré's inequalities are another important subject studied in Chapter 12. Parallel to Chapter 7, 'Functions of bounded variation' are dealt with in Chapter 13, that is, $\mathrm{BV}(\Omega)$, $\Omega \subset \mathbb{R}^N$; an appropriate selection of their fundamentals is presented. The co-area formula is now proved, relating the total variation measure $|Du|$ of $u \in \mathrm{BV}(\Omega)$ with the perimeter of its superlevel sets. Further results include the Sobolev-Gagliardo-Nirenberg theorem and the isoperimetric inequality. As already mentioned, to discuss traces of Sobolev spaces one needs the concept of Besov spaces $B^{s,p,\theta}(\mathbb{R}^N)$, $0 < s < 1$, $1 \leq p, \theta \leq \infty$ (since this domain for the parameters will be sufficient): It can be found in Chapter 14, 'Besov spaces'. The author explains at the very beginning of this textbook why he cannot follow the usual approach to introduce them by interpolation or by Fourier-analytical methods. Obviously, to keep the book as self-contained as possible, as well as to avoid relying on a sound functional-analytical background, there is no suitable direct approach. To circumvent these structural difficulties, the third famous approach (due to Besov and the Russian school) is presented (referring to [O. V. Besov, V. P. Il′in and S. M. Nikol′skiĭ, *Integral representations of functions and imbedding theorems. Vol. I*, Translated from the Russian, Winston, Washington, D.C., 1978; MR0519341 (80f:46030a); *Vol. II*, Winston, Washington, D.C., 1979; MR0521808 (80f:46030b)]): the definition by differences, together with some famous inequalities and integral identities. The spaces $\mathrm{BV}(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$ are located within the scale $B^{s,p,\theta}(\mathbb{R}^N)$ and the role of the smoothness parameter s as well as the fine index θ is discussed. In general there is no monotonicity in p, but as a counterpart for the Sobolev embedding the same interplay between smoothness parameters s_1, s_2 and integrability p_1, p_2 can be observed (along lines with the same differential dimension). There are counterparts of Morrey's embedding in the supercritical case $s > \frac{N}{p}$ and embeddings between the Sobolev scale and the Besov scale. To deal with trace results another extension is necessary: to reasonably fill the gaps of the smoothness parameters between $s = 0$ and $s = 1$ (or $s = k \in \mathbb{N}$, in general) and to introduce fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$, $0 < s < 1$, as $W^{s,p}(\mathbb{R}^N) = B^{s,p,p}(\mathbb{R}^N)$ (the notation $W^{s,p}$ is rather unfortunate, since it has become clear that the more suitable *fractional* Sobolev spaces are the spaces $H^{s,p}$, but they are out of the scope of this textbook). Now everything is prepared to explain traces of functions $u \in W^{1,p}(\mathbb{R}^N_+)$ in Chapter 15, essentially concentrating on the half-space (with some subsequent extension to the setting of bounded domains). First the notion of a trace operator mapping $W^{1,p}(\mathbb{R}^N_+)$ or $\mathrm{BV}(\mathbb{R}^N_+)$ into $L^q(\mathbb{R}^{N-1})$ has to be clarified, before results of the type $\mathrm{Tr}(W^{1,p}(\mathbb{R}^N_+)) = B^{1-\frac{1}{p},p,p}(\mathbb{R}^{N-1}) = W^{s-\frac{1}{p},p}(\mathbb{R}^{N-1})$ can be shown. Chapter 16, 'Sobolev spaces: symmetrization', returns partly to the notion u^* and modifications of this, in the context of both Sobolev and BV spaces.

This comprehensive textbook is very well written and well structured. It will certainly serve as a valued reference work for graduate students and researchers working in related fields (while offering some interesting points for specialists, too). Unfortunately, due to the intention of the book to start from a very basic level and not assume any special knowledge in functional analysis, Fourier analysis, etc., essential features of Sobolev spaces are ignored (as admitted from the outset by the author). In particular, since Sobolev spaces of higher order as well as Fourier-analytic approaches are mentioned at most only in the exercises and remarks, application to the study of elliptic differential equations is (nearly) impossible. Further, well-established background or application fields like signal theory, interpolation and approximation theory, and harmonic analysis are mainly left out of the text. However, to be fair, in view of the already considerable length and the given starting point of the work, further digressions on these topics would have gone far beyond the scope of any readable textbook. In addition, good books already exist on such advanced topics, which students and researchers can seek out if inspired by the current text. Doubtless the benefit of this detailed explanation of ideas, concepts, proofs, and problems is its immediate accessibility, which makes it recommendable both for graduate students and for mathematicians with a wider field of interest, since prerequisites for reading this book are standard undergraduate courses.

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