1 Riesz Potential and Enbeddings Theorems

Given $0 < \alpha < N$ and a function $u \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$, the *Riesz potential* of u is defined by

$$I_\alpha(u)(x) := \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-\alpha}}\, dy, \quad x \in \mathbb{R}^N.$$

We begin by finding an exponent q such that

$$\|I_\alpha(u)\|_{L^q(\mathbb{R}^N)} \leq c\, \|u\|_{L^p(\mathbb{R}^N)} \tag{1}$$

for all $u \in L^p\left(\mathbb{R}^N\right)$. Assume for simplicity that $u \in C_c\left(\mathbb{R}^N\right)$, so that $I_\alpha(u)$ is well-defined, and for $r > 0$ define the rescaled function

$$u_r(x) := u(rx), \quad x \in \mathbb{R}^N.$$

If (1) holds for u_r, we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} \frac{u(ry)}{|x-y|^{N-\alpha}}\, dy\right|^q dx\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} \frac{u_r(y)}{|x-y|^{N-\alpha}}\, dy\right|^q dx\right)^{\frac{1}{q}} \\
&\leq c\left(\int_{\mathbb{R}^N} |u_r(x)|^p\, dx\right)^{\frac{1}{p}} \\
&= c\left(\int_{\mathbb{R}^N} |u(rx)|^p\, dx\right)^{\frac{1}{p}},
\end{aligned}$$

or, equivalently, after the change of variables $z := rx$, $w := ry$,

$$\left(\int_{\mathbb{R}^N} \frac{1}{r^N} \left|\int_{\mathbb{R}^N} \frac{r^{N-\alpha}}{r^N} \frac{u(w)}{|z-w|^{N-\alpha}}\, dw\right|^q dz\right)^{\frac{1}{q}} \leq c\left(\frac{1}{r^N}\int_{\mathbb{R}^N} |u(z)|^p\, dz\right)^{\frac{1}{p}},$$

that is,

$$\left(\int_{\mathbb{R}^N} |I_\alpha(u)(z)|^q\, dz\right)^{\frac{1}{q}} \leq c r^{\alpha - \frac{N}{p} + \frac{N}{q}} \left(\int_{\mathbb{R}^N} |u(z)|^p\, dz\right)^{\frac{1}{p}}.$$

If $\alpha - \frac{N}{p} + \frac{N}{q} > 0$, let $r \to 0^+$ to conclude that $u \equiv 0$, while if $\alpha - \frac{N}{p} + \frac{N}{q} < 0$, let $r \to \infty$ to conclude again that $u \equiv 0$. Hence, the only possible case is when

$$\frac{N}{q} = \frac{N}{p} - \alpha.$$

So in order for q to be positive, we need $\alpha p < N$, in which case,

$$q := \frac{Np}{N - \alpha p}.$$

1.1 The Subcritical Case $1 \le p < \frac{N}{\alpha}$

Theorem 1 *Let $0 < \alpha < N$, $1 \le p < \frac{N}{\alpha}$,*

$$q := \frac{Np}{N - \alpha p},$$

and let $u \in L^p(\mathbb{R}^N)$. Then

(i) $I_\alpha(u)(x)$ is well-defined and real valued for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$,

(ii) if $p = 1$, then for any $t > 0$,

$$\mathcal{L}^N\left(\left\{x \in \mathbb{R}^N : I_\alpha(|u|)(x) > t\right\}\right) \le \frac{C(\alpha, N)}{t^q} \|u\|_{L^1(\mathbb{R}^N)}^q,$$

(iii) if $p > 1$, then

$$\|I_\alpha(|u|)\|_{L^q(\mathbb{R}^N)} \le C(\alpha, N, p) \|u\|_{L^p(\mathbb{R}^N)}. \tag{2}$$

Part (iii) has already been proved in Proposition C.31, but we repeat the proofs since we will need to keep track of the constants. The proof is due to Hedberg [1] and uses the maximal function of u (see Definition C.27). We refer to [3] for an alternative proof and for more information on the Riesz potential.

We begin with a preliminary lemma, which is due to Tartar.

Lemma 2 *Let $u \in L^p(\mathbb{R}^N)$, $1 \le p \le \infty$, and let $v \in L^1(\mathbb{R}^N)$ be such that*

$$|v(x)| \le g(x)$$

for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$, where $g \in L^1(\mathbb{R}^N)$ is a radial function of the form $g(x) = f(|x|)$, with $f : [0, \infty) \to [0, \infty)$ decreasing. Then

$$\left|\int_{\mathbb{R}^N} v(x-y) u(y)\, dy\right| \le \|g\|_{L^1(\mathbb{R}^N)} \mathrm{M}(u)(x)$$

for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$.

Proof. By the hypotheses on v,

$$\left|\int_{\mathbb{R}^N} v(x-y) u(y)\, dy\right| \le \int_{\mathbb{R}^N} g(x-y) |u(y)|\, dy.$$

Step 1: Assume first that $f = \chi_{[0,r)}$, so that $g = \chi_{B(0,r)}$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} g(x-y) |u(y)|\, dy &= \int_{B(x,r)} |u(y)|\, dy \le \mathcal{L}^N(B(x,r)) \mathrm{M}(u)(x) \\ &= \|g\|_{L^1(\mathbb{R}^N)} \mathrm{M}(u)(x). \end{aligned}$$

Step 2: Next, consider the case in which

$$f = \sum_{i=1}^{n} a_i \chi_{[r_{i-1}, r_i)},$$

where $0 =: r_0 < r_1 < \cdots < r_n$ and $a_1 > a_2 > \cdots > a_n$. Set $c_i := a_i - a_{i+1} > 0$, $i = 1, \ldots, n$, where $a_{n+1} := 0$. Then we can write

$$f = \sum_{i=1}^{n} c_i \chi_{[0, r_i)}$$

and

$$\int_{\mathbb{R}} f(t)\, dt = \sum_{i=1}^{n} a_i (r_i - r_{i-1}) = \sum_{i=1}^{n} c_i r_i.$$

In turn,

$$g = \sum_{i=1}^{n} c_i \chi_{B(0, r_i)}$$

and so

$$\begin{aligned}\int_{\mathbb{R}^N} g(x-y)\, |u(y)|\, dy &\le \sum_{i=1}^{n} c_i \int_{\mathbb{R}^N} \chi_{B(0,r_i)}(x-y)\, |u(y)|\, dy \\ &\le \sum_{i=1}^{n} c_i \mathcal{L}^N (B(x, r_i)) (\mathrm{M}(u)(x)) = \|g\|_{L^1(\mathbb{R}^N)} \mathrm{M}(u)(x),\end{aligned}$$

where in the second inequality we have used Step 1.

Step 3: The general case follows by observing that every increasing function $f : [0, \infty) \to [0, \infty)$ can be approximated from below by an increasing sequence of simple functions of the type given in Step 2. ■

We turn to the proof of Theorem 1.

Proof of Theorem 1. Fix $r > 0$ and for $x \in \mathbb{R}^N$ write

$$\begin{aligned}I_\alpha(|u|)(x) &= \int_{B(x,r)} \frac{|u(y)|}{|x-y|^{N-\alpha}}\, dy \\ &\quad + \int_{\mathbb{R}^N \setminus B(x,r)} \frac{|u(y)|}{|x-y|^{N-\alpha}}\, dy =: I + II.\end{aligned}$$

To estimate I, we apply the previous lemma, taking $g(x) = f(|x|)$, where

$$f(t) := \begin{cases} t^{\alpha - N} & \text{if } 0 < t < r, \\ 0 & \text{otherewise.} \end{cases}$$

Then

$$\|g\|_{L^1(\mathbb{R}^N)} = \beta_N \int_0^r t^{\alpha-N} t^{N-1}\, dt = \frac{\beta_N}{\alpha} r^\alpha,$$

and so

$$I \leq \|g\|_{L^1(\mathbb{R}^N)} \operatorname{M}(u)(x) = \frac{\beta_N}{\alpha} r^\alpha \operatorname{M}(u)(x) .$$

On the other hand, using Hölder's inequality for $p > 1$,

$$\begin{aligned} II &\leq \left(\beta_N \int_r^\infty t^{(\alpha - N)p' + N - 1} \, dt \right)^{1/p'} \|u\|_{L^p(\mathbb{R}^N)} \\ &= \left(\frac{\beta_N (p-1)}{\alpha p - N} \right)^{1/p'} r^{\alpha - \frac{N}{p}} \|u\|_{L^p(\mathbb{R}^N)} . \end{aligned}$$

For $p = 1$, we use instead the fact that $\frac{1}{|x-y|^{N-\alpha}} \leq \frac{1}{r^{N-\alpha}}$ to obtain the simpler inequality

$$II \leq r^{\alpha - N} \|u\|_{L^1(\mathbb{R}^N)} .$$

Hence, we have proved that

$$I_\alpha (|u|)(x) \leq \frac{\beta_N}{\alpha} r^\alpha \operatorname{M}(u)(x) + C(\alpha, N, p) \, r^{\alpha - \frac{N}{p}} \|u\|_{L^p(\mathbb{R}^N)} .$$

Fix $\varepsilon > 0$ and choose

$$r := \left(\frac{\|u\|_{L^p(\mathbb{R}^N)}}{\operatorname{M}(u)(x) + \varepsilon} \right)^{\frac{p}{N}} .$$

Then

$$I_\alpha (|u|)(x) \leq \left(\frac{\beta_N}{\alpha} + C(\alpha, N, p) \right) \|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}} \left(\operatorname{M}(u)(x) + \varepsilon \right)^{1 - \frac{\alpha p}{N}} .$$

Since $p < \frac{N}{\alpha}$, letting $\varepsilon \to 0^+$ gives

$$I_\alpha (|u|)(x) \leq \left(\frac{\beta_N}{\alpha} + C(\alpha, N, p) \right) \|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}} \left(\operatorname{M}(u)(x) \right)^{1 - \frac{\alpha p}{N}} . \tag{3}$$

Note that in view of Theorem C.29(i), the previous inequality implies that $I_\alpha (u)(x)$ is well-defined and real valued for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$.

To prove part (ii), assume that $p = 1$. Then by (3), if $I_\alpha (|u|)(x) > t$, then

$$\begin{aligned} t < I_\alpha (|u|)(x) &\leq \left(\frac{\beta_N}{\alpha} + 1 \right) \|u\|_{L^1(\mathbb{R}^N)}^{\frac{\alpha}{N}} \left(\operatorname{M}(u)(x) \right)^{1 - \frac{\alpha}{N}} = \\ &= \left(\frac{\beta_N}{\alpha} + 1 \right) \|u\|_{L^1(\mathbb{R}^N)}^{\frac{\alpha}{N}} \left(\operatorname{M}(u)(x) \right)^{\frac{1}{q}} , \end{aligned}$$

and so

$$\left\{ x \in \mathbb{R}^N : \, I_\alpha (|u|) > t \right\} \subset \left\{ x \in \mathbb{R}^N : \, \operatorname{M}(u)(x) > \left(\frac{t}{\left(\frac{\beta_N}{\alpha} + 1 \right) \|u\|_{L^1}^{\frac{\alpha}{N}}} \right)^q \right\} .$$

It follows from Theorem C.29(ii) that

$$\mathcal{L}^N\left(\left\{x\in\mathbb{R}^N : I_\alpha(|u|)>t\right\}\right)\leq\frac{3^N\left(\frac{\beta_N}{\alpha}+1\right)^q}{t^q}\|u\|_{L^1(\mathbb{R}^N)}^q.$$

Finally, if $p>1$, then taking the L^q norm on both sides of (3) gives

$$\|I_\alpha(|u|)\|_{L^q(\mathbb{R}^N)}\leq\left(\frac{\beta_N}{\alpha}+C(\alpha,N,p)\right)\|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}}\left(\int_{\mathbb{R}^N}(\mathrm{M}(u)(x))^{\left(1-\frac{\alpha p}{N}\right)q}\,dx\right)^{\frac{1}{q}}.$$

Since $\left(1-\frac{\alpha p}{N}\right)q=p$, by Theorem C.29(iii), we have

$$\begin{aligned}\|I_\alpha(|u|)\|_{L^q(\mathbb{R}^N)}&\leq C(N,p,q)\|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}}\|u\|_{L^p(\mathbb{R}^N)}^{\frac{p}{q}}\\&=C(N,p,q)\|u\|_{L^p(\mathbb{R}^N)}.\end{aligned}$$

This concludes the proof. ■

Example 3 *The inequality (2) does not hold for $p=1$. To see this, consider a standard mollifier φ_ε. Then*

$$I_\alpha(\varphi_\varepsilon)(x)=\int_{\mathbb{R}^N}\frac{\varphi_\varepsilon(y)}{|x-y|^{N-\alpha}}\,dy=(v*\varphi_\varepsilon)(x),$$

where

$$v(x):=\frac{1}{|x|^{N-\alpha}}.$$

Hence, by Theorem C.19(ii), $I_\alpha(\varphi_\varepsilon)(x)\to\frac{1}{|x|^{N-\alpha}}$ for $\mathcal{L}^N$-a.e. $x\in\mathbb{R}^N$. If (2) holds, then

$$\|I_\alpha(\varphi_\varepsilon)\|_{L^{\frac{N}{N-\alpha}}(\mathbb{R}^N)}\leq C(\alpha,N)\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^N)}.$$

By Fatou's lemma and Theorem C.19(iv),

$$\begin{aligned}\left(\int_{\mathbb{R}^N}\frac{1}{|x|^{(N-\alpha)\frac{N}{N-\alpha}}}\,dx\right)^{\frac{N-\alpha}{N}}&\leq\liminf_{\varepsilon\to0^+}\|I_\alpha(\varphi_\varepsilon)\|_{L^{\frac{N}{N-\alpha}}(\mathbb{R}^N)}\\&\leq C(\alpha,N)\lim_{\varepsilon\to0^+}\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^N)}\\&=C(\alpha,N)\int_{\mathbb{R}^N}\frac{1}{|x|^{N-\alpha}}\,dx,\end{aligned}$$

which is a contradiction, since the integral on the left-hand side is infinite and the one on the right-hand side is finite.

As a corollary of Theorem 1, we obtain an alternative proof of the Sobolev–Gagliardo–Niremberg theorem in the case $1<p<N$.

Theorem 4 (Sobolev–Gagliardo–Nirenberg's embedding theorem) *Let $1 \leq p < N$. Then there exists a constant $C = C(N, p) > 0$ such that for every function $u \in L^{1,p}(\mathbb{R}^N)$ vanishing at infinity,*

$$\left(\int_{\mathbb{R}^N} |u(x)|^{p^*} \, dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx\right)^{\frac{1}{p}}. \tag{4}$$

In particular, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p \leq q \leq p^*$.

Proof. Assume that $1 < p < N$ and as in the proof of Theorem 11.2, that $u \in L^{p^*}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ with $\nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)$. By Exercise 16.20, for $x \in \mathbb{R}^N$,

$$|u(x)| \leq \frac{1}{\beta_N} \int_{\mathbb{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}} \, dx = \frac{1}{\beta_N} I_1(|\nabla u|)(x). \tag{5}$$

It is enough to apply Theorem 1 with $\alpha = 1$. ■

1.2 The Critical Case

Next we discuss the critical case $p = \frac{\alpha}{N}$. When $u \in L^{\frac{\alpha}{N}}(\mathbb{R}^N)$, then $I_\alpha(u)(x)$ is not finite for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$.

Exercise 5 *Prove that the function*

$$u(x) := \frac{1}{|x|^\alpha \log|x|} \chi_{\mathbb{R}^N \setminus B(0,2)}, \quad x \in \mathbb{R}^N,$$

belongs to $L^{\frac{\alpha}{N}}(\mathbb{R}^N)$ but $I_\alpha(u)(x) = \infty$ for all $x \in \mathbb{R}^N$.

Note that the problem is the behavior at ∞. To overcome this problem, there are two alternatives: One should either restrict attention to functions $u \in L^{\frac{\alpha}{N}}(\mathbb{R}^N)$ with compact support, or modify the Riesz potential by considering one of the following variants

$$\hat{I}_\alpha(u)(x) := \int_{\mathbb{R}^N} \left[\frac{1}{|x-y|^{N-\alpha}} - \frac{1 - \chi_{B(0,1)}(y)}{|x-y|^{N-\alpha}}\right] u(y) \, dy, \quad x \in \mathbb{R}^N.$$

or, for $0 < \alpha < 1$,

$$\tilde{I}_\alpha(u)(x) := \int_{\mathbb{R}^N} \left[\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x_0-y|^{N-\alpha}}\right] u(y) \, dy, \quad x \in \mathbb{R}^N.$$

In what follows, we consider functions with compact support. When $p \to \left(\frac{\alpha}{N}\right)^+$, we have that $q = \frac{Np}{N-\alpha p} \to \infty$, however if $u \in L^{\frac{\alpha}{N}}(\mathbb{R}^N)$, then one cannot conclude that $I_\alpha(u)$ belongs to $L^\infty(\mathbb{R}^N)$.

Exercise 6 *Prove that for $\varepsilon > 0$ sufficiently small the function*

$$u(x) := \frac{1}{|x|^{\alpha}\left(\log\frac{1}{|x|}\right)^{\frac{\alpha}{N}(1+\varepsilon)}}\chi_{B\left(0,\frac{1}{2}\right)}, \quad x \in \mathbb{R}^N,$$

belongs to $L^{\frac{\alpha}{N}}\left(\mathbb{R}^N\right)$ but $I_\alpha(u)(0) = \infty$.

Theorem 7 *Let $0 < \alpha < N$ and let $u \in L^{\frac{N}{\alpha}}\left(\mathbb{R}^N\right) \setminus \{0\}$ have support in a ball $B(x_0, R)$. Then for every $\gamma \in \left(0, \frac{N}{\beta_N}\right)$ there exists a constant $C_\gamma = C_\gamma(N) > 0$ such that*

$$\int_{B(x_0,R)} \exp\gamma\left(\frac{I_\alpha(|u|)(x)}{\|u\|_{L^{\frac{N}{\alpha}}}}\right)^{\frac{N}{N-\alpha}} dx \le C_\gamma R^N. \tag{6}$$

First proof. Without loss of generality, we may assume that $\|u\|_{L^{\frac{N}{\alpha}}} = 1$. We proceed as in the first part of the proof of Theorem 1, with the only difference that we take $x \in B(x_0, R)$. The estimate for I does not change, while to estimate II, note that $B(x_0, R) \subset B(x, 2R)$, so that, if $0 < r \le 2R$,

$$\begin{aligned} II &= \int_{B(x,2R)\setminus B(x,r)} \frac{|u(y)|}{|x-y|^{N-\alpha}}\, dy \\ &\le \left(\beta_N \int_r^{2R} t^{-1}\, dt\right)^{\frac{N-\alpha}{N}} \|u\|_{L^{\frac{N}{\alpha}}} = \left(\beta_N \log\frac{2R}{r}\right)^{\frac{N-\alpha}{N}} \end{aligned}$$

where we used Hölder's inequality and the fact that $\left(\frac{N}{\alpha}\right)' = \frac{N}{N-\alpha}$. On the other hand, if $r > 2R$, then $II = 0$. Hence, we have proved that

$$I_\alpha(|u|)(x) \le \frac{\beta_N}{\alpha} r^\alpha \,\mathrm{M}(u)(x) + \left(\beta_N \log\frac{2R}{r}\right)^{\frac{N-\alpha}{N}}$$

for $x \in B(x_0, R)$ and $0 < r \le 2R$, and

$$I_\alpha(|u|)(x) \le \frac{\beta_N}{\alpha} r^\alpha \,\mathrm{M}(u)(x)$$

for $x \in B(x_0, R)$ and $r > 2R$. Fix $\varepsilon, \delta > 0$ and choose

$$r := \min\left\{\left(\frac{\alpha}{\beta_N}\frac{\delta}{\mathrm{M}(u)(x) + \varepsilon}\right)^{\frac{1}{\alpha}}, 2R\right\}.$$

Then

$$I_\alpha(|u|)(x) \le \delta + \left(\beta_N \log^+\left[\left(\frac{\alpha}{\beta_N\delta}\right)^{\frac{1}{\alpha}} 2R\left(\mathrm{M}(u)(x)+\varepsilon\right)^{\frac{1}{\alpha}}\right]\right)^{\frac{N-\alpha}{N}}$$

$$= \delta + \left(\frac{\beta_N}{N} \log^+\left[\left(\frac{\alpha}{\beta_N\delta}\right)^{\frac{N}{\alpha}} (2R)^N\left(\mathrm{M}(u)(x)+\varepsilon\right)^{\frac{N}{\alpha}}\right]\right)^{\frac{N-\alpha}{N}}$$

$$\le \delta + \left(\frac{\beta_N}{N} \log\left[1+\left(\frac{\alpha}{\beta_N\delta}\right)^{\frac{N}{\alpha}} (2R)^N\left(\mathrm{M}(u)(x)+\varepsilon\right)^{\frac{N}{\alpha}}\right]\right)^{\frac{N-\alpha}{N}}.$$

Since $\gamma < \frac{N}{\beta_N}$, there exists $\rho > 0$ so large that

$$\gamma < \frac{N}{\beta_N}\left(\frac{\rho}{1+\rho}\right)^{\frac{N}{N-\alpha}}.$$

If $I_\alpha(|u|)(x) \ge (1+\rho)\delta$, then

$$I_\alpha(|u|)(x) - \delta \ge I_\alpha(|u|)(x) - \frac{1}{1+\rho} I_\alpha(|u|)(x) = \frac{\rho}{1+\rho} I_\alpha(|u|)(x),$$

and so

$$\gamma\left(I_\alpha(|u|)(x)\right)^{\frac{N}{N-\alpha}} < \frac{N}{\beta_N}\left(\frac{\rho}{1+\rho}\right)^{\frac{N}{N-\alpha}\left(I_\alpha(|u|)(x)\right)^{\frac{N}{N-\alpha}}} \le \frac{N}{\beta_N}\left|I_\alpha(|u|)(x)-\delta\right|^{\frac{N}{N-\alpha}}$$

$$\le \log\left[1+\left(\frac{\alpha}{\beta_N\delta}\right)^{\frac{N}{\alpha}} (2R)^N\left(\mathrm{M}(u)(x)+\varepsilon\right)^{\frac{N}{\alpha}}\right]$$

In turn,

$$\exp\gamma\left|I_\alpha(u)(x)\right|^{\frac{N}{N-\alpha}} \le 1+\left(\frac{\alpha}{\beta_N\delta}\right)^{\frac{N}{\alpha}} (2R)^N\left(\mathrm{M}(u)(x)+\varepsilon\right)^{\frac{N}{\alpha}}.$$

Letting $\varepsilon \to 0^+$ gives

$$\exp\gamma\left(I_\alpha(|u|)(x)\right)^{\frac{N}{N-\alpha}} \le 1+\left(\frac{\alpha}{\beta_N\delta}\right)^{\frac{N}{\alpha}} (2R)^N\left(\mathrm{M}(u)(x)\right)^{\frac{N}{\alpha}}.$$

On the other hand, if $I_\alpha(|u|)(x) < (1+\rho)\delta$, then

$$\exp\gamma\left(I_\alpha(|u|)(x)\right)^{\frac{N}{N-\alpha}} \le \exp\gamma\left((1+\rho)\delta\right)^{\frac{N}{N-\alpha}}.$$

Hence,

$$\int_{B(x_0,R)} \exp\gamma\left(I_\alpha(|u|)(x)\right)^{\frac{N}{N-\alpha}}\, dx$$

$$\le C_\gamma R^N + \left(\frac{\alpha}{\beta_N\delta}\right)^{\frac{N}{\alpha}} (2R)^N \int_{B(x_0,R)} \left(\mathrm{M}(u)(x)\right)^{\frac{N}{\alpha}}\, dx.$$

The result now follows from Theorem C.29(iii). ■

We now present a second proof, which does not rely on maximal functions, but does not give as sharp a constant γ. The following two lemmas are taken from a paper of Serrin [2].

Lemma 8 *Let $0<\alpha<N$, let $p\geq 1$, and let $u\in L^{p}\left(\mathbb{R}^{N}\right)\setminus\{0\}$ have support in a ball $B\left(x_{0},R\right)$. Then*

$$\left\|I_{\alpha}\left(|u|\right)\right\|_{L^{1}\left(B\left(x_{0},R\right)\right)}\leq\frac{\beta_{N}}{\alpha}\left(\alpha_{N}\right)^{\frac{1}{p^{\prime}}}\left(2R\right)^{\alpha+\frac{N}{p^{\prime}}}\left\|u\right\|_{L^{p}\left(B\left(x_{0},R\right)\right)}.$$

Proof. Fix $x\in B\left(x_{0},R\right)$. Using the fact that $B\left(x_{0},R\right)\subset B\left(x,2R\right)$, we have

$$\begin{aligned}\int_{B\left(x_{0},R\right)}\frac{1}{\left|x-y\right|^{N-\alpha}}\,dy&\leq\int_{B\left(x,2R\right)}\frac{1}{\left|x-y\right|^{N-\alpha}}\,dy\\&=\beta_{N}\int_{0}^{2R}r^{\alpha-1}\,dr=\frac{\beta_{N}}{\alpha}\left(2R\right)^{\alpha}.\end{aligned}$$

Hence, also by Tonelli's theorem,

$$\begin{aligned}\int_{B\left(x_{0},R\right)}I_{\alpha}\left(|u|\right)\left(x\right)\,dx&=\int_{B\left(x_{0},R\right)}\left|u\left(y\right)\right|\int_{B\left(x_{0},R\right)}\frac{1}{\left|x-y\right|^{N-\alpha}}\,dxdy.\\&\leq\frac{\beta_{N}}{\alpha}\left(2R\right)^{\alpha}\int_{B\left(x_{0},R\right)}\left|u\left(y\right)\right|\,dy\\&\leq\frac{\beta_{N}}{\alpha}\left(2R\right)^{\alpha}\left(\alpha_{N}R^{N}\right)^{\frac{1}{p^{\prime}}}\left\|u\right\|_{L^{p}\left(B\left(x_{0},R\right)\right)},\end{aligned}$$

where in the last inequality we have used Hölder's inequality. ■

Lemma 9 *Let $0<\alpha<N$, let $p>\frac{N}{\alpha}$, and let $u\in L^{p}\left(\mathbb{R}^{N}\right)\setminus\{0\}$ have support in a ball $B\left(x_{0},R\right)$. Then*

$$\left|I_{\alpha}\left(|u|\right)\left(x\right)\right|\leq\frac{N\left(p-1\right)}{p\alpha-N}\alpha_{N}^{\frac{1}{p^{\prime}}}\left(2R\right)^{\alpha-\frac{N}{p}}\left\|u\right\|_{L^{p}\left(B\left(x_{0},R\right)\right)}$$

for all $x\in B\left(x_{0},R\right)$.

Proof. Fix $0<\varepsilon<2R$ and $x\in B\left(x_{0},R\right)$ and for $t\in\left[\varepsilon,2R\right]$, define

$$\phi\left(t\right):=\int_{B\left(x,t\right)\setminus B\left(x,\varepsilon\right)}\left|u\left(y\right)\right|\,dy.$$

Using polar coordinates and Fubini's theorem we have that

$$\phi\left(t\right)=\int_{\varepsilon}^{t}r^{N-1}\int_{S^{N-1}}\left|u\left(y\left(r,\omega\right)\right)\right|\,d\mathcal{H}^{N-1}\left(\omega\right)dr.$$

Note that this shows that ϕ is absolutely continuous in $\left[\varepsilon,2R\right]$. Similarly, we have that the function

$$F\left(t\right):=\int_{B\left(x,t\right)\setminus B\left(x,\varepsilon\right)}\frac{u\left(y\right)}{\left|x-y\right|^{N-\alpha}}\,dy=\int_{\varepsilon}^{t}\frac{r^{N-1}}{r^{N-\alpha}}\int_{S^{N-1}}\left|u\left(y\left(r,\omega\right)\right)\right|\,d\mathcal{H}^{N-1}\left(\omega\right)dr$$

is absolutely continuous in $[\varepsilon, 2R]$, with

$$F'(t) = t^{\alpha - N}\phi'(t)$$

for $\mathcal{L}^1$-a.e. $t \in [\varepsilon, 2R]$. By the fundamental theorem of calculus and integration by parts, we have

$$\begin{aligned} F(2R) = F(2R) - F(\varepsilon) &= \int_\varepsilon^{2R} F'(r)\, dr = \int_\varepsilon^{2R} r^{\alpha-N}\phi'(r)\, dr \\ &= (2R)^{\alpha-N}\phi(2R) - (\alpha - N)\int_\varepsilon^{2R} r^{\alpha-N-1}\phi(r)\, dr. \end{aligned}$$

On the other hand, by Hölder's inequality,

$$\phi(r) = \int_{B(x,r)\setminus B(x,\varepsilon)} |u(y)|\, dy \le \|u\|_{L^p}\left(\alpha_N r^N\right)^{\frac{1}{p'}},$$

and so

$$\begin{aligned} F(2R) &\le \alpha_N^{\frac{1}{p'}} 2^{\alpha-N} R^{\alpha-N+\frac{N}{p'}}\|u\|_{L^p} + (N-\alpha)\,\alpha_N^{\frac{1}{p'}}\|u\|_{L^p}\int_\varepsilon^{2R} r^{\alpha-N-1+\frac{N}{p'}}\, dr \\ &\le \alpha_N^{\frac{1}{p'}}(2R)^{\alpha-\frac{N}{p}}\|u\|_{L^p} + (N-\alpha)\,\alpha_N^{\frac{1}{p'}}\frac{(2R)^{\alpha-\frac{N}{p}}}{\alpha - \frac{N}{p}}\|u\|_{L^p} \\ &= \frac{N(p-1)}{p\alpha - N}\alpha_N^{\frac{1}{p'}}(2R)^{\alpha-\frac{N}{p}}\|u\|_{L^p}. \end{aligned}$$

Using the fact that $B(x_0, R) \subset B(x, 2R)$, we have that

$$\int_{B(x_0,R)\setminus B(x,\varepsilon)} \frac{|u(y)|}{|x-y|^{N-\alpha}}\, dy = F(2R) \le \frac{N(p-1)}{p\alpha - N}\alpha_N^{\frac{1}{p'}}(2R)^{\alpha-\frac{N}{p}}\|u\|_{L^p}.$$

Letting $\varepsilon \to 0^+$ and using Lebesgue monotone convergence theorem gives the desired result. ■

Second proof. The proof follows essentially Theorem 2 in [4]. Without loss of generality, we may assume that $\|u\|_{L^{\frac{N}{\alpha}}} = 1$. Let $p = \frac{N}{\alpha}$ and write

$$N - \alpha = \frac{N-\theta_1}{p} + \frac{N-\theta_2}{p'}, \tag{7}$$

where $0 < \theta_1, \theta_2 < N$. Then, given $q > 1$, for $f \in L^{q'}(B(x_0, R))$, we have

$$\frac{|f(x)|\,|u(y)|}{|x-y|^{N-\alpha}} = \frac{|f(x)|^{\frac{1}{p}}\,|u(y)|}{|x-y|^{\frac{N-\theta_1}{p}}}\frac{|f(x)|^{\frac{1}{p'}}}{|x-y|^{\frac{N-\theta_2}{p'}}}.$$

By Hölder's inequality, we have

$$\int_{B(x_0,R)}\int_{B(x_0,R)}\frac{|f(x)|\,|u(y)|}{|x-y|^{N-\alpha}}\,dxdy \le \left(\int_{B(x_0,R)}\int_{B(x_0,R)}\frac{|f(x)|}{|x-y|^{N-\theta_2}}\,dxdy\right)^{\frac{1}{p'}}$$
$$\times\left(\int_{B(x_0,R)}|u(y)|^p\int_{B(x_0,R)}\frac{|f(x)|}{|x-y|^{N-\theta_1}}\,dxdy\right)^{\frac{1}{p}}$$
$$=: I_1\times I_2.$$

By Lemma 8 (where p and α there are replaced here by q' and θ_2),

$$I_1 \le \left(\frac{\beta_N}{\theta_2}(\alpha_N)^{\frac{1}{q}}(2R)^{\theta_2+\frac{N}{q}}\|f\|_{L^{q'}}\right)^{\frac{1}{p'}}.$$

On the other hand, if $q' > \frac{N}{\theta_1}$, by Lemma 9 (where p and α there are replaced here by q' and θ_1), we have that

$$\int_{B(x_0,R)}\frac{|f(x)|}{|x-y|^{N-\theta_1}}\,dx \le \frac{N(q'-1)}{q'\theta_1-N}\alpha_N^{\frac{1}{q}}(2R)^{\theta_1-\frac{N}{q'}}\|f\|_{L^{q'}}.$$

Hence,

$$\int_{B(x_0,R)}\int_{B(x_0,R)}\frac{|f(x)|\,|u(y)|}{|x-y|^{N-\alpha}}\,dxdy \le (\alpha_N)^{\frac{1}{q}}(2R)^{\frac{N}{q}}\left(\frac{\beta_N}{\theta_2}\right)^{\frac{1}{p'}}$$
$$\times\left(\frac{N(q'-1)}{q'\theta_1-N}\right)^{\frac{1}{p}}\|f\|_{L^{q'}}\|u\|_{L^p},$$

where we have used the fact that

$$\frac{\theta_2}{p'}+\frac{N}{qp'}+\frac{\theta_1}{p}-\frac{N}{pq'}=\frac{N}{q}$$

by (7). Taking the supremum over all $f\in L^{q'}(B(x_0,R))$, we get

$$\int_{B(x_0,R)}(I_\alpha(|u|)(x))^q\,dx \le \alpha_N(2R)^N\left(\frac{\beta_N}{\theta_2}\right)^{\frac{q}{p'}}\left(\frac{N(q'-1)}{q'\theta_1-N}\right)^{\frac{q}{p}}\|u\|_{L^p}. \quad (8)$$

Taking

$$\theta_2 := \frac{\alpha}{q},$$

we have that $\theta_2<\alpha<N$, while from (7),

$$N>\theta_1=\frac{N}{q'}+\frac{\alpha}{q}>\frac{N}{q'}.$$

Moreover, (8) becomes

$$\int_{B(x_0,R)}(I_\alpha(|u|)(x))^q\,dx \le \alpha_N(2R)^N\left(\frac{\beta_N q}{\alpha}\right)^{\frac{N-\alpha}{N}q}\left(\frac{N}{\alpha}\right)^{\frac{\alpha}{N}q}\|u\|_{L^p}.$$

Taking $q = \frac{Nk}{N-\alpha}$, where $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\int_{B(x_0,R)} \sum_{k=1}^{n} \frac{1}{k!}\gamma^k \left(I_\alpha\left(|u|\right)(x)\right)^{\frac{Nk}{N-\alpha}}\, dx \\
&\qquad \leq \alpha_N \left(2R\right)^N \|u\|_{L^p} \sum_{k=1}^{n} \frac{k^k}{k!}\left[\gamma \frac{\beta_N}{N-\alpha}\left(\frac{N}{\alpha}\right)^{\frac{N}{N-\alpha}}\right]^k \\
&\qquad =: \alpha_N \left(2R\right)^N \|u\|_{L^p} \sum_{k=1}^{n} x_k.
\end{aligned}$$

Since

$$\frac{x_{k+1}}{x_k} = \left(1+\frac{1}{k}\right)^k \gamma \frac{\beta_N}{N-\alpha}\left(\frac{N}{\alpha}\right)^{\frac{N}{N-\alpha}} \to e\gamma \frac{\beta_N}{N-\alpha}\left(\frac{N}{\alpha}\right)^{\frac{N}{N-\alpha}},$$

it follows that if

$$\gamma < \frac{N-\alpha}{e\beta_N}\left(\frac{\alpha}{N}\right)^{\frac{N}{N-\alpha}},$$

then the series converges. Hence,

$$\int_{B(x_0,R)} \exp \gamma \left(I_\alpha\left(|u|\right)(x)\right)^{\frac{N}{N-\alpha}}\, dx \leq C_\gamma R^N \|u\|_{L^p}$$

■

Using Theorem 7, we can prove Trudinger's embedding theorem. We recall that

$$\gamma_N := N\beta_N^{\frac{1}{N-1}}, \tag{9}$$

Theorem 10 *Suppose $N \geq 2$ and let $u \in W^{1,N}\left(\mathbb{R}^N\right)\backslash\{0\}$ have support in a ball $B\left(x_0,R\right)$. Then for every $\gamma \in \left(0,\gamma_N\right)$ there exists a constant $C = C\left(N,\gamma\right) > 0$ such that*

$$\int_{B(x_0,R)} \exp\left(\gamma \frac{|u(x)|^{N'}}{\|\nabla u\|_{L^N}^{N'}}\right) dx \leq C_\gamma R^N.$$

Proof. By (5),

$$\gamma \left|u(x)\right|^{N'} \leq \gamma \frac{1}{\beta_N^{N'}}\left(I_1\left(|\nabla u|\right)(x)\right)^{N'}.$$

Hence if $\gamma < \gamma_N = N\beta_N^{\frac{1}{N-1}}$, then $\gamma\frac{1}{\beta_N^{N'}} < \frac{N}{\beta_N}$, and so we are in a position to apply Theorem 7 with $a = 1$, to conclude that

$$\int_{B(x_0,R)} \exp\left(\gamma \frac{|u(x)|^{N'}}{\|\nabla u\|_{L^N}^{N'}}\right) dx \leq \int_{B(x_0,R)} \exp \gamma \frac{1}{\beta_N^{N'}}\left(\frac{I_1\left(|\nabla u|\right)(x)}{\||\nabla u|\|_{L^N}}\right)^{N'} dx \leq C_\gamma R^N.$$

■

Remark 11 *I am unable to find a simple proof of Theorem 11.29, which does not make use of symmetrization. Both proofs of Theorem 7 rely strongly on the fact that u has compact support.*

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