

1 Riesz Potential and Embeddings Theorems

Given $0 < \alpha < N$ and a function $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, the *Riesz potential* of u is defined by

$$I_\alpha(u)(x) := \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-\alpha}} dy, \quad x \in \mathbb{R}^N.$$

We begin by finding an exponent q such that

$$\|I_\alpha(u)\|_{L^q(\mathbb{R}^N)} \leq c \|u\|_{L^p(\mathbb{R}^N)} \quad (1)$$

for all $u \in L^p(\mathbb{R}^N)$. Assume for simplicity that $u \in C_c(\mathbb{R}^N)$, so that $I_\alpha(u)$ is well-defined, and for $r > 0$ define the rescaled function

$$u_r(x) := u(rx), \quad x \in \mathbb{R}^N.$$

If (1) holds for u_r , we get

$$\begin{aligned} \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \frac{u(r y)}{|x-y|^{N-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \frac{u_r(y)}{|x-y|^{N-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \\ &\leq c \left(\int_{\mathbb{R}^N} |u_r(x)|^p dx \right)^{\frac{1}{p}} \\ &= c \left(\int_{\mathbb{R}^N} |u(rx)|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

or, equivalently, after the change of variables $z := rx$, $w := ry$,

$$\left(\int_{\mathbb{R}^N} \frac{1}{r^N} \left| \int_{\mathbb{R}^N} \frac{r^{N-\alpha}}{r^N} \frac{u(w)}{|z-w|^{N-\alpha}} dw \right|^q dz \right)^{\frac{1}{q}} \leq c \left(\frac{1}{r^N} \int_{\mathbb{R}^N} |u(z)|^p dz \right)^{\frac{1}{p}},$$

that is,

$$\left(\int_{\mathbb{R}^N} |I_\alpha(u)(z)|^q dz \right)^{\frac{1}{q}} \leq c r^{\alpha - \frac{N}{p} + \frac{N}{q}} \left(\int_{\mathbb{R}^N} |u(z)|^p dz \right)^{\frac{1}{p}}.$$

If $\alpha - \frac{N}{p} + \frac{N}{q} > 0$, let $r \rightarrow 0^+$ to conclude that $u \equiv 0$, while if $\alpha - \frac{N}{p} + \frac{N}{q} < 0$, let $r \rightarrow \infty$ to conclude again that $u \equiv 0$. Hence, the only possible case is when

$$\frac{N}{q} = \frac{N}{p} - \alpha.$$

So in order for q to be positive, we need $\alpha p < N$, in which case,

$$q := \frac{Np}{N - \alpha p}.$$

1.1 The Subcritical Case $1 \leq p < \frac{N}{\alpha}$

Theorem 1 *Let $0 < \alpha < N$, $1 \leq p < \frac{N}{\alpha}$,*

$$q := \frac{Np}{N - \alpha p},$$

and let $u \in L^p(\mathbb{R}^N)$. Then

- (i) $I_\alpha(u)(x)$ is well-defined and real valued for \mathcal{L}^N -a.e. $x \in \mathbb{R}^N$,
- (ii) if $p = 1$, then for any $t > 0$,

$$\mathcal{L}^N(\{x \in \mathbb{R}^N : I_\alpha(|u|)(x) > t\}) \leq \frac{C(\alpha, N)}{t^q} \|u\|_{L^1(\mathbb{R}^N)}^q,$$

- (iii) if $p > 1$, then

$$\|I_\alpha(|u|)\|_{L^q(\mathbb{R}^N)} \leq C(\alpha, N, p) \|u\|_{L^p(\mathbb{R}^N)}. \quad (2)$$

Part (iii) has already been proved in Proposition C.31, but we repeat the proofs since we will need to keep track of the constants. The proof is due to Hedberg [1] and uses the maximal function of u (see Definition C.27). We refer to [3] for an alternative proof and for more information on the Riesz potential.

We begin with a preliminary lemma, which is due to Tartar.

Lemma 2 *Let $u \in L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, and let $v \in L^1(\mathbb{R}^N)$ be such that*

$$|v(x)| \leq g(x)$$

for \mathcal{L}^N -a.e. $x \in \mathbb{R}^N$, where $g \in L^1(\mathbb{R}^N)$ is a radial function of the form $g(x) = f(|x|)$, with $f : [0, \infty) \rightarrow [0, \infty)$ decreasing. Then

$$\left| \int_{\mathbb{R}^N} v(x-y) u(y) dy \right| \leq \|g\|_{L^1(\mathbb{R}^N)} M(u)(x)$$

for \mathcal{L}^N -a.e. $x \in \mathbb{R}^N$.

Proof. By the hypotheses on v ,

$$\left| \int_{\mathbb{R}^N} v(x-y) u(y) dy \right| \leq \int_{\mathbb{R}^N} g(x-y) |u(y)| dy.$$

Step 1: Assume first that $f = \chi_{[0,r]}$, so that $g = \chi_{B(0,r)}$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} g(x-y) |u(y)| dy &= \int_{B(x,r)} |u(y)| dy \leq \mathcal{L}^N(B(x,r)) M(u)(x) \\ &= \|g\|_{L^1(\mathbb{R}^N)} M(u)(x). \end{aligned}$$

Step 2: Next, consider the case in which

$$f = \sum_{i=1}^n a_i \chi_{[r_{i-1}, r_i]},$$

where $0 =: r_0 < r_1 < \dots < r_n$ and $a_1 > a_2 > \dots > a_n$. Set $c_i := a_i - a_{i+1} > 0$, $i = 1, \dots, n$, where $a_{n+1} := 0$. Then we can write

$$f = \sum_{i=1}^n c_i \chi_{[0, r_i]}$$

and

$$\int_{\mathbb{R}} f(t) dt = \sum_{i=1}^n a_i (r_i - r_{i-1}) = \sum_{i=1}^n c_i r_i.$$

In turn,

$$g = \sum_{i=1}^n c_i \chi_{B(0, r_i)}$$

and so

$$\begin{aligned} \int_{\mathbb{R}^N} g(x-y) |u(y)| dy &\leq \sum_{i=1}^n c_i \int_{\mathbb{R}^N} \chi_{B(0, r_i)}(x-y) |u(y)| dy \\ &\leq \sum_{i=1}^n c_i \mathcal{L}^N(B(x, r_i)) (M(u)(x)) = \|g\|_{L^1(\mathbb{R}^N)} M(u)(x), \end{aligned}$$

where in the second inequality we have used Step 1.

Step 3: The general case follows by observing that every increasing function $f : [0, \infty) \rightarrow [0, \infty)$ can be approximated from below by an increasing sequence of simple functions of the type given in Step 2. ■

We turn to the proof of Theorem 1.

Proof of Theorem 1. Fix $r > 0$ and for $x \in \mathbb{R}^N$ write

$$\begin{aligned} I_\alpha(|u|)(x) &= \int_{B(x, r)} \frac{|u(y)|}{|x-y|^{N-\alpha}} dy \\ &\quad + \int_{\mathbb{R}^N \setminus B(x, r)} \frac{|u(y)|}{|x-y|^{N-\alpha}} dy =: I + II. \end{aligned}$$

To estimate I , we apply the previous lemma, taking $g(x) = f(|x|)$, where

$$f(t) := \begin{cases} t^{\alpha-N} & \text{if } 0 < t < r, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|g\|_{L^1(\mathbb{R}^N)} = \beta_N \int_0^r t^{\alpha-N} t^{N-1} dt = \frac{\beta_N}{\alpha} r^\alpha,$$

and so

$$I \leq \|g\|_{L^1(\mathbb{R}^N)} \mathbf{M}(u)(x) = \frac{\beta_N}{\alpha} r^\alpha \mathbf{M}(u)(x).$$

On the other hand, using Hölder's inequality for $p > 1$,

$$\begin{aligned} II &\leq \left(\beta_N \int_r^\infty t^{(\alpha-N)p'+N-1} dt \right)^{1/p'} \|u\|_{L^p(\mathbb{R}^N)} \\ &= \left(\frac{\beta_N (p-1)}{\alpha p - N} \right)^{1/p'} r^{\alpha - \frac{N}{p}} \|u\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

For $p = 1$, we use instead the fact that $\frac{1}{|x-y|^{N-\alpha}} \leq \frac{1}{r^{N-\alpha}}$ to obtain the simpler inequality

$$II \leq r^{\alpha-N} \|u\|_{L^1(\mathbb{R}^N)}.$$

Hence, we have proved that

$$I_\alpha(|u|)(x) \leq \frac{\beta_N}{\alpha} r^\alpha \mathbf{M}(u)(x) + C(\alpha, N, p) r^{\alpha - \frac{N}{p}} \|u\|_{L^p(\mathbb{R}^N)}.$$

Fix $\varepsilon > 0$ and choose

$$r := \left(\frac{\|u\|_{L^p(\mathbb{R}^N)}}{\mathbf{M}(u)(x) + \varepsilon} \right)^{\frac{p}{N}}.$$

Then

$$I_\alpha(|u|)(x) \leq \left(\frac{\beta_N}{\alpha} + C(\alpha, N, p) \right) \|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}} (\mathbf{M}(u)(x) + \varepsilon)^{1 - \frac{\alpha p}{N}}.$$

Since $p < \frac{N}{\alpha}$, letting $\varepsilon \rightarrow 0^+$ gives

$$I_\alpha(|u|)(x) \leq \left(\frac{\beta_N}{\alpha} + C(\alpha, N, p) \right) \|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}} (\mathbf{M}(u)(x))^{1 - \frac{\alpha p}{N}}. \quad (3)$$

Note that in view of Theorem C.29(i), the previous inequality implies that $I_\alpha(u)(x)$ is well-defined and real valued for \mathcal{L}^N -a.e. $x \in \mathbb{R}^N$.

To prove part (ii), assume that $p = 1$. Then by (3), if $I_\alpha(|u|)(x) > t$, then

$$\begin{aligned} t < I_\alpha(|u|)(x) &\leq \left(\frac{\beta_N}{\alpha} + 1 \right) \|u\|_{L^1(\mathbb{R}^N)}^{\frac{\alpha}{N}} (\mathbf{M}(u)(x))^{1 - \frac{\alpha}{N}} = \\ &= \left(\frac{\beta_N}{\alpha} + 1 \right) \|u\|_{L^1(\mathbb{R}^N)}^{\frac{\alpha}{N}} (\mathbf{M}(u)(x))^{\frac{1}{q}}, \end{aligned}$$

and so

$$\{x \in \mathbb{R}^N : I_\alpha(|u|) > t\} \subset \left\{ x \in \mathbb{R}^N : \mathbf{M}(u)(x) > \left(\frac{t}{\left(\frac{\beta_N}{\alpha} + 1 \right) \|u\|_{L^1}^{\frac{\alpha}{N}}} \right)^q \right\}.$$

It follows from Theorem C.29(ii) that

$$\mathcal{L}^N (\{x \in \mathbb{R}^N : I_\alpha (|u|) > t\}) \leq \frac{3^N \left(\frac{\beta_N}{\alpha} + 1\right)^q}{t^q} \|u\|_{L^1(\mathbb{R}^N)}^q.$$

Finally, if $p > 1$, then taking the L^q norm on both sides of (3) gives

$$\|I_\alpha (|u|)\|_{L^q(\mathbb{R}^N)} \leq \left(\frac{\beta_N}{\alpha} + C(\alpha, N, p)\right) \|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}} \left(\int_{\mathbb{R}^N} (\mathbf{M}(u)(x))^{(1-\frac{\alpha p}{N})q} dx\right)^{\frac{1}{q}}.$$

Since $(1 - \frac{\alpha p}{N})q = p$, by Theorem C.29(iii), we have

$$\begin{aligned} \|I_\alpha (|u|)\|_{L^q(\mathbb{R}^N)} &\leq C(N, p, q) \|u\|_{L^p(\mathbb{R}^N)}^{\frac{\alpha p}{N}} \|u\|_{L^p(\mathbb{R}^N)}^{\frac{p}{q}} \\ &= C(N, p, q) \|u\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

This concludes the proof. ■

Example 3 *The inequality (2) does not hold for $p = 1$. To see this, consider a standard mollifier φ_ε . Then*

$$I_\alpha (\varphi_\varepsilon)(x) = \int_{\mathbb{R}^N} \frac{\varphi_\varepsilon(y)}{|x-y|^{N-\alpha}} dy = (v * \varphi_\varepsilon)(x),$$

where

$$v(x) := \frac{1}{|x|^{N-\alpha}}.$$

Hence, by Theorem C.19(ii), $I_\alpha (\varphi_\varepsilon)(x) \rightarrow \frac{1}{|x|^{N-\alpha}}$ for \mathcal{L}^N -a.e. $x \in \mathbb{R}^N$. If (2) holds, then

$$\|I_\alpha (\varphi_\varepsilon)\|_{L^{\frac{N}{N-\alpha}}(\mathbb{R}^N)} \leq C(\alpha, N) \|\varphi_\varepsilon\|_{L^1(\mathbb{R}^N)}.$$

By Fatou's lemma and Theorem C.19(iv),

$$\begin{aligned} \left(\int_{\mathbb{R}^N} \frac{1}{|x|^{(N-\alpha)\frac{N}{N-\alpha}}} dx\right)^{\frac{N-\alpha}{N}} &\leq \liminf_{\varepsilon \rightarrow 0^+} \|I_\alpha (\varphi_\varepsilon)\|_{L^{\frac{N}{N-\alpha}}(\mathbb{R}^N)} \\ &\leq C(\alpha, N) \lim_{\varepsilon \rightarrow 0^+} \|\varphi_\varepsilon\|_{L^1(\mathbb{R}^N)} \\ &= C(\alpha, N) \int_{\mathbb{R}^N} \frac{1}{|x|^{N-\alpha}} dx, \end{aligned}$$

which is a contradiction, since the integral on the left-hand side is infinite and the one on the right-hand side is finite.

As a corollary of Theorem 1, we obtain an alternative proof of the Sobolev–Gagliardo–Nirenberg theorem in the case $1 < p < N$.

Theorem 4 (Sobolev–Gagliardo–Nirenberg’s embedding theorem) *Let $1 \leq p < N$. Then there exists a constant $C = C(N, p) > 0$ such that for every function $u \in L^{1,p}(\mathbb{R}^N)$ vanishing at infinity,*

$$\left(\int_{\mathbb{R}^N} |u(x)|^{p^*} dx \right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}. \quad (4)$$

In particular, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p \leq q \leq p^$.*

Proof. Assume that $1 < p < N$ and as in the proof of Theorem 11.2, that $u \in L^{p^*}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ with $\nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)$. By Exercise 16.20, for $x \in \mathbb{R}^N$,

$$|u(x)| \leq \frac{1}{\beta_N} \int_{\mathbb{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}} dx = \frac{1}{\beta_N} I_1(|\nabla u|)(x). \quad (5)$$

It is enough to apply Theorem 1 with $\alpha = 1$. ■

1.2 The Critical Case

Next we discuss the critical case $p = \frac{\alpha}{N}$. When $u \in L^{\frac{\alpha}{N}}(\mathbb{R}^N)$, then $I_\alpha(u)(x)$ is not finite for \mathcal{L}^N -a.e. $x \in \mathbb{R}^N$.

Exercise 5 *Prove that the function*

$$u(x) := \frac{1}{|x|^\alpha \log|x|} \chi_{\mathbb{R}^N \setminus B(0,2)}, \quad x \in \mathbb{R}^N,$$

belongs to $L^{\frac{\alpha}{N}}(\mathbb{R}^N)$ but $I_\alpha(u)(x) = \infty$ for all $x \in \mathbb{R}^N$.

Note that the problem is the behavior at ∞ . To overcome this problem, there are two alternatives: One should either restrict attention to functions $u \in L^{\frac{\alpha}{N}}(\mathbb{R}^N)$ with compact support, or modify the Riesz potential by considering one of the following variants

$$\hat{I}_\alpha(u)(x) := \int_{\mathbb{R}^N} \left[\frac{1}{|x-y|^{N-\alpha}} - \frac{1 - \chi_{B(0,1)}(y)}{|x-y|^{N-\alpha}} \right] u(y) dy, \quad x \in \mathbb{R}^N.$$

or, for $0 < \alpha < 1$,

$$\tilde{I}_\alpha(u)(x) := \int_{\mathbb{R}^N} \left[\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x_0-y|^{N-\alpha}} \right] u(y) dy, \quad x \in \mathbb{R}^N.$$

In what follows, we consider functions with compact support. When $p \rightarrow \left(\frac{\alpha}{N}\right)^+$, we have that $q = \frac{Np}{N-\alpha p} \rightarrow \infty$, however if $u \in L^{\frac{\alpha}{N}}(\mathbb{R}^N)$, then one cannot conclude that $I_\alpha(u)$ belongs to $L^\infty(\mathbb{R}^N)$.

Exercise 6 Prove that for $\varepsilon > 0$ sufficiently small the function

$$u(x) := \frac{1}{|x|^\alpha \left(\log \frac{1}{|x|}\right)^{\frac{\alpha}{N}(1+\varepsilon)}} \chi_{B(0, \frac{1}{2})}, \quad x \in \mathbb{R}^N,$$

belongs to $L^{\frac{\alpha}{N}}(\mathbb{R}^N)$ but $I_\alpha(u)(0) = \infty$.

Theorem 7 Let $0 < \alpha < N$ and let $u \in L^{\frac{N}{\alpha}}(\mathbb{R}^N) \setminus \{0\}$ have support in a ball $B(x_0, R)$. Then for every $\gamma \in \left(0, \frac{N}{\beta_N}\right)$ there exists a constant $C_\gamma = C_\gamma(N) > 0$ such that

$$\int_{B(x_0, R)} \exp \gamma \left(\frac{I_\alpha(|u|)(x)}{\|u\|_{L^{\frac{N}{\alpha}}}} \right)^{\frac{N-\alpha}{N}} dx \leq C_\gamma R^N. \quad (6)$$

First proof. Without loss of generality, we may assume that $\|u\|_{L^{\frac{N}{\alpha}}} = 1$. We proceed as in the first part of the proof of Theorem 1, with the only difference that we take $x \in B(x_0, R)$. The estimate for I does not change, while to estimate II , note that $B(x_0, R) \subset B(x, 2R)$, so that, if $0 < r \leq 2R$,

$$\begin{aligned} II &= \int_{B(x, 2R) \setminus B(x, r)} \frac{|u(y)|}{|x-y|^{N-\alpha}} dy \\ &\leq \left(\beta_N \int_r^{2R} t^{-1} dt \right)^{\frac{N-\alpha}{N}} \|u\|_{L^{\frac{N}{\alpha}}} = \left(\beta_N \log \frac{2R}{r} \right)^{\frac{N-\alpha}{N}} \end{aligned}$$

where we used Hölder's inequality and the fact that $\left(\frac{N}{\alpha}\right)' = \frac{N}{N-\alpha}$. On the other hand, if $r > 2R$, then $II = 0$. Hence, we have proved that

$$I_\alpha(|u|)(x) \leq \frac{\beta_N}{\alpha} r^\alpha M(u)(x) + \left(\beta_N \log \frac{2R}{r} \right)^{\frac{N-\alpha}{N}}$$

for $x \in B(x_0, R)$ and $0 < r \leq 2R$, and

$$I_\alpha(|u|)(x) \leq \frac{\beta_N}{\alpha} r^\alpha M(u)(x)$$

for $x \in B(x_0, R)$ and $r > 2R$. Fix $\varepsilon, \delta > 0$ and choose

$$r := \min \left\{ \left(\frac{\alpha}{\beta_N} \frac{\delta}{M(u)(x) + \varepsilon} \right)^{\frac{1}{\alpha}}, 2R \right\}.$$

Then

$$\begin{aligned}
I_\alpha(|u|)(x) &\leq \delta + \left(\beta_N \log^+ \left[\left(\frac{\alpha}{\beta_N \delta} \right)^{\frac{1}{\alpha}} 2R (M(u)(x) + \varepsilon)^{\frac{1}{\alpha}} \right] \right)^{\frac{N-\alpha}{N}} \\
&= \delta + \left(\frac{\beta_N}{N} \log^+ \left[\left(\frac{\alpha}{\beta_N \delta} \right)^{\frac{N}{\alpha}} (2R)^N (M(u)(x) + \varepsilon)^{\frac{N}{\alpha}} \right] \right)^{\frac{N-\alpha}{N}} \\
&\leq \delta + \left(\frac{\beta_N}{N} \log \left[1 + \left(\frac{\alpha}{\beta_N \delta} \right)^{\frac{N}{\alpha}} (2R)^N (M(u)(x) + \varepsilon)^{\frac{N}{\alpha}} \right] \right)^{\frac{N-\alpha}{N}}.
\end{aligned}$$

Since $\gamma < \frac{N}{\beta_N}$, there exists $\rho > 0$ so large that

$$\gamma < \frac{N}{\beta_N} \left(\frac{\rho}{1+\rho} \right)^{\frac{N}{N-\alpha}}.$$

If $I_\alpha(|u|)(x) \geq (1+\rho)\delta$, then

$$I_\alpha(|u|)(x) - \delta \geq I_\alpha(|u|)(x) - \frac{1}{1+\rho} I_\alpha(|u|)(x) = \frac{\rho}{1+\rho} I_\alpha(|u|)(x),$$

and so

$$\begin{aligned}
\gamma (I_\alpha(|u|)(x))^{\frac{N}{N-\alpha}} &< \frac{N}{\beta_N} \left(\frac{\rho}{1+\rho} \right)^{\frac{N}{N-\alpha} (I_\alpha(|u|)(x))^{\frac{N}{N-\alpha}}} \leq \frac{N}{\beta_N} |I_\alpha(|u|)(x) - \delta|^{\frac{N}{N-\alpha}} \\
&\leq \log \left[1 + \left(\frac{\alpha}{\beta_N \delta} \right)^{\frac{N}{\alpha}} (2R)^N (M(u)(x) + \varepsilon)^{\frac{N}{\alpha}} \right]
\end{aligned}$$

In turn,

$$\exp \gamma |I_\alpha(u)(x)|^{\frac{N}{N-\alpha}} \leq 1 + \left(\frac{\alpha}{\beta_N \delta} \right)^{\frac{N}{\alpha}} (2R)^N (M(u)(x) + \varepsilon)^{\frac{N}{\alpha}}.$$

Letting $\varepsilon \rightarrow 0^+$ gives

$$\exp \gamma (I_\alpha(|u|)(x))^{\frac{N}{N-\alpha}} \leq 1 + \left(\frac{\alpha}{\beta_N \delta} \right)^{\frac{N}{\alpha}} (2R)^N (M(u)(x))^{\frac{N}{\alpha}}.$$

On the other hand, if $I_\alpha(|u|)(x) < (1+\rho)\delta$, then

$$\exp \gamma (I_\alpha(|u|)(x))^{\frac{N}{N-\alpha}} \leq \exp \gamma ((1+\rho)\delta)^{\frac{N}{N-\alpha}}.$$

Hence,

$$\begin{aligned}
&\int_{B(x_0, R)} \exp \gamma (I_\alpha(|u|)(x))^{\frac{N}{N-\alpha}} dx \\
&\leq C_\gamma R^N + \left(\frac{\alpha}{\beta_N \delta} \right)^{\frac{N}{\alpha}} (2R)^N \int_{B(x_0, R)} (M(u)(x))^{\frac{N}{\alpha}} dx.
\end{aligned}$$

The result now follows from Theorem C.29(iii). ■

We now present a second proof, which does not rely on maximal functions, but does not give as sharp a constant γ . The following two lemmas are taken from a paper of Serrin [2].

Lemma 8 *Let $0 < \alpha < N$, let $p \geq 1$, and let $u \in L^p(\mathbb{R}^N) \setminus \{0\}$ have support in a ball $B(x_0, R)$. Then*

$$\|I_\alpha(|u|)\|_{L^1(B(x_0, R))} \leq \frac{\beta_N}{\alpha} (\alpha_N)^{\frac{1}{p'}} (2R)^{\alpha + \frac{N}{p'}} \|u\|_{L^p(B(x_0, R))}.$$

Proof. Fix $x \in B(x_0, R)$. Using the fact that $B(x_0, R) \subset B(x, 2R)$, we have

$$\begin{aligned} \int_{B(x_0, R)} \frac{1}{|x-y|^{N-\alpha}} dy &\leq \int_{B(x, 2R)} \frac{1}{|x-y|^{N-\alpha}} dy \\ &= \beta_N \int_0^{2R} r^{\alpha-1} dr = \frac{\beta_N}{\alpha} (2R)^\alpha. \end{aligned}$$

Hence, also by Tonelli's theorem,

$$\begin{aligned} \int_{B(x_0, R)} I_\alpha(|u|)(x) dx &= \int_{B(x_0, R)} |u(y)| \int_{B(x_0, R)} \frac{1}{|x-y|^{N-\alpha}} dx dy \\ &\leq \frac{\beta_N}{\alpha} (2R)^\alpha \int_{B(x_0, R)} |u(y)| dy \\ &\leq \frac{\beta_N}{\alpha} (2R)^\alpha (\alpha_N R^N)^{\frac{1}{p'}} \|u\|_{L^p(B(x_0, R))}, \end{aligned}$$

where in the last inequality we have used Hölder's inequality. ■

Lemma 9 *Let $0 < \alpha < N$, let $p > \frac{N}{\alpha}$, and let $u \in L^p(\mathbb{R}^N) \setminus \{0\}$ have support in a ball $B(x_0, R)$. Then*

$$|I_\alpha(|u|)(x)| \leq \frac{N(p-1)}{p\alpha - N} \alpha_N^{\frac{1}{p'}} (2R)^{\alpha - \frac{N}{p}} \|u\|_{L^p(B(x_0, R))}$$

for all $x \in B(x_0, R)$.

Proof. Fix $0 < \varepsilon < 2R$ and $x \in B(x_0, R)$ and for $t \in [\varepsilon, 2R]$, define

$$\phi(t) := \int_{B(x, t) \setminus B(x, \varepsilon)} |u(y)| dy.$$

Using polar coordinates and Fubini's theorem we have that

$$\phi(t) = \int_\varepsilon^t r^{N-1} \int_{S^{N-1}} |u(y(r, \omega))| d\mathcal{H}^{N-1}(\omega) dr.$$

Note that this shows that ϕ is absolutely continuous in $[\varepsilon, 2R]$. Similarly, we have that the function

$$F(t) := \int_{B(x, t) \setminus B(x, \varepsilon)} \frac{u(y)}{|x-y|^{N-\alpha}} dy = \int_\varepsilon^t \frac{r^{N-1}}{r^{N-\alpha}} \int_{S^{N-1}} |u(y(r, \omega))| d\mathcal{H}^{N-1}(\omega) dr$$

is absolutely continuous in $[\varepsilon, 2R]$, with

$$F'(t) = t^{\alpha-N} \phi'(t)$$

for \mathcal{L}^1 -a.e. $t \in [\varepsilon, 2R]$. By the fundamental theorem of calculus and integration by parts, we have

$$\begin{aligned} F(2R) &= F(2R) - F(\varepsilon) = \int_{\varepsilon}^{2R} F'(r) dr = \int_{\varepsilon}^{2R} r^{\alpha-N} \phi'(r) dr \\ &= (2R)^{\alpha-N} \phi(2R) - (\alpha - N) \int_{\varepsilon}^{2R} r^{\alpha-N-1} \phi(r) dr. \end{aligned}$$

On the other hand, by Hölder's inequality,

$$\phi(r) = \int_{B(x,r) \setminus B(x,\varepsilon)} |u(y)| dy \leq \|u\|_{L^p} (\alpha_N r^N)^{\frac{1}{p'}},$$

and so

$$\begin{aligned} F(2R) &\leq \alpha_N^{\frac{1}{p'}} 2^{\alpha-N} R^{\alpha-N+\frac{N}{p'}} \|u\|_{L^p} + (N - \alpha) \alpha_N^{\frac{1}{p'}} \|u\|_{L^p} \int_{\varepsilon}^{2R} r^{\alpha-N-1+\frac{N}{p'}} dr \\ &\leq \alpha_N^{\frac{1}{p'}} (2R)^{\alpha-\frac{N}{p}} \|u\|_{L^p} + (N - \alpha) \alpha_N^{\frac{1}{p'}} \frac{(2R)^{\alpha-\frac{N}{p}}}{\alpha - \frac{N}{p}} \|u\|_{L^p} \\ &= \frac{N(p-1)}{p\alpha - N} \alpha_N^{\frac{1}{p'}} (2R)^{\alpha-\frac{N}{p}} \|u\|_{L^p}. \end{aligned}$$

Using the fact that $B(x_0, R) \subset B(x, 2R)$, we have that

$$\int_{B(x_0, R) \setminus B(x, \varepsilon)} \frac{|u(y)|}{|x-y|^{N-\alpha}} dy = F(2R) \leq \frac{N(p-1)}{p\alpha - N} \alpha_N^{\frac{1}{p'}} (2R)^{\alpha-\frac{N}{p}} \|u\|_{L^p}.$$

Letting $\varepsilon \rightarrow 0^+$ and using Lebesgue monotone convergence theorem gives the desired result. ■

Second proof. The proof follows essentially Theorem 2 in [4]. Without loss of generality, we may assume that $\|u\|_{L^{\frac{N}{\alpha}}} = 1$. Let $p = \frac{N}{\alpha}$ and write

$$N - \alpha = \frac{N - \theta_1}{p} + \frac{N - \theta_2}{p'}, \quad (7)$$

where $0 < \theta_1, \theta_2 < N$. Then, given $q > 1$, for $f \in L^{q'}(B(x_0, R))$, we have

$$\frac{|f(x)| |u(y)|}{|x-y|^{N-\alpha}} = \frac{|f(x)|^{\frac{1}{p}} |u(y)|}{|x-y|^{\frac{N-\theta_1}{p}}} \frac{|f(x)|^{\frac{1}{p'}}}{|x-y|^{\frac{N-\theta_2}{p'}}}.$$

By Hölder's inequality, we have

$$\begin{aligned} \int_{B(x_0, R)} \int_{B(x_0, R)} \frac{|f(x)| |u(y)|}{|x-y|^{N-\alpha}} dx dy &\leq \left(\int_{B(x_0, R)} \int_{B(x_0, R)} \frac{|f(x)|}{|x-y|^{N-\theta_2}} dx dy \right)^{\frac{1}{p'}} \\ &\quad \times \left(\int_{B(x_0, R)} |u(y)|^p \int_{B(x_0, R)} \frac{|f(x)|}{|x-y|^{N-\theta_1}} dx dy \right)^{\frac{1}{p}} \\ &=: I_1 \times I_2. \end{aligned}$$

By Lemma 8 (where p and α there are replaced here by q' and θ_2),

$$I_1 \leq \left(\frac{\beta_N}{\theta_2} (\alpha_N)^{\frac{1}{q}} (2R)^{\theta_2 + \frac{N}{q}} \|f\|_{L^{q'}} \right)^{\frac{1}{p'}}.$$

On the other hand, if $q' > \frac{N}{\theta_1}$, by Lemma 9 (where p and α there are replaced here by q' and θ_1), we have that

$$\int_{B(x_0, R)} \frac{|f(x)|}{|x-y|^{N-\theta_1}} dx \leq \frac{N(q'-1)}{q'\theta_1 - N} \alpha_N^{\frac{1}{q}} (2R)^{\theta_1 - \frac{N}{q}} \|f\|_{L^{q'}}.$$

Hence,

$$\begin{aligned} \int_{B(x_0, R)} \int_{B(x_0, R)} \frac{|f(x)| |u(y)|}{|x-y|^{N-\alpha}} dx dy &\leq (\alpha_N)^{\frac{1}{q}} (2R)^{\frac{N}{q}} \left(\frac{\beta_N}{\theta_2} \right)^{\frac{1}{p'}} \\ &\quad \times \left(\frac{N(q'-1)}{q'\theta_1 - N} \right)^{\frac{1}{p}} \|f\|_{L^{q'}} \|u\|_{L^p}, \end{aligned}$$

where we have used the fact that

$$\frac{\theta_2}{p'} + \frac{N}{qp'} + \frac{\theta_1}{p} - \frac{N}{pq'} = \frac{N}{q}$$

by (7). Taking the supremum over all $f \in L^{q'}(B(x_0, R))$, we get

$$\int_{B(x_0, R)} (I_\alpha(|u|)(x))^q dx \leq \alpha_N (2R)^N \left(\frac{\beta_N}{\theta_2} \right)^{\frac{q}{p'}} \left(\frac{N(q'-1)}{q'\theta_1 - N} \right)^{\frac{q}{p}} \|u\|_{L^p}^q. \quad (8)$$

Taking

$$\theta_2 := \frac{\alpha}{q},$$

we have that $\theta_2 < \alpha < N$, while from (7),

$$N > \theta_1 = \frac{N}{q'} + \frac{\alpha}{q} > \frac{N}{q'}.$$

Moreover, (8) becomes

$$\int_{B(x_0, R)} (I_\alpha(|u|)(x))^q dx \leq \alpha_N (2R)^N \left(\frac{\beta_N q}{\alpha} \right)^{\frac{N-\alpha}{N} q} \left(\frac{N}{\alpha} \right)^{\frac{\alpha}{N} q} \|u\|_{L^p}^q.$$

Taking $q = \frac{Nk}{N-\alpha}$, where $k \in \mathbb{N}$, we obtain

$$\begin{aligned} & \int_{B(x_0, R)} \sum_{k=1}^n \frac{1}{k!} \gamma^k (I_\alpha(|u|)(x))^{\frac{Nk}{N-\alpha}} dx \\ & \leq \alpha_N (2R)^N \|u\|_{L^p} \sum_{k=1}^n \frac{k^k}{k!} \left[\gamma \frac{\beta_N}{N-\alpha} \left(\frac{N}{\alpha} \right)^{\frac{N}{N-\alpha}} \right]^k \\ & =: \alpha_N (2R)^N \|u\|_{L^p} \sum_{k=1}^n x_k. \end{aligned}$$

Since

$$\frac{x_{k+1}}{x_k} = \left(1 + \frac{1}{k}\right)^k \gamma \frac{\beta_N}{N-\alpha} \left(\frac{N}{\alpha}\right)^{\frac{N}{N-\alpha}} \rightarrow e\gamma \frac{\beta_N}{N-\alpha} \left(\frac{N}{\alpha}\right)^{\frac{N}{N-\alpha}},$$

it follows that if

$$\gamma < \frac{N-\alpha}{e\beta_N} \left(\frac{\alpha}{N}\right)^{\frac{N}{N-\alpha}},$$

then the series converges. Hence,

$$\int_{B(x_0, R)} \exp \gamma (I_\alpha(|u|)(x))^{\frac{N}{N-\alpha}} dx \leq C_\gamma R^N \|u\|_{L^p}$$

■

Using Theorem 7, we can prove Trudinger's embedding theorem. We recall that

$$\gamma_N := N\beta_N^{\frac{1}{N-1}}, \quad (9)$$

Theorem 10 *Suppose $N \geq 2$ and let $u \in W^{1, N}(\mathbb{R}^N) \setminus \{0\}$ have support in a ball $B(x_0, R)$. Then for every $\gamma \in (0, \gamma_N)$ there exists a constant $C = C(N, \gamma) > 0$ such that*

$$\int_{B(x_0, R)} \exp \left(\gamma \frac{|u(x)|^{N'}}{\|\nabla u\|_{L^N}^{N'}} \right) dx \leq C_\gamma R^N.$$

Proof. By (5),

$$\gamma |u(x)|^{N'} \leq \gamma \frac{1}{\beta_N^{N'}} (I_1(|\nabla u|)(x))^{N'}.$$

Hence if $\gamma < \gamma_N = N\beta_N^{\frac{1}{N-1}}$, then $\gamma \frac{1}{\beta_N^{N'}} < \frac{N}{\beta_N}$, and so we are in a position to apply Theorem 7 with $a = 1$, to conclude that

$$\int_{B(x_0, R)} \exp \left(\gamma \frac{|u(x)|^{N'}}{\|\nabla u\|_{L^N}^{N'}} \right) dx \leq \int_{B(x_0, R)} \exp \gamma \frac{1}{\beta_N^{N'}} \left(\frac{I_1(|\nabla u|)(x)}{\|\nabla u\|_{L^N}} \right)^{N'} dx \leq C_\gamma R^N.$$

■

Remark 11 *I am unable to find a simple proof of Theorem 11.29, which does not make use of symmetrization. Both proofs of Theorem 7 rely strongly on the fact that u has compact support.*

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