# 1 Yet Another Proof of Theorem 1.21

**Theorem 1 (Theorem 1.21)** *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be monotone. Then $f$ is differentiable for $\mathcal{L}^1$ a.e. $x \in I$.*

We begin by showing that in Besicovitch's derivation theorem (Theorem B.119) once could replace balls with "nicely" shrinking sets.

**Lemma 2** *Let $\mu, \nu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0, \infty]$ be two Radon measures. Then there exists a Borel set $E$ with $\mu(E) = 0$, such that for every $x \in \mathbb{R}^N \setminus E$ and for every family of Borel subsets $\{E_{x,r}\}_{r>0}$ such that $E_{x,r} \subset \overline{B(x,r)}$ and*

$$\mu(E_{x,r}) > \alpha\mu\left(\overline{B(x,r)}\right)$$

*for some constant $\alpha > 0$ independent of $r > 0$,*

$$\frac{d\nu_{ac}}{d\mu}(x) = \lim_{r\to 0^+} \frac{\nu(E_{x,r})}{\mu(E_{x,r})} \in \mathbb{R},$$

*where $\nu = \nu_{ac} + \nu_s$, with $\nu_{ac} \ll \mu$ and $\nu_s \perp \mu$.*

**Proof.** By Besicovitch's derivation theorem (Theorem B.119) there exists a Borel set $M \subset \mathbb{R}^N$, with $\mu(M) = 0$, such that for every $x \in \mathbb{R}^N \setminus M$,

$$\frac{d\nu_{ac}}{d\mu}(x) = \lim_{r\to 0^+} \frac{\nu\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} \in \mathbb{R} \tag{1}$$

and

$$\lim_{r\to 0^+} \frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} = 0.$$

Since $\nu$ is finite on compact sets, so is $\nu_{ac}$. Hence, $\frac{d\nu_{ac}}{d\mu}$ is a locally integrable function. It follows that for every Lebesgue point $x \in \mathbb{R}^N \setminus M$,

$$\begin{aligned}
&\limsup_{r\to 0^+} \frac{1}{\mu(E_{x,r})} \int_{E_{x,r}} \left| \frac{d\nu_{ac}}{d\mu}(y) - \frac{d\nu_{ac}}{d\mu}(x) \right| d\mu(y) \\
&\quad \le \limsup_{r\to 0^+} \frac{1}{\mu(E_{x,r})} \int_{\overline{B(x,r)}} \left| \frac{d\nu_{ac}}{d\mu}(y) - \frac{d\nu_{ac}}{d\mu}(x) \right| d\mu(y) \\
&\quad \le \frac{1}{\alpha} \lim_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} \left| \frac{d\nu_{ac}}{d\mu}(y) - \frac{d\nu_{ac}}{d\mu}(x) \right| d\mu(y) = 0,
\end{aligned}$$

which implies that

$$\frac{d\nu_{ac}}{d\mu}(x) = \lim_{r\to 0^+} \frac{\nu(E_{x,r})}{\mu(E_{x,r})}.$$

On the other hand, we have that

$$\limsup_{r\to 0^+}\frac{\nu_s\left(E_{x,r}\right)}{\mu\left(E_{x,r}\right)}\leq \limsup_{r\to 0^+}\frac{\nu_s\left(\overline{B\left(x,r\right)}\right)}{\mu\left(E_{x,r}\right)}$$
$$\leq \frac{1}{\alpha}\lim_{r\to 0^+}\frac{\nu_s\left(\overline{B\left(x,r\right)}\right)}{\mu\left(\overline{B\left(x,r\right)}\right)}=0.$$

Hence, the result follows since $\nu=\nu_{ac}+\nu_s$. ■

Note that the sets $E_{x,r}$ need not contain $x$.

**Proof of Theorem 1.** Without loss of generality, we can assume that $f$ is increasing.
**Step 1:** Assume that $f$ is right-continuous. Let $\mu_f:\mathcal{B}\left(I\right)\to[0,\infty]$ be the the Lebesgue-Stieltjes measure generated by $f$. For every $x\in I^\circ$ consider the family of sets $E_{x,r}:=(x,x+r]$ and $F_{x,r}:=(x-r,x]$. Note that $\mathcal{L}^1\left(E_{x,r}\right)=\mathcal{L}^1\left(F_{x,r}\right)=\frac{1}{2}\mathcal{L}^1\left([x-r,x+r]\right)$. Hence, by the previous lemma, there exists a Borel set $E\subset I$, with $\mathcal{L}^1\left(E\right)=0$, such that for any $x\in I\setminus E$,

$$\frac{d\left(\mu_f\right)_{ac}}{d\mathcal{L}^1}\left(x\right)=\lim_{r\to 0^+}\frac{\mu_f\left((x,x+r]\right)}{\mathcal{L}^1\left((x,x+r]\right)}=\lim_{r\to 0^+}\frac{\mu_f\left((x-r,x]\right)}{\mathcal{L}^1\left((x-r,x]\right)}.$$

But by Theorem 5.3, we have that

$$f\left(x+r\right)-f\left(x\right)=\mu_f\left((x,x+r]\right),$$
$$f\left(x\right)-f\left(x-r\right)=\mu_f\left((x-r,x]\right),$$

and so

$$\lim_{r\to 0^+}\frac{f\left(x+r\right)-f\left(x\right)}{r}=\lim_{r\to 0^+}\frac{f\left(x\right)-f\left(x-r\right)}{r}=\frac{d\left(\mu_f\right)_{ac}}{d\mathcal{L}^1}\left(x\right)\in\mathbb{R},$$

which shows that $f$ is differentiable for $\mathcal{L}^1$ a.e. $x\in I$.
**Step 2:** Consider the function $g:I\to\mathbb{R}$ defined by $g\left(x\right):=f^+\left(x\right)$. Then $g$ is increasing and right-continuous. Moreover, $g\left(x\right)=f\left(x\right)$ for all but countably many points (the discontinuity points of $f$). Moreover, $f^-\left(x\right)=g^-\left(x\right)$ for all $x\in I$. By the first step, $g$ is differentiable for $\mathcal{L}^1$ a.e. $x\in I$.

Let $h:=g-f\geq 0$. Let $\{x_n\}$ be the family of discontinuity points of $f$. Then $h\left(x\right)=0$ for all $x\neq x_n$. Moreover, for every $[a,b]\subset I$ we have that

$$\sum_{x_n\in[a,b]}h\left(x_n\right)=\sum_{x_n\in[a,b]}\left(f^+\left(x_n\right)-f\left(x_n\right)\right)\leq f\left(b\right)-f\left(a\right).$$

Thus, the measure

$$\nu:=\sum_n h\left(x_n\right)\delta_{x_n}$$

is finite on compact sets. Since $\nu \perp \mathcal{L}^1$, by Besicovitch's derivation theorem (Theorem B.119) we have that there exists a Borel set $E_1 \subset I$, with $\mathcal{L}^1(E_1) = 0$, such that for any $x \in I \setminus E_1$,

$$\lim_{r \to 0^+} \frac{\nu\left(\overline{B(x,r)}\right)}{\mathcal{L}^1\left(\overline{B(x,r)}\right)} = 0.$$

But

$$\left|\frac{h(x+t) - h(t)}{t}\right| \leq \frac{|h(x+t)| + |h(t)|}{|t|} \leq 4\frac{\nu\left(\overline{B(x,2|t|)}\right)}{\mathcal{L}^1\left(\overline{B(x,2|t|)}\right)} \to 0$$

as $t \to 0$. Hence, $h'(x) = 0$ for $\mathcal{L}^1$ a.e. $x \in I$. Together with the previous part, this implies that there exists in $\mathbb{R}$, $f'(x) = g'(x) - h'(x)$ for $\mathcal{L}^1$ a.e. $x \in I$. ■