1 On Theorems 11.23 and 11.29

In this section we present an alternative proof of Theorem 11.23, which can be used to give a form of Theorem 11.29, which does not make use of spherically symmetric rearrangements. The proof is adapted from [1].

Theorem 1 *The space* $W^{1,N}\left(\mathbb{R}^N\right)$ *is continuously embedded in the space* $L^q\left(\mathbb{R}^N\right)$ *for all* $N < q < \infty$, *with*

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C_1 q^{1-1/N+1/q} \|u\|_{W^{1,N}(\mathbb{R}^N)}$$

for all $u \in W^{1,N}\left(\mathbb{R}^N\right)$, *where* $C_1 > 0$ *depends only on* N.

Proof. Step 1: Assume first that $u \in C_c^\infty\left(\mathbb{R}^N\right)$. Then for $x \in \mathbb{R}^N$ and $z \in S^{N-1}$, by the fundamental theorem of calculus applied to the function $g(r) := u(x + rz)$, $t \geq 0$,

$$u(x) = -\int_0^\infty \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x + rz)\, z_i \, dr.$$

By averaging in z over S^{N-1} and using spherical coordinates, we get

$$|u(x)| \leq \frac{1}{\beta_N} \int_{\mathbb{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}} \, dy = \int_{B(x,d)} \frac{|\nabla u(y)|}{|x-y|^{N-1}} \, dy,$$

where d is the diameter of the support of u and $\beta_N := N\alpha_N$.

Let $N < q < \infty$. By Young's inequality (see Theorem C.16), with $\frac{1}{N} + \frac{1}{s} = 1 + \frac{1}{q}$,

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}^N)} &\leq \frac{1}{\beta_N}\left(\int_{B(0,d)} \frac{1}{|w|^{(N-1)s}} \, dw\right)^{1/s} \|\nabla u\|_{L^N(\mathbb{R}^N;\mathbb{R}^N)} \\
&= \frac{1}{\beta_N^{1-1/s}\left(N-(N-1)s\right)^{1/s}} d^{[N-(N-1)s]/s} \|\nabla u\|_{L^N(\mathbb{R}^N;\mathbb{R}^N)} \\
&= \frac{q^{1/s} d^{N/q}}{\beta_N^{1/N-1/q}(Ns)^{1/s}} \|\nabla u\|_{L^N(\mathbb{R}^N;\mathbb{R}^N)}.
\end{aligned}$$

Step 2: For every $z \in \mathbb{Z}^N$ consider the open cube $Q(k, 2)$ centered at k and of side-length 2. Then for every $1 < r \leq 2$,

$$\mathbb{R}^N = \bigcup_{z \in \mathbb{Z}^N} Q(z, r),$$

and so reasoning as in the proof of Theorem C.21 we can construct a smooth partition of unity $\{\psi_z\}_{z\in\mathbb{Z}^N}$ subordinated to $\{Q(z,2)\}_{z\in\mathbb{Z}^N}$ with the property that $\|\nabla\psi_z\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} \le C(N)$. Since for every $x \in \mathbb{R}^N$, there are at most 2^N cubes $Q(z,r)$ that contain x, it follows from Exercise 12.14 that

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}^N)} &= \left\| \sum_{z\in\mathbb{Z}^N} (u\psi_z) \right\|_{L^q(\mathbb{R}^N)} \le 2^{N/q'} \left(\sum_{z\in\mathbb{Z}^N} \|u\psi_z\|^q_{L^q(\mathbb{R}^N)} \right)^{1/q} \\
&\le \frac{2^{N/q'} q^{1/s} \left(2\sqrt{N}\right)^{N/q}}{\beta_N^{1/N-1/q} (Ns)^{1/s}} \left(\sum_{z\in\mathbb{Z}^N} \|\nabla(u\psi_z)\|^q_{L^N(\mathbb{R}^N;\mathbb{R}^N)} \right)^{1/q},
\end{aligned}$$

where in the last inequality we have used Step 1 with $d = 2\sqrt{N}$.

Using the fact that

$$\left(\sum_{z\in\mathbb{Z}^N} a_z^q \right)^{1/q} \le \left(\sum_{z\in\mathbb{Z}^N} a_z^N \right)^{1/N}$$

for $a_z \ge 0$, we have that

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}^N)} &\le \frac{2^N q^{1/s} N^{N/(2q)}}{\beta_N^{1/N-1/q} (Ns)^{1/s}} \left(\sum_{z\in\mathbb{Z}^N} \|\nabla(u\psi_z)\|^N_{L^N(\mathbb{R}^N;\mathbb{R}^N)} \right)^{1/N} \\
&\le \frac{2^N q^{1/s} N^{N/(2q)}}{\beta_N^{1/N-1/q} (Ns)^{1/s}} \left(\sum_{z\in\mathbb{Z}^N} 2^{N-1} \int_{Q(z,2)} \left(\psi_z |\nabla u|^N + |u|^N |\nabla\psi_z|^N \right) dx \right)^{1/N} \\
&\le \frac{2^N q^{1/s} N^{N/(2q)}}{\beta_N^{1/N-1/q} (Ns)^{1/s}} \left(2^{N-1} \int_{\mathbb{R}^N} \left(|\nabla u|^N + 2^N C(N)^N |u|^N \right) dx \right)^{1/N} \\
&\le C_1 q^{1/s} \|u\|_{W^{1,N}(\mathbb{R}^N)},
\end{aligned}$$

where $C_1 := 2^{N+1-1/N} \left(1 + 2^N C(N)^N\right)^{1/N} N^{1/2}$ and we have used the facts that $q > N$, $s > 1$ and $\beta_N > 1$.

A standard mollification argument concludes the proof. ■

Next we use the previous proof to give a form of Theorem 11.29, which does not make use of spherically symmetric rearrangements. The price to pay is that it does not give the sharp constant γ_N and that $\|u\|_{W^{1,N}(\mathbb{R}^N)}$ replaces $\|\nabla u\|_{L^N(\mathbb{R}^N;\mathbb{R}^N)}$ in (1.17) and (1.18).

Theorem 2 *Suppose $N \ge 2$. Then there exist two constants c_1, $c_2 > 0$ depending only on N such that*

$$\int_{\mathbb{R}^N} \exp_{N-1}\left(c_1 \frac{|u(x)|^{N'}}{\|u\|^{N'}_{W^{1,N}(\mathbb{R}^N)}} \right) dx \le c_2 \tag{1}$$

for all $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$. In particular, if $\Phi(s) := \exp_{N-1}\left(c_1 s^{N'}\right)$, $s \ge 0$, then

$$\|u\|_\Phi \le \|u\|_{W^{1,N}(\mathbb{R}^N)} \tag{2}$$

for all $u \in W^{1,N}\left(\mathbb{R}^N\right)$.

Proof. By replacing u with $u/\left\|u\right\|_{W^{1,N}(\mathbb{R}^N)}$, we can assume that $\left\|u\right\|_{W^{1,N}(\mathbb{R}^N)} = 1$. Then for $c > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} \exp_{N-1}\left(c\left|u\left(x\right)\right|^{N'}\right) dx &= \sum_{n=N-1}^{\infty} \frac{c^n}{n!} \int_{\mathbb{R}^N} \left|u\left(x\right)\right|^{nN'} dx \\ &\leq \sum_{n=N-1}^{\infty} \frac{c^n C_1^{nN'}}{n!} \left(nN'\right)^{n+1}, \end{aligned}$$

where we have used the previous theorem with $q = nN'$. By the ratio test,

$$\frac{\frac{c^{n+1}C_1^{(n+1)N'}}{(n+1)!}\left(\left(n+1\right)N'\right)^{n+2}}{\frac{c^n C_1^{nN'}}{n!}\left(nN'\right)^{n+1}} = cC_1^{N'}\left(1+\frac{1}{n}\right)^{n+1} \to cC_1^{N'}e < 1,$$

provided $c < 1/\left(C_1^{N'}e\right)$. Hence, the series converges. ■

References

[1] V. G. Maz′ja, and S. V. Poborchi, Differentiable Functions on Bad Domains, World Scientific, 1997.