1 An Simpler Proof of Theorem 11.51

In this section we present a simpler proof of Theorem 11.51.

Theorem 1 (Change of variables) *Let $\Omega, \Omega' \subset \mathbb{R}^N$ be open sets, let $\Psi : \Omega' \to \Omega$ be invertible, with Ψ and Ψ^{-1} Lipschitz functions, and let $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. Then $u \circ \Psi \in W^{1,p}(\Omega')$ and for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$-a.e. $y \in \Omega'$,*

$$\frac{\partial (u \circ \Psi)}{\partial y_i}(y) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(\Psi(y)) \frac{\partial \Psi_j}{\partial y_i}(y).$$

Proof. By the Meyers-Serrin theorem, Theorem 10.15, there exists a sequence $\{u_n\} \subset W^{1,p}(\Omega) \cap C^\infty(\Omega)$ converging to u in $W^{1,p}(\Omega)$. By extracting a subsequence, we can assume that $u_n \to u$ and $\nabla u_n \to \nabla u$ pointwise $\mathcal{L}^N$-a.e. in Ω. For $y \in \Omega'$ set

$$v_n(y) := u_n(\Psi(y)). \tag{1}$$

By Rademacher's theorem the Lipschitz continuous function Ψ is differentiable $\mathcal{L}^N$-a.e. in Ω'. Since $u \in C^\infty(\Omega)$, we conclude that for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$-a.e. $y \in \Omega'$,

$$\frac{\partial v_n}{\partial y_i}(y) = \sum_{j=1}^{N} \frac{\partial u_n}{\partial x_j}(\Psi(y)) \frac{\partial \Psi_j}{\partial y_i}(y). \tag{2}$$

Hence,

$$\left| \frac{\partial v_n}{\partial y_i}(y) \right| \leq \sum_{j=1}^{N} \operatorname{Lip} \Psi_j \left| \frac{\partial u_n}{\partial x_j}(\Psi(y)) \right|,$$

and so

$$\begin{aligned}
\int_{\Omega'} |\nabla v_n(y)|^p \, dy &\leq C \int_{\Psi^{-1}(\Omega)} |\nabla u_n(\Psi(y))|^p \, dy \\
&\leq C \int_{\Psi^{-1}(\Omega)} |\nabla u_n(\Psi(y))|^p \, |J\Psi(x)| \, dy \\
&= C \int_{\Omega} |\nabla u_n(x)|^p \, dx,
\end{aligned}$$

where we have used the fact that $|J\Psi|$ is bounded from below by a positive constant (since Ψ^{-1} is Lipschitz) and Theorem 8.21. The previous inequality implies that

$$\int_{\Omega'} |\nabla v_j(y) - \nabla v_l(y)|^p \, dy \leq C \int_{\Omega} |\nabla u_j(y) - \nabla u_l(y)|^p \, dx$$

for all $j, l \in \mathbb{N}$. Similarly,

$$\int_{\Omega'} |v_j(y) - v_l(y)|^p \, dy \leq C \int_{\Omega} |u_j(y) - u_l(y)|^p \, dx$$

for all $j, l \in \mathbb{N}$. Since $u_n \to u$ in $W^{1,p}(\Omega)$, it follows that $\{v_n\}$ is a Cauchy sequence in $W^{1,p}(\Omega')$, and so it converges to a function $v \in W^{1,p}(\Omega')$. By extracting a subsequence, we can assume that v_n and ∇v_n converge, respectively, to v and ∇v pointwise $\mathcal{L}^N$-a.e. in Ω'. Since Ψ^{-1} has the (N) property and $u_n \to u$ and $\nabla u_n \to \nabla u$ pointwise $\mathcal{L}^N$-a.e. in Ω, it follows that $u_n \circ \Psi \to u \circ \Psi$ and $(\nabla u_n) \circ \Psi \to (\nabla u) \circ \Psi$ pointwise $\mathcal{L}^N$-a.e. in Ω'. In view of (1) and (2), we conclude that $v(y) = u(\Psi(y))$ for $\mathcal{L}^N$-a.e. $y \in \Omega$ and that for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$-a.e. $y \in \Omega'$,

$$\frac{\partial (u \circ \Psi)}{\partial y_i}(y) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(\Psi(y)) \frac{\partial \Psi_j}{\partial y_i}(y). \tag{3}$$

Since $u \circ \Psi = v$ in Ω', it follows that $u \circ \Psi \in W^{1,p}(\Omega')$. ■