1 A Simpler Proof of Theorem 15.8 (iii)

We present a simpler proof of Theorem 15.8 (iii).

Theorem 1 *Let $f : \mathbb{R}^{N-1} \to \mathbb{R}$ be a Lipschitz function, $N \geq 2$, and let*

$$\Omega := \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \, x_N > f(x') \right\}. \tag{1}$$

There exists a continuous linear operator

$$\operatorname{Tr} : W^{1,1}(\Omega) \to L^1\left(\partial\Omega, \mathcal{H}^{N-1}\right)$$

such that

(i) $\operatorname{Tr}(u) = u$ *on* $\partial\Omega$ *for all* $u \in W^{1,1}(\Omega) \cap C\left(\overline{\Omega}\right)$,

(ii) for all $u \in W^{1,1}(\Omega)$,

$$\int_{\partial\Omega} |\operatorname{Tr}(u)| \, d\mathcal{H}^{N-1} \leq \sqrt{1 + |\operatorname{Lip} f|^2} \int_\Omega \left| \frac{\partial u}{\partial x_N}(x) \right| dx,$$

(iii) for all $\psi \in C_c^1\left(\mathbb{R}^N\right)$, $u \in W^{1,1}(\Omega)$, *and* $i = 1, \ldots, N$,

$$\int_\Omega u \frac{\partial \psi}{\partial x_i} \, dx = -\int_\Omega \psi \frac{\partial u}{\partial x_i} \, dx + \int_{\partial\Omega} \psi \operatorname{Tr}(u) \, \nu_i \, d\mathcal{H}^{N-1},$$

where ν *is the outward unit normal to* $\partial\Omega$, *that is, for* $\mathcal{L}^{N-1}$*-a.e.* $x' \in \mathbb{R}^{N-1}$,

$$\nu(x', f(x')) = \left(\frac{\nabla_{x'} f(x')}{\sqrt{1 + |\nabla_{x'} f(x')|^2}}, \frac{-1}{\sqrt{1 + |\nabla_{x'} f(x')|^2}} \right). \tag{2}$$

Proof. To prove part (iii) observe that since the boundary of Ω is Lipschitz, we can use Theorem 10.29 to construct a sequence of functions $u_n \in C_c^1\left(\mathbb{R}^N\right)$ such that $u_n \to u$ in $W^{1,1}(\Omega)$. It follows by part (ii) that $\operatorname{Tr}(u_n) \to \operatorname{Tr}(u)$ in $L^1\left(\partial\Omega, \mathcal{H}^{N-1}\right)$. By the divergence theorem for every $\phi \in C_c^1\left(\mathbb{R}^N\right)$ we have that

$$\int_\Omega u_n \frac{\partial \phi}{\partial x_N} \, dx + \int_\Omega \phi \frac{\partial u_n}{\partial x_N} \, dx = \int_{\partial\Omega} \phi u_n \nu_N \, d\mathcal{H}^{N-1}.$$

Letting $n \to \infty$ gives

$$\int_\Omega u \frac{\partial \phi}{\partial x_N} \, dx + \int_\Omega \phi \frac{\partial u}{\partial x_N} \, dx = \int_{\partial\Omega} \phi \operatorname{Tr}(u) \, \nu_N \, d\mathcal{H}^{N-1}. \tag{3}$$

This concludes the proof. ■