1 An Alternative Proof of Theorem 3.30

Theorem 1 *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$. Then u belongs to $AC(I)$ if and only if u is differentiable $\mathcal{L}^1$-a.e. in I, u' belongs to $L^1_{\mathrm{loc}}(I)$ and is equi-integrable, and the fundamental theorem of calculus is valid, that is, for all $x, x_0 \in I$,*

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\, dt. \tag{1}$$

Lemma 2 *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be an absolutely continuous function such that $u'(x) = 0$ for $\mathcal{L}^1$ a.e. $x \in I$. Then u is constant.*

Proof. Given $\varepsilon > 0$, let $\delta > 0$ be the number given in the definition of absolute continuity. Let $a, b \in I$ with $a < b$. We claim that $u(a) = u(b)$. Let $E := \{x \in (a,b) : u'(x) = 0\}$. Then $\mathcal{L}^1(E) = b - a$. For every $x \in E$, we have that

$$\lim_{y \to x} \frac{u(y) - u(x)}{y - x} = u'(x) = 0,$$

and so there exists $\eta_x > 0$ such that $[x - \eta_x, x + \eta_x] \subset (a,b)$ and

$$\left| \frac{u(y) - u(x)}{y - x} \right| \le \frac{\varepsilon}{b - a} \tag{2}$$

for all $y \in I$ with $|x - y| \le \eta_x$. The family

$$\{[x - r, x + r] : x \in E,\ 0 < r \le \eta_x\}$$

is a fine cover of E, and so by the Vitali covering theorem (see Theorem 1.31), there exists a countable disjoint family $\{[x_n - r_n, x_n + r_n]\}$ such that

$$\mathcal{L}^1\left(E \setminus \bigcup_n [x_n - r_n, x_n + r_n]\right) = 0.$$

Let ℓ be so large that

$$\mathcal{L}^1\left(E \setminus \bigcup_{n=1}^{\ell} [x_n - r_n, x_n + r_n]\right) \le \delta.$$

Then

$$\begin{aligned}\mathcal{L}^1([a,b]) = \mathcal{L}^1(E) &\le \mathcal{L}^1\left(E \setminus \bigcup_{n=1}^{\ell} [x_n - r_n, x_n + r_n]\right) + \sum_{n=1}^{\ell} 2r_n \\ &\le \delta + \sum_{n=1}^{\ell} 2r_n.\end{aligned}$$

Without loss of generality assume that $x_1 < x_2 < \cdots < x_\ell$. Then by the previous inequality, it follows that sum of the length of the intervals $[a, x_1 - r_1]$,

$[x_1 + r_1, x_2 - r_2]$, ..., $[x_\ell + r_\ell, b]$ is less than or equal δ. Since u is absolutely continuous, we have that

$$|u(a) - u(x_1 - r_1)| + \sum_{n=1}^{\ell-1} |u(x_{n+1} - r_{n+1}) - u(x_n + r_n)| + |u(b) - u(x_\ell + r_\ell)| \le \varepsilon.$$

On the other hand by (2),

$$\begin{aligned} |u(x_n + r_n) - u(x_n - r_n)| &\le |u(x_n + r_n) - u(x_n)| + |u(x_n) - u(x_n - r_n)| \\ &\le \frac{\varepsilon 2 r_n}{b-a} \end{aligned}$$

and so

$$\begin{aligned} |u(a) - u(b)| \le |u(a) - u(x_1 - r_1)| &+ \sum_{n=1}^{\ell-1} |u(x_{n+1} - r_{n+1}) - u(x_n + r_n)| \\ &+ \sum_{n=1}^{\ell} |u(x_n + r_n) - u(x_n + r_n)| + |u(b) - u(x_\ell + r_\ell)| \\ &\le \varepsilon + \sum_{n=1}^{\ell} \frac{\varepsilon 2 r_n}{b-a} \le 3\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0^+$ gives $u(a) = u(b)$. Hence, u is constant. ■

Lemma 3 *Let $I \subset \mathbb{R}$ be an interval and let $v \in L^1_{\text{loc}}(I)$, with v equi-integrable. Fix $x_0 \in I$ and let*

$$u(x) := \int_{x_0}^{x} v(t)\, dt, \quad x \in I.$$

Then the function u is absolutely continuous in I and differentiable for $\mathcal{L}^1$ a.e. $x \in I$ with $u'(x) = v(x)$.

Proof. Since, v is equi-integrable, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subseteq I$ is a Lebesgue measurable set with $\mathcal{L}^1(E) \le \delta$, then

$$\int_E |v(t)|\, dt \le \varepsilon.$$

Hence, for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subset I$ and

$$\sum_{k=1}^{\ell} (b_k - a_k) \le \delta,$$

we have

$$\begin{aligned} \sum_{k=1}^{\ell} |u(b_k) - u(a_k)| &= \sum_{k=1}^{\ell} \left| \int_{a_k}^{b_k} v(t)\, dt \right| \le \sum_{k=1}^{\ell} \int_{a_k}^{b_k} |v(t)|\, dt \\ &= \int_{\bigcup_{k=1}^{\ell}(a_k, b_k)}^{b_k} |v(t)|\, dt \le \varepsilon. \end{aligned}$$

Hence, u is absolutely continuous.

Next fix a representative of v and let $x_0 \in I$ be a Lebesgue point of v, that is,

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{B(x_0,\varepsilon)} |v(t) - v(x_0)| \, dt = 0.$$

Then for $h \neq 0$,

$$\begin{aligned} \left| \frac{u(x_0+h) - u(x_0)}{h} - v(x_0) \right| &= \left| \frac{1}{h} \int_{x_0}^{x_0+h} (v(t) - v(x_0)) \, dt \right| \\ &\leq \frac{2}{2|h|} \int_{x_0-|h|}^{x_0+|h|} |v(t) - v(x_0)| \, dt \to 0 \end{aligned}$$

as $h \to 0$, which shows that u is differentiable at x_0 with $u'(x_0) = v(x_0)$. ■

Proof of Theorem 1. Assume that u is differentiable $\mathcal{L}^1$-a.e. in I, u' belongs to $L^1_{\text{loc}}(I)$ and is equi-integrable, and the fundamental theorem of calculus is valid. Fix $x_0 \in I$ and define

$$g(x) := \int_{x_0}^{x} u'(t) \, dt.$$

Then by the previous lemma g is absolutely continuous. In turn, since constant functions are absolutely continuous, it follows that the function $u = u(x_0) + g$ is absolutely continuous.

Conversely, assume that u is absolutely continuous. Let's prove that the fundamental theorem of calculus is valid. Fix $x, x_0 \in I$, with, say $x_0 < x$. We have seen last week that $u' \in L^1_{\text{loc}}(I)$. It follows that $u' \in L^1([x_0, x])$ and so by the previous lemma again the function

$$g(y) := \int_{x_0}^{y} u'(t) \, dt, \quad y \in [x_0, x],$$

belongs to $AC([x_0, x])$ with $g'(y) = u'(y)$ for $\mathcal{L}^1$ a.e. $y \in [x_0, x]$. Since $u - g \in AC([x_0, x])$ and

$$(u-g)'(y) = u'(y) - u'(y) = 0$$

for $\mathcal{L}^1$ a.e. $y \in [x_0, x]$, by Lemma 2, we have that $u - g$ is constant in $[x_0, x]$. Thus, there exists $c \in \mathbb{R}$ such that

$$(u-g)(y) = c$$

for all $y \in [x_0, x]$, that is,

$$u(y) = c + \int_{x_0}^{y} u'(t) \, dt$$

for all $y \in [x_0, x]$. Taking $y = x_0$ gives (1).

It remains to show that u' is equi-integrable. We begin by observing that by Exercise 3.7, u can be extended to $\overline{I}$ as an absolutely continuous function $\overline{u}$. Fix $\varepsilon > 0$ and let $\delta > 0$ be as in Definition 3.1 for $\overline{u}$. Consider a Lebesgue measurable set $E \subset I$, with $\mathcal{L}^1(E) \leq \frac{\delta}{2}$. By the outer regularity of the Lebesgue measure we may find an open set $A \supset E$ such that $\mathcal{L}^1(A) < \delta$. Decompose A into a countable family $\{J_k\}$ of pairwise disjoint intervals. By replacing each J_k with $J_k \cap I$, we may assume that $J_k \subset I$. Let $\overline{J_k} = [a_k, b_k]$. Using the fact that $\mathcal{L}^1(A) < \delta$, we have that

$$\sum_k |b_k - a_k| < \delta.$$

Consider a partition $P_k = \left\{x_0^{(k)}, \dots, x_{m_k}^{(k)}\right\}$ of $[a_k, b_k]$. Since

$$\sum_k \sum_{i=1}^{m_k} \left| x_i^{(k)} - x_{i-1}^{(k)} \right| = \sum_k |b_k - a_k| < \delta,$$

it follows from the fact that $\overline{u} \in AC\left(\overline{I}\right)$ that

$$\sum_k \sum_{i=1}^{m_k} \left| \overline{u}\left(x_i^{(k)}\right) - \overline{u}\left(x_{i-1}^{(k)}\right) \right| \leq \varepsilon.$$

Taking the supremum over every partition P_k of $[a_k, b_k]$ for each k, we have that

$$\sum_k \operatorname{Var}_{[a_k, b_k]} \overline{u} \leq \varepsilon.$$

Hence, by Corollary 2.23 applied to each interval $[a_k, b_k]$,

$$\int_E |u'| \, dx = \int_E |\overline{u}'| \, dx \leq \sum_k \int_{[a_k, b_k]} |\overline{u}'| \, dx \leq \sum_k \operatorname{Var}_{[a_k, b_k]} \overline{u} \leq \varepsilon.$$

This concludes the proof. ■