1 An Alternative Proof of Theorem 3.30

**Theorem 1** Let \( I \subset \mathbb{R} \) be an interval and let \( u : I \to \mathbb{R} \). Then \( u \) belongs to \( AC(I) \) if and only if \( u \) is differentiable \( L^1 \)-a.e. in \( I \), \( u' \) belongs to \( L^1_{\text{loc}}(I) \) and is equi-integrable, and the fundamental theorem of calculus is valid, that is, for all \( x, x_0 \in I \),

\[
    u(x) = u(x_0) + \int_{x_0}^{x} u'(t) \, dt.
\]

**Lemma 2** Let \( I \subset \mathbb{R} \) be an interval and let \( u : I \to \mathbb{R} \) be an absolutely continuous function such that \( u'(x) = 0 \) for \( L^1 \) a.e. \( x \in I \). Then \( u \) is constant.

**Proof.** Given \( \varepsilon > 0 \), let \( \delta > 0 \) be the number given in the definition of absolute continuity. Let \( a, b \in I \) with \( a < b \). We claim that \( u(a) = u(b) \). Let \( E := \{x \in (a,b) : u'(x) = 0 \} \). Then \( L^1(E) = b - a \). For every \( x \in E \), we have that

\[
    \lim_{y \to x} \frac{u(y) - u(x)}{y - x} = u'(x) = 0,
\]

and so there exists \( \eta_x > 0 \) such that \([x - \eta_x, x + \eta_x] \subset (a,b) \) and

\[
    \left| \frac{u(y) - u(x)}{y - x} \right| \leq \frac{\varepsilon}{b - a}
\]

for all \( y \in I \) with \( |y - x| \leq \eta_x \). The family

\[
    \{[x - r, x + r] : x \in E, 0 < r \leq \eta_x \}
\]

is a fine cover of \( E \), and so by the Vitali covering theorem (see Theorem 1.31), there exists a countable disjoint family \( \{[x_n - r_n, x_n + r_n] \} \) such that

\[
    L^1 \left( E \setminus \bigcup_n [x_n - r_n, x_n + r_n] \right) = 0.
\]

Let \( \ell \) be so large that

\[
    L^1 \left( E \setminus \bigcup_{n=1}^{\ell} [x_n - r_n, x_n + r_n] \right) \leq \delta.
\]

Then

\[
    L^1([a,b]) = L^1(E) \leq L^1 \left( E \setminus \bigcup_{n=1}^{\ell} [x_n - r_n, x_n + r_n] \right) + \sum_{n=1}^{\ell} 2r_n
\]

\[
    \leq \delta + \sum_{n=1}^{\ell} 2r_n.
\]

Without loss of generality assume that \( x_1 < x_2 < \cdots < x_\ell \). Then by the previous inequality, it follows that sum of the length of the intervals \([a, x_1 - r] \),
\[ [x_1 + r_1, x_2 - r_2], \ldots, [x_\ell + r_\ell, b] \] is less than or equal to \( \delta \). Since \( u \) is absolutely continuous, we have that

\[
|u(a) - u(x_1 - r_1)| + \sum_{n=1}^{\ell-1} |u(x_{n+1} - r_{n+1}) - u(x_n + r_n)| + |u(b) - u(x_\ell + r_\ell)| \leq \varepsilon.
\]

On the other hand by (2),

\[
|u(x_n + r_n) - u(x_n - r_n)| \leq |u(x_n + r_n) - u(x_n)| + |u(x_n) - u(x_n - r_n)|
\]

\[
\leq \frac{\varepsilon 2 r_n}{b - a}
\]

and so

\[
|u(a) - u(b)| \leq |u(a) - u(x_1 - r_1)| + \sum_{n=1}^{\ell-1} |u(x_{n+1} - r_{n+1}) - u(x_n + r_n)|
\]

\[
+ \sum_{n=1}^\ell |u(x_n + r_n) - u(x_n + r_n)| + |u(b) - u(x_\ell + r_\ell)|
\]

\[
\leq \varepsilon + \sum_{n=1}^\ell \frac{\varepsilon 2 r_n}{b - a} \leq 3\varepsilon.
\]

Letting \( \varepsilon \to 0^+ \) gives \( u(a) = u(b) \). Hence, \( u \) is constant. \( \blacksquare \)

**Lemma 3** Let \( I \subseteq \mathbb{R} \) be an interval and let \( v \in L^1_{\text{loc}}(I) \), with \( v \) equi-integrable. Fix \( x_0 \in I \) and let

\[
u(x) := \int_{x_0}^x v(t) \, dt, \quad x \in I.
\]

Then the function \( u \) is absolutely continuous in \( I \) and differentiable for \( L^1 \) a.e. \( x \in I \) with \( u'(x) = v(x) \).

**Proof.** Since, \( v \) is equi-integrable, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( E \subseteq I \) is a Lebesgue measurable set with \( L^1(E) \leq \delta \), then

\[
\int_E |v(t)| \, dt \leq \varepsilon.
\]

Hence, for every finite number of nonoverlapping intervals \( (a_k, b_k), k = 1, \ldots, \ell, \) with \( [a_k, b_k] \subseteq I \) and

\[
\sum_{k=1}^\ell (b_k - a_k) \leq \delta,
\]

we have

\[
\sum_{k=1}^\ell |u(b_k) - u(a_k)| = \sum_{k=1}^\ell \left| \int_{a_k}^{b_k} v(t) \, dt \right| \leq \sum_{k=1}^\ell \int_{a_k}^{b_k} |v(t)| \, dt
\]

\[
= \int_{\bigcup_{k=1}^\ell (a_k, b_k)} |v(t)| \, dt \leq \varepsilon.
\]
Hence, $u$ is absolutely continuous.

Next fix a representative of $v$ and let $x_0 \in I$ be a Lebesgue point of $v$, that is,
\[
\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{B(x_0, \varepsilon)} |v(t) - v(x_0)| \, dt = 0.
\]
Then for $h \neq 0$,
\[
\left| \frac{u(x_0 + h) - u(x_0)}{h} - v(x_0) \right| = \left| \frac{1}{h} \int_{x_0}^{x_0+h} (v(t) - v(x_0)) \, dt \right|
\leq \frac{2}{|h|} \int_{x_0-|h|}^{x_0+|h|} |v(t) - v(x_0)| \, dt \to 0
\]
as $h \to 0$, which shows that $u$ is differentiable at $x_0$ with $u'(x_0) = v(x_0)$.

**Proof of Theorem 1.** Assume that $u$ is differentiable $L^1$-a.e. in $I$, $u'$ belongs to $L^1_{\text{loc}}(I)$ and is equi-integrable, and the fundamental theorem of calculus is valid. Fix $x_0 \in I$ and define
\[
g(x) := \int_{x_0}^x u'(t) \, dt.
\]
Then by the previous lemma $g$ is absolutely continuous. In turn, since constant functions are absolutely continuous, it follows that the function $u = u(x_0) + g$ is absolutely continuous.

Conversely, assume that $u$ is absolutely continuous. Let’s prove that the fundamental theorem of calculus is valid. Fix $x, x_0 \in I$, with, say $x_0 < x$. We have seen last week that $u' \in L^1_{\text{loc}}(I)$. It follows that $u' \in L^1([x_0, x])$ and so by the previous lemma again the function
\[
g(y) := \int_{x_0}^y u'(t) \, dt, \quad y \in [x_0, x],
\]
belongs to $AC([x_0, x])$ with $g'(y) = u'(y)$ for $L^1$ a.e. $y \in [x_0, x]$. Since $u - g \in AC([x_0, x])$ and
\[
(u-g)'(y) = u'(y) - u'(y) = 0
\]
for $L^1$ a.e. $y \in [x_0, x]$, by Lemma 2, we have that $u - g$ is constant in $[x_0, x]$. Thus, there exists $c \in \mathbb{R}$ such that
\[
(u-g)(y) = c
\]
for all $y \in [x_0, x]$, that is,
\[
u(y) = c + \int_{x_0}^y u'(t) \, dt
\]
for all $y \in [x_0, x]$. Taking $y = x_0$ gives (1).
It remains to show that \( u' \) is equi-integrable. We begin by observing that by Exercise 3.7, \( u \) can be extended to \( \mathcal{T} \) as an absolutely continuous function \( \overline{\pi} \). Fix \( \varepsilon > 0 \) and let \( \delta > 0 \) be as in Definition 3.1 for \( \overline{\pi} \). Consider a Lebesgue measurable set \( E \subset I \), with \( L^1(E) \leq \frac{\delta}{2} \). By the outer regularity of the Lebesgue measure we may find an open set \( A \supset E \) such that \( L^1(A) < \delta \). Decompose \( A \) into a countable family \( \{ J_k \} \) of pairwise disjoint intervals. By replacing each \( J_k \) with \( J_k \cap I \), we may assume that \( J_k \subset I \). Let \( J_k = [a_k, b_k] \). Using the fact that \( L^1(A) < \delta \), we have that

\[
\sum_k |b_k - a_k| < \delta.
\]

Consider a partition \( P_k = \{x_0^{(k)}, \ldots, x_m^{(k)}\} \) of \([a_k, b_k]\). Since

\[
\sum_k \sum_{i=1}^{m_k} \left| x_i^{(k)} - x_{i-1}^{(k)} \right| = \sum_k |b_k - a_k| < \delta,
\]

it follows from the fact that \( \overline{\pi} \in AC(\mathcal{T}) \) that

\[
\sum_k \sum_{i=1}^{m_k} \left| \overline{\pi}(x_i^{(k)}) - \overline{\pi}(x_{i-1}^{(k)}) \right| \leq \varepsilon.
\]

Taking the supremum over every partition \( P_k \) of \([a_k, b_k]\) for each \( k \), we have that

\[
\sum_k \text{Var}_{[a_k, b_k]} \overline{\pi} \leq \varepsilon.
\]

Hence, by Corollary 2.23 applied to each interval \([a_k, b_k]\),

\[
\int_E |u'| \, dx = \int_E |\overline{\pi}'| \, dx \leq \sum_k \int_{[a_k, b_k]} |\overline{\pi}'| \, dx \leq \sum_k \text{Var}_{[a_k, b_k]} \overline{\pi} \leq \varepsilon.
\]

This concludes the proof. \( \blacksquare \)