1 An Alternative Proof of Theorem 7.2

In this section we present an alternative proof of Theorem 7.2, taken from [1].

Theorem 1 (Theorem 7.2) *Let $\Omega \subset \mathbb{R}$ be an open set. If $u : \Omega \to \mathbb{R}$ is integrable and belongs to $BPV(\Omega)$, then $u \in BV(\Omega)$ and*

$$|Du|(\Omega) \leq \operatorname{Var} u.$$

Conversely, if $u \in BV(\Omega)$, then u admits a right continuous representative $\overline{u}$ in $BPV(\Omega)$ such that

$$\operatorname{Var} \overline{u} = |Du|(\Omega).$$

Proof. Step 1: Let Ω be in interval and let $u : \Omega \to \mathbb{R}$ be an integrable function with finite pointwise variation. Let $\varphi \in C_c^\infty(\Omega)$ with $\|\varphi\|_\infty \leq 1$ and let $[a, b] \subset \Omega$ contain the support of φ. Since φ' is uniformly continuous, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|\varphi'(x) - \varphi'(y)| \leq \varepsilon$$

for all $x, y \in \Omega$ with $|x - y| \leq \delta$. Since u is the difference of two increasing functions, it is continuous except on a countable number of points. Moreover it is bounded in $[a, b]$ by some contant M. Hence, it is Riemann integrable in $[a, b]$. In turn $u\varphi'$ is Riemann integrable. Hence, using Riemann sums, there exists a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

with $x_i - x_{i-1} \leq \delta$ for all $i = 1, \ldots, n$ such that

$$\left| \int_a^b u\varphi' \, dx - \sum_{i=1}^n u(x_i)\, \varphi'(x_i)\,(x_i - x_{i-1}) \right| \leq \varepsilon.$$

In turn,

$$\begin{aligned} \left| \int_a^b u\varphi' \, dx \right| &\leq \left| \int_a^b u\varphi' \, dx - \sum_{i=1}^n u(x_i)\, \varphi'(x_i)\,(x_i - x_{i-1}) \right| + \left| \sum_{i=1}^n u(x_i)\, \varphi'(x_i)\,(x_i - x_{i-1}) \right| \\ &\leq \varepsilon + \left| \sum_{i=1}^n u(x_i)\, \varphi'(x_i)\,(x_i - x_{i-1}) \right|. \end{aligned}$$

By the mean value theorem,

$$\varphi(x_i) - \varphi(x_{i-1}) = \varphi'(c_i)\,(x_i - x_{i-1})$$

for some $c_i \in [x_i, x_{i-1}]$. Hence,

$$\left|\sum_{i=1}^{n} u(x_i)\varphi'(x_i)(x_i - x_{i-1})\right| = \left|\sum_{i=1}^{n} u(x_i)\left[\varphi'(x_i) - \varphi'(c_i) + \varphi'(c_i)\right](x_i - x_{i-1})\right|$$
$$\leq \sum_{i=1}^{n} |u(x_i)|\,|\varphi'(x_i) - \varphi'(c_i)|\,(x_i - x_{i-1}) + \left|\sum_{i=1}^{n} u(x_i)\varphi'(c_i)(x_i - x_{i-1})\right|$$
$$\leq M\varepsilon(b-a) + \left|\sum_{i=1}^{n} u(x_i)(\varphi(x_i) - \varphi(x_{i-1}))\right|.$$

Now

$$\begin{aligned}\left|\sum_{i=1}^{n} u(x_i)(\varphi(x_i) - \varphi(x_{i-1}))\right| &= \left|\sum_{i=1}^{n} u(x_i)\varphi(x_i) - \sum_{k=0}^{n-1} u(x_{k+1})\varphi(x_k)\right| \\ &= \left|\sum_{i=1}^{n} (u(x_i) - u(x_{i+1}))\varphi(x_i)\right| \\ &\leq \sum_{i=1}^{n} |u(x_i) - u(x_{i+1})| \leq \operatorname{Var} u,\end{aligned}$$

where we have used the fact that $\varphi(b) = 0$ and that $\|\varphi\|_\infty \leq 1$.

In conclusion, we have show that

$$\left|\int_a^b u\varphi'\,dx\right| \leq \varepsilon + M\varepsilon(b-a) + \operatorname{Var} u.$$

Letting $\varepsilon \to 0^+$ and recalling that $u\varphi' = 0$ outside $[a,b]$ gives

$$\left|\int_\Omega u\varphi'\,dx\right| \leq \operatorname{Var} u.$$

Taking the supremum over all such φ gives

$$|Du|(\Omega) \leq \operatorname{Var} u.$$

Step 2: Conversely, let $u \in BV(\Omega)$ and assume first that $\mu := Du$ is a (positive) measure, $\mu : \mathcal{B}(\Omega) \to [0,\infty)$ and define

$$w(x) := \mu(\Omega \cap (-\infty, x)).$$

Note that w is left continuous and increasing. To see this fix $x \in \Omega$ and let $x_n \nearrow x$. Then

$$\begin{aligned}\lim_{n\to\infty} w(x_n) &= \lim_{n\to\infty} \mu(\Omega \cap (-\infty, x_n)) = \mu\left(\bigcup_{n=1}^{\infty} I \cap (-\infty, x_n)\right) \\ &= \mu(I \cap (-\infty, x)) = w(x).\end{aligned}$$

Let $\varphi \in C_c^\infty(\Omega)$, then

$$\begin{aligned}
\int_\Omega w\varphi' \, dx &= \int_\Omega \mu\left(\Omega \cap (-\infty, x)\right) \varphi'(x) \, dx \\
&= \int_\Omega \varphi'(x) \left[\int_\Omega \chi_{(-\infty,x)}(y) \, d\mu(y)\right] dx \\
&= \int_{\Omega\times\Omega} h(x,y) \, d\left(\mu \times \mathcal{L}^1\right)(y,x),
\end{aligned}$$

where we have considered the product measure $\mu \times \mathcal{L}^1$ and the function h is defined by

$$h(x,y) := \begin{cases} \varphi'(x) & \text{if } y < x, \\ 0 & \text{if } y \geq x. \end{cases}$$

By Fubini's theorem

$$\begin{aligned}
\int_{\Omega\times\Omega} h(x,y) \, d\mu &= \int_\Omega \left[\int_\Omega \chi_{(y,\infty)}(x) \varphi'(x) \, dx\right] d\mu(y) \\
&= \int_\Omega \left[\int_y^\infty \varphi'(x) \, dx\right] d\mu(y) = \int_\Omega \left(\varphi(\infty) - \varphi(y)\right) d\mu(y).
\end{aligned}$$

Hence,

$$\int_\Omega w\varphi' \, dx = -\int_\Omega \varphi \, d\mu.$$

On the other hand,

$$\int_\Omega u\varphi' \, dx = -\int_\Omega \varphi \, d\mu.$$

Thus, by subtracting these two equalities, we get

$$\int_\Omega (u-w)\varphi' \, dx = 0$$

for all $C_c^\infty(\Omega)$. This implies that $u - w$ is a constant, say, $u(x) - w(x) = c$ for $\mathcal{L}^1$ a.e. $x \in \Omega$. Define

$$v(x) := w(x) + c = \mu\left(\Omega \cap (-\infty, x)\right) + c.$$

Then v is left-continuous and increasing. Hence,

$$\operatorname{Var} v = \sup v - \inf v = \mu(\Omega) = |Du|(\Omega).$$

Step 3: Finally, let $u \in BV(\Omega)$ and let $\lambda := Du$. Write $\lambda = \lambda^+ - \lambda^-$ and define

$$w^\pm(x) = \lambda^\pm\left(\Omega \cap (-\infty, x)\right)$$

and $w := w^+ - w^-$. Reasoning as before, we find that

$$\begin{aligned}
\int_\Omega w\varphi' \, dx &= \int_\Omega w^\pm \varphi' \, dx - \int_\Omega w^- \varphi' \, dx \\
&= -\int_\Omega \varphi \, d\lambda^+ + \int_\Omega \varphi \, d\lambda^- = -\int_\Omega \varphi \, d\lambda,
\end{aligned}$$

and so

$$\int_{\Omega} (u - w)\, \varphi'\, dx = 0$$

for all $C_c^\infty(\Omega)$, which $u(x) - w(x) = c$ for $\mathcal{L}^1$ a.e. $x \in \Omega$. Define

$$v(x) := w(x) + c = \lambda(\Omega \cap (-\infty, x)) + c.$$

Then

$$\begin{aligned} \operatorname{Var} v &= \operatorname{Var}\left(w^+ - w^- + c\right) \le \operatorname{Var}\left(w^+\right) + \operatorname{Var}\left(w^-\right) \\ &= \lambda^+(\Omega) + \lambda^-(\Omega) = \lambda(\Omega) = |Du|(\Omega). \end{aligned}$$

Observe that since $\int_\Omega \varphi'\, dx = 0$, we have that

$$\int_\Omega v\varphi'\, dx = \int_\Omega (w + c)\, \varphi'\, dx = \int_\Omega w\varphi'\, dx = -\int_\Omega \varphi\, d\lambda.$$

Hence, $v \in BV(\Omega)$ and $Dv = Du$. Thus, by Step 1,

$$|Dv|(\Omega) = |Du|(\Omega) \le \operatorname{Var} v,$$

which shows that $|Du|(\Omega) = \operatorname{Var} v$. ■

References

[1] G. Dal Maso, *Functions of bounded variation,* Lecture Notes (unpublished), SISSA, Trieste, 2002.