# 1 Traces of Functions in $W^{2,1}\left(\mathbb{R}^N_+\right)$

Since $W^{2,1}\left(\mathbb{R}^N_+\right) \subset W^{1,1}\left(\mathbb{R}^N_+\right)$, the trace operator

$$\mathrm{Tr} : W^{2,1}\left(\mathbb{R}^N_+\right) \to L^1\left(\mathbb{R}^{N-1}\right)$$

is a well-defined continuous linear operator. In what follows we show that if $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$, then its trace $\mathrm{Tr}\,(u)$ belongs to the Besov space $B^{1,1}\left(\mathbb{R}^{N-1}\right)$.

We begin by showing the inclusion $\mathrm{Tr}\left(W^{2,1}\left(\mathbb{R}^N_+\right)\right) \subset W^{1,1}\left(\mathbb{R}^{N-1}\right)$.

**Theorem 1** *Let $N \geq 2$. Then $\mathrm{Tr}\left(W^{2,1}\left(\mathbb{R}^N_+\right)\right) \subset W^{1,1}\left(\mathbb{R}^{N-1}\right)$ and there exists a constant $C = C(N) > 0$ such that for all $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$,*

$$\|\mathrm{Tr}\,(u)\|_{W^{1,1}(\mathbb{R}^{N-1})} \leq C(N)\,\|u\|_{W^{2,1}(\mathbb{R}^N)}\,. \tag{1}$$

*Moreover, for $i = 1, \ldots, N-1$, $\mathrm{Tr}\left(\frac{\partial u}{\partial x_i}\right)$ is the distributional partial derivative of $\mathrm{Tr}\,(u)$ with respect to $x_i$.*

**Proof.** Since for every $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$ and every $i = 1, \ldots, N$, $\frac{\partial u}{\partial x_i} \in W^{1,1}\left(\mathbb{R}^N_+\right)$, by Theorem 15.1, we have that $\mathrm{Tr}\left(\frac{\partial u}{\partial x_i}\right) \in L^1\left(\mathbb{R}^{N-1}\right)$ and

$$\int_{\mathbb{R}^{N-1}} \left|\mathrm{Tr}\left(\frac{\partial u}{\partial x_i}\right)(x')\right| dx' \leq \int_{\mathbb{R}^N_+} \left|\frac{\partial^2 u}{\partial x_N \partial x_i}(x)\right| dx. \tag{2}$$

We claim that for $i = 1, \ldots, N-1$, $\mathrm{Tr}\left(\frac{\partial u}{\partial x_i}\right)$ is the distributional partial derivative of $\mathrm{Tr}\,(u)$ with respect to $x_i$. To see this, note that if $u \in C^2\left(\overline{\mathbb{R}^N_+}\right)$, for every $\varphi \in C^1_c\left(\mathbb{R}^{N-1}\right)$ and every $i = 1, \ldots, N-1$, we have that

$$\int_{\mathbb{R}^{N-1}} u(x', 0)\,\frac{\partial \varphi}{\partial x_i}(x')\,dx = -\int_{\mathbb{R}^{N-1}} \varphi(x')\,\frac{\partial u}{\partial x_i}(x', 0)\,dx. \tag{3}$$

If $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$, construct a sequence of functions $\{u_n\} \subset C^2_c\left(\overline{\mathbb{R}^N_+}\right)$ such that $u_n \to u$ in $W^{2,1}\left(\mathbb{R}^N_+\right)$. By the continuity of the operator Tr, it follows that $u_n(\cdot, 0) \to \mathrm{Tr}\,(u)$ and $\frac{\partial u_n}{\partial x_i}(\cdot, 0) \to \mathrm{Tr}\left(\frac{\partial u}{\partial x_i}\right)$ in $L^1\left(\mathbb{R}^{N-1}\right)$. Replacing $u$ with $u_n$ in (3) and letting $n \to \infty$, we obtain

$$\int_{\mathbb{R}^{N-1}} \mathrm{Tr}\,(u)(x')\,\frac{\partial \varphi}{\partial x_i}(x')\,dx = -\int_{\mathbb{R}^{N-1}} \mathrm{Tr}\left(\frac{\partial u}{\partial x_i}\right)(x')\,\frac{\partial u}{\partial x_i}(x', 0)\,dx.$$

This shows that $\mathrm{Tr}\,(u) \in W^{1,1}\left(\mathbb{R}^{N-1}\right)$. The bound (1) now follows from (2). ■

Next we show that $\mathrm{Tr}\left(W^{2,1}\left(\mathbb{R}^N_+\right)\right) \subset B^{1,1}\left(\mathbb{R}^{N-1}\right)$.

**Theorem 2** *Let $N \geq 2$. Then $\mathrm{Tr}\left(W^{2,1}\left(\mathbb{R}^N_+\right)\right) \subset B^{1,1}\left(\mathbb{R}^{N-1}\right)$ and there exists a constant $C = C(N) > 0$ such that for all $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$,*

$$|\mathrm{Tr}\,(u)|_{B^{1,1}(\mathbb{R}^{N-1})} \leq C \int_{\mathbb{R}^N_+} \left|\nabla^2 u(x)\right| dx. \tag{4}$$

**Proof.** Assume that $u \in C^2\left(\overline{\mathbb{R}^N_+}\right)$. Then taking $\theta = p = 1$ in Step 2 of the proof of Theorem 14 in the file "Higher order Besov spaces", we get (4). A density argument shows that the same result continues to hold for an arbitrary $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$. ■

We now prove that

$$\operatorname{Tr}\left(W^{2,1}\left(\mathbb{R}^N_+\right)\right) = B^{1,1}\left(\mathbb{R}^{N-1}\right).$$

**Theorem 3** *Let $N \geq 2$ and let $f \in B^{1,1}\left(\mathbb{R}^{N-1}\right)$. Then there exists a function $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$ such that $\operatorname{Tr}(u) = f$ and*

$$\|u\|_{W^{2,1}\left(\mathbb{R}^N_+\right)} \leq C\, \|f\|_{B^{1,1}(\mathbb{R}^{N-1})}, \tag{5}$$

*where $C = C(N) > 0$.*

**Proof.** Consider a cut-off function $\varphi \in C_c^\infty(B(0,1))$ such that $\varphi$ is even in each variable $x_i$, $i = 1, \ldots, N-1$, and $\int_{\mathbb{R}^{N-1}} \varphi(x')\, dx' = 1$. For $x' \in \mathbb{R}^{N-1}$ and $x_N > 0$ define

$$\begin{aligned} v(x', x_N) :&= \frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \varphi\left(\frac{x'-y'}{x_N}\right) f(y')\, dy' \\ &= \int_{\mathbb{R}^{N-1}} \varphi(z')\, f(x' - x_N z')\, dy'. \end{aligned}$$

Since $B^{1,1}\left(\mathbb{R}^{N-1}\right) \subset W^{1,1}\left(\mathbb{R}^{N-1}\right)$ by Corollary 15 in the file "Higher order Besov spaces" (taking $\theta = p = 1$), following the proof of Theorem 15.21 and using Theorems B.53 and C.20, for $x' \in \mathbb{R}^{N-1}$ and $x_N > 0$, we have

$$\begin{aligned} \frac{\partial v}{\partial x_i}(x) &= \frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \varphi\left(\frac{x'-y'}{x_N}\right) \frac{\partial f}{\partial x_i}(y')\, dy', \\ \frac{\partial v}{\partial x_N}(x) &= -\frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \varphi\left(\frac{x'-y'}{x_N}\right) \sum_{i=1}^{N-1} \left(\frac{x_i - y}{x_N}\right) \frac{\partial f}{\partial x_i}(y')\, dy', \end{aligned}$$

for every $i = 1, \ldots, N-1$. Hence, by Theorem C.19 we have that for all $x_N > 0$,

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} |v(x', x_N)|\, dx' &\leq \int_{\mathbb{R}^{N-1}} |f(x')|\, dx', \\ \int_{\mathbb{R}^{N-1}} \left|\frac{\partial v}{\partial x_i}(x', x_N)\right| dx' &\leq \int_{\mathbb{R}^{N-1}} \left|\frac{\partial u}{\partial x_i}(x')\right| dx', \\ \int_{\mathbb{R}^{N-1}} \left|\frac{\partial v}{\partial x_N}(x', x_N)\right| dx' &\leq C \int_{\mathbb{R}^{N-1}} |\nabla_{x'} u(x')|\, dx', \end{aligned} \tag{6}$$

where in the last inequality we have used also the fact that $\operatorname{supp}\varphi \subset B(0,1)$. Moreover, taking $\theta = p = 1$ in Theorem 14 in the file "Higher order Besov spaces", we get

$$\left\|\frac{\partial^2 v}{\partial x_i \partial x_j}\right\|_{L^1\left(\mathbb{R}^N_+\right)} \leq C\, |f|_{B^{1,1}(\mathbb{R}^{N-1})} \tag{7}$$

for all $i, j = 1, \ldots, N$. Define

$$u(x) := e^{-x_N} v(x), \quad x \in \mathbb{R}^N_+.$$

Then

$$\begin{aligned}
\frac{\partial u}{\partial x_i} &= e^{-x_N} \frac{\partial v}{\partial x_i}, \quad \frac{\partial u}{\partial x_N} = -e^{-x_N} v + e^{-x_N} \frac{\partial v}{\partial x_N}, \\
\frac{\partial^2 u}{\partial x_j \partial x_i} &= e^{-x_N} \frac{\partial^2 v}{\partial x_j \partial x_i}, \quad \frac{\partial^2 u}{\partial x_N \partial x_i} = -e^{-x_N} \frac{\partial v}{\partial x_i} + e^{-x_N} \frac{\partial^2 v}{\partial x_N \partial x_j}, \\
\frac{\partial^2 u}{\partial x_N^2} &= e^{-x_N} v - 2e^{-x_N} \frac{\partial v}{\partial x_N} + e^{-x_N} \frac{\partial^2 v}{\partial x_N^2},
\end{aligned} \tag{8}$$

for all $i, j = 1, \ldots, N-1$. Hence, as in the proof of Theorem 15.21 and by (6), we have that

$$\begin{aligned}
\int_{\mathbb{R}^N_+} |u| \, dx \le \int_{\mathbb{R}^{N-1}} |f| \, dx', &\quad \int_{\mathbb{R}^N_+} \left| \frac{\partial u}{\partial x_i} \right| dx \le \int_{\mathbb{R}^{N-1}} \left| \frac{\partial f}{\partial x_i} \right| dx', \\
\int_{\mathbb{R}^N_+} \left| \frac{\partial u}{\partial x_N} \right| dx \le \int_{\mathbb{R}^{N-1}} |f| \, dx' &+ C \int_{\mathbb{R}^{N-1}} |\nabla_{x'} f| \, dx',
\end{aligned}$$

for $i = 1, \ldots, N-1$, while by (7) and the estimates we just obtained for the partial derivatives of $u$,

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| dx \le C |f|_{B^{1,1}(\mathbb{R}^{N-1})}, \\
&\int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx \le \int_{\mathbb{R}^{N-1}} \left| \frac{\partial f}{\partial x_i} \right| dx' + C |f|_{B^{1,1}(\mathbb{R}^{N-1})} \\
&\int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N^2} \right| dx \le 2 \int_{\mathbb{R}^{N-1}} |f| \, dx' + C \int_{\mathbb{R}^{N-1}} |\nabla_{x'} f| \, dx' + C |f|_{B^{1,1}(\mathbb{R}^{N-1})}.
\end{aligned}$$

This concludes the proof. ■

Next we study the normal derivative. In the proof of Theorem 1 we have seen that if $u \in W^{2,1}(\mathbb{R}^N_+)$, then $\operatorname{Tr}\left(\frac{\partial u}{\partial x_N}\right) \in L^1(\mathbb{R}^{N-1})$ with

$$\int_{\mathbb{R}^{N-1}} \left| \operatorname{Tr}\left(\frac{\partial u}{\partial x_N}\right)(x') \right| dx' \le \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N \partial x_i}(x) \right| dx. \tag{9}$$

**Theorem 4** *Let $g \in L^1(\mathbb{R}^{N-1})$, $N \ge 2$. Then for every $0 < \varepsilon < 1$ there exists* *a function* $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$ *such that* $\operatorname{Tr}(u) = 0$, $\operatorname{Tr}\left(\frac{\partial u}{\partial x_N}\right) = g$, *and*

$$\int_{\mathbb{R}^N_+} |u| \, dx \le \varepsilon \int_{\mathbb{R}^{N-1}} |g| \, dx', \quad \int_{\mathbb{R}^N_+} |\nabla u| \, dx \le \varepsilon \int_{\mathbb{R}^{N-1}} |g| \, dx',$$

$$\int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx \le \varepsilon \int_{\mathbb{R}^{N-1}} |g| \, dx', \quad \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N^2} \right| dx \le (1+\varepsilon) \int_{\mathbb{R}^{N-1}} |g| \, dx',$$

$$\int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx \le \varepsilon \int_{\mathbb{R}^{N-1}} |g| \, dx'$$

*for all* $i, j = 1, \ldots, N-1$.

**Proof.** If $g = 0$, it suffices to take $u = 0$. Thus, assume that $g \neq 0$. By Theorem C.23 there exists a sequence $\{g_n\} \subset C_c^\infty\left(\mathbb{R}^{N-1}\right)$ such that $g_n \to g$ in $L^1\left(\mathbb{R}^{N-1}\right)$. For each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that for all $n \ge N_k$,

$$\|g_n - g\|_{L^1(\mathbb{R}^{N-1})} \le \frac{\varepsilon}{2^{k+1}} \|g\|_{L^1(\mathbb{R}^{N-1})}.$$

Let $n_k := \max\{N_k, N_{k-1} + 1\}$ and define $h_k := g_{n_k}$ and $h_0 :\equiv 0$. The sequence $\{h_k\}$ satisfies the inequalities

$$\begin{aligned} \|h_{k+1} - h_k\|_{L^1(\mathbb{R}^{N-1})} &\le \frac{\varepsilon}{2^k} \|g\|_{L^1(\mathbb{R}^{N-1})} \quad \text{for all } k \in \mathbb{N}, \\ \|h_k\|_{L^1(\mathbb{R}^{N-1})} &\le (1+\varepsilon) \|g\|_{L^1(\mathbb{R}^{N-1})} \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{10}$$

Construct a strictly decreasing sequence $\{t_k\} \subset (0,1)$, $k \in \mathbb{N}_0$, such that $t_k \to 0$ and

$$|t_{k+1} - t_k| \le \frac{\varepsilon}{2^k} \frac{\|g\|_{L^1(\mathbb{R}^{N-1})}}{\|\nabla_{x'} h_{k+1}\|_{W^{1,1}} + \|\nabla_{x'} h_k\|_{W^{1,1}} + 1}, \quad t_0 \le \frac{\varepsilon}{4}. \tag{11}$$

For $x \in \mathbb{R}^N_+$ define

$$w(x) := \begin{cases} 0 & \text{if } x_N \ge t_0, \\ \dfrac{t_k - x_N}{t_k - t_{k+1}} h_{k+1}(x') + \dfrac{x_N - t_{k+1}}{t_k - t_{k+1}} h_k(x') & \text{if } t_{k+1} \le x_N \le t_k. \end{cases}$$

As in the proof of Theorem 15.6, we have that $w \in W^{1,1}\left(\mathbb{R}^N_+\right)$, $\operatorname{Tr}(w) = g$, and

$$\begin{aligned} &\int_{\mathbb{R}^N_+} |w| \, dx \le \varepsilon \int_{\mathbb{R}^{N-1}} |g| \, dx', \quad \int_{\mathbb{R}^N_+} \left| \frac{\partial w}{\partial x_i} \right| dx \le \varepsilon \int_{\mathbb{R}^{N-1}} |g| \, dx' \\ &\int_{\mathbb{R}^N_+} \left| \frac{\partial w}{\partial x_N} \right| dx \le (1+2\varepsilon) \int_{\mathbb{R}^{N-1}} |g| \, dx', \end{aligned} \tag{12}$$

$i = 1, \ldots, N-1$. Moreover, for $t_{k+1} < x_N < t_k$,

$$\left| \frac{\partial^2 w}{\partial x_i \partial x_j}(x) \right| \le \left| \frac{\partial^2 h_{k+1}}{\partial x_i \partial x_j}(x') \right| + \left| \frac{\partial^2 h_k}{\partial x_i \partial x_j}(x') \right| \quad \text{for all } i, j = 1, \ldots, N-1. \tag{13}$$

For $x \in \mathbb{R}^N_+$ define

$$v(x', x_N) := e^{-x_N} \int_0^{x_N} w(x', s)\ ds, \quad u(x', x_N) := e^{-x_N} v(x', x_N).$$

By Theorem 10.35 and the fact that $w \in W^{1,1}(\mathbb{R}^N_+)$, we have that $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^N_+)$. Since $w \in L^1(\mathbb{R}^N_+)$, we have that

$$\lim_{x_N \to 0^+} \int_{\mathbb{R}^{N-1}} |u(x', x_N)|\ dx' = \lim_{x_N \to 0^+} e^{-x_N} \int_0^{x_N} \int_{\mathbb{R}^{N-1}} |w(x', s)|\ dx' ds = 0,$$

which, reasoning as in (15.6), implies that $\mathrm{Tr}(u) = 0$. Moreover, since

$$\frac{\partial u}{\partial x_N}(x) = -e^{-x_N} \int_0^{x_N} w(x', s)\ ds + e^{-x_N} w(x) = -u(x) + e^{-x_N} w(x) \quad (14)$$

for all $x \in \mathbb{R}^N_+$, it follows that $\mathrm{Tr}\left(\frac{\partial u}{\partial x_N}\right) = \mathrm{Tr}(w) = g$. By (8), and (12),

$$\begin{aligned}\int_{\mathbb{R}^N_+} |u(x)|\ dx &= \int_0^\infty \int_{\mathbb{R}^{N-1}} \left| e^{-x_N} \int_0^{x_N} w(x', s)\ ds \right| dx' dx_N \\ &\le \int_0^\infty e^{-x_N} dx_N \int_{\mathbb{R}^{N-1}} \int_0^\infty |w(x', s)|\ ds dx' \le \varepsilon \int_{\mathbb{R}^{N-1}} |g|\ dx',\end{aligned}$$

while

$$\begin{aligned}\int_{\mathbb{R}^N_+} \left| \frac{\partial u}{\partial x_N}(x) \right| dx &\le \int_{\mathbb{R}^N_+} |u(x)|\ dx + \int_{\mathbb{R}^N_+} \left| e^{-x_N} w(x) \right| dx \\ &\le 2\varepsilon \int_{\mathbb{R}^{N-1}} |g|\ dx'\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N_+} \left| \frac{\partial u}{\partial x_i}(x) \right| dx &\le \int_{\mathbb{R}^N_+} e^{-x_N} \int_0^{x_N} \left| \frac{\partial w}{\partial x_i}(x', s) \right| ds dx \\ &\le \int_0^\infty e^{-x_N} dx_N \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial w}{\partial x_i}(x', s) \right| ds dx' \\ &\le \varepsilon \int_{\mathbb{R}^{N-1}} |g|\ dx'.\end{aligned}$$

Similarly, by (12) and (14),

$$\begin{aligned}\int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N^2} \right| dx &\le \int_{\mathbb{R}^N_+} \left| \frac{\partial w}{\partial x_N} \right| dx + 2 \int_{\mathbb{R}^N_+} |u|\ dx + \int_{\mathbb{R}^N_+} \left| \frac{\partial w}{\partial x_N} \right| dx \\ &\le (1 + 5\varepsilon) \int_{\mathbb{R}^{N-1}} |g|\ dx'\end{aligned}$$

$$\int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx \le \int_{\mathbb{R}^N_+} \left| \frac{\partial w}{\partial x_i} \right| dx + \int_{\mathbb{R}^N_+} \left| \frac{\partial u}{\partial x_i} \right| dx \le 2\varepsilon \int_{\mathbb{R}^{N-1}} |g|\ dx'$$

$i = 1, \cdots, N-1$, while for $i, j = 1, \cdots, N-1$, by (13),

$$\int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2 u}{\partial x_j \partial x_i} (x', s) \right| dx' dx_N = \int_{\mathbb{R}^{N-1}} \int_0^\infty e^{-x_N} \int_0^{x_N} \left| \frac{\partial^2 w}{\partial x_j \partial x_i} (x', s) \right| ds dx' dx_N$$
$$\leq \int_{\mathbb{R}^{N-1}} \int_0^\infty e^{-x_N} \sum_{k=0}^\infty \int_{t_{k+1}}^{t_k} \left| \frac{\partial^2 h_{k+1}}{\partial x_j \partial x_i} (x') \right| + \left| \frac{\partial^2 h_k}{\partial x_j \partial x_i} (x') \right| ds dx_N dx'$$
$$\leq \sum_{k=0}^\infty \left( \left\| \nabla^2_{x'} h_{k+1} \right\|_{L^1(\mathbb{R}^{N-1})} + \left\| \nabla^2_{x'} h_k \right\|_{L^1(\mathbb{R}^{N-1})} \right) |t_{k+1} - t_k|$$
$$\leq \varepsilon \|g\|_{L^1(\mathbb{R}^{N-1})} .$$

This concludes the proof. ■

**Corollary 5** *Let $N \geq 2$, let $g \in B^{1,1}\left(\mathbb{R}^{N-1}\right)$, and let $f \in L^1\left(\mathbb{R}^{N-1}\right)$. Then there exists $u \in W^{2,1}\left(\mathbb{R}^N_+\right)$ such that $\mathrm{Tr}\,(u) = f$ and $\mathrm{Tr}\left(\frac{\partial u}{\partial x_N}\right) = g$.*

**Proof.** By Theorem 2 there exists $w \in W^{2,1}\left(\mathbb{R}^N_+\right)$ such that $\mathrm{Tr}\,(w) = f$ and

$$\|w\|_{W^{2,1}\left(\mathbb{R}^N_+\right)} \leq C \|f\|_{B^{1,1}(\mathbb{R}^{N-1})} ,$$

while by Theorem 4 there exists $v \in W^{2,1}\left(\mathbb{R}^N_+\right)$ such that $\mathrm{Tr}\,(v) = 0$, $\mathrm{Tr}\left(\frac{\partial v}{\partial x_N}\right) = g - \mathrm{Tr}\left(\frac{\partial w}{\partial x_N}\right)$, and

$$\|v\|_{W^{2,1}\left(\mathbb{R}^N_+\right)} \leq C \left\| g - \mathrm{Tr}\left(\frac{\partial w}{\partial x_N}\right) \right\|_{L^1(\mathbb{R}^{N-1})} \leq C \left( \|g\|_{L^1(\mathbb{R}^{N-1})} + \|f\|_{B^{1,1}(\mathbb{R}^{N-1})} \right),$$

where in the last inequality we have used (9). ■

## 2 Traces of Functions in $W^{k,p}\left(\mathbb{R}^N_+\right)$

Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $k \geq 2$. An induction argument shows that the linear mapping

$$u \in W^{k,p}\left(\mathbb{R}^N\right) \mapsto \gamma u := \left(\mathrm{Tr}_0\,(u), \cdots, \mathrm{Tr}_{k-1}\,(u)\right),$$

where $\mathrm{Tr}_0\,(u) := \mathrm{Tr}\,(u)$ and $\mathrm{Tr}_j\,(u) := \mathrm{Tr}\left(\frac{\partial^j u}{\partial x_N^j}\right)$ for $j = 1, \ldots, k-1$, is a well-defined continuous linear operator. Then we have the following result.

**Theorem 6** *Let $1 \leq p \leq \infty$, $k \in \mathbb{N}$, $k \geq 2$. Then*

$$\gamma\left(W^{k,p}\left(\mathbb{R}^N_+\right)\right) = \prod_{j=0}^{k-1} B^{k-j-\frac{1}{p},p}\left(\mathbb{R}^{N-1}\right)$$

*if* $1 < p \leq \infty$, *while*

$$\gamma\left(W^{k,1}\left(\mathbb{R}_{+}^{N}\right)\right)=\left(\prod_{j=0}^{k-2} B^{k-j-1,1}\left(\mathbb{R}^{N-1}\right)\right)\times L^{1}\left(\mathbb{R}^{N-1}\right)$$

*if* $p = 1$.

**Proof.** The proof follows as in the previous section, using Theorem 15 in the file "Higher order Besov spaces". Alternatively, one could use an induction argument of $k$ using Theorem 18 in the file "Higher order Besov spaces" and separating the cases $p = 1$ and $p > 1$. We omit the details. ■

# 3 Notes

For more information on the space $B^{1,p,\theta}\left(\mathbb{R}^{N-1}\right)$, we refer to [1], [2], [3], and [5]. Theorems 2 and 3 are adapted from [5], see also [3] for a different proof. Theorem 4 is based on a paper of Demengel [4].

# References


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