

1 Traces of Functions in $W^{2,1}(\mathbb{R}_+^N)$

Since $W^{2,1}(\mathbb{R}_+^N) \subset W^{1,1}(\mathbb{R}_+^N)$, the trace operator

$$\text{Tr} : W^{2,1}(\mathbb{R}_+^N) \rightarrow L^1(\mathbb{R}^{N-1})$$

is a well-defined continuous linear operator. In what follows we show that if $u \in W^{2,1}(\mathbb{R}_+^N)$, then its trace $\text{Tr}(u)$ belongs to the Besov space $B^{1,1}(\mathbb{R}^{N-1})$.

We begin by showing the inclusion $\text{Tr}(W^{2,1}(\mathbb{R}_+^N)) \subset W^{1,1}(\mathbb{R}^{N-1})$.

Theorem 1 *Let $N \geq 2$. Then $\text{Tr}(W^{2,1}(\mathbb{R}_+^N)) \subset W^{1,1}(\mathbb{R}^{N-1})$ and there exists a constant $C = C(N) > 0$ such that for all $u \in W^{2,1}(\mathbb{R}_+^N)$,*

$$\|\text{Tr}(u)\|_{W^{1,1}(\mathbb{R}^{N-1})} \leq C(N) \|u\|_{W^{2,1}(\mathbb{R}_+^N)}. \quad (1)$$

Moreover, for $i = 1, \dots, N-1$, $\text{Tr}\left(\frac{\partial u}{\partial x_i}\right)$ is the distributional partial derivative of $\text{Tr}(u)$ with respect to x_i .

Proof. Since for every $u \in W^{2,1}(\mathbb{R}_+^N)$ and every $i = 1, \dots, N$, $\frac{\partial u}{\partial x_i} \in W^{1,1}(\mathbb{R}_+^N)$, by Theorem 15.1, we have that $\text{Tr}\left(\frac{\partial u}{\partial x_i}\right) \in L^1(\mathbb{R}^{N-1})$ and

$$\int_{\mathbb{R}^{N-1}} \left| \text{Tr}\left(\frac{\partial u}{\partial x_i}\right)(x') \right| dx' \leq \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N \partial x_i}(x) \right| dx. \quad (2)$$

We claim that for $i = 1, \dots, N-1$, $\text{Tr}\left(\frac{\partial u}{\partial x_i}\right)$ is the distributional partial derivative of $\text{Tr}(u)$ with respect to x_i . To see this, note that if $u \in C^2(\overline{\mathbb{R}_+^N})$, for every $\varphi \in C_c^1(\mathbb{R}^{N-1})$ and every $i = 1, \dots, N-1$, we have that

$$\int_{\mathbb{R}^{N-1}} u(x', 0) \frac{\partial \varphi}{\partial x_i}(x') dx = - \int_{\mathbb{R}^{N-1}} \varphi(x') \frac{\partial u}{\partial x_i}(x', 0) dx. \quad (3)$$

If $u \in W^{2,1}(\mathbb{R}_+^N)$, construct a sequence of functions $\{u_n\} \subset C_c^2(\overline{\mathbb{R}_+^N})$ such that $u_n \rightarrow u$ in $W^{2,1}(\mathbb{R}_+^N)$. By the continuity of the operator Tr , it follows that $u_n(\cdot, 0) \rightarrow \text{Tr}(u)$ and $\frac{\partial u_n}{\partial x_i}(\cdot, 0) \rightarrow \text{Tr}\left(\frac{\partial u}{\partial x_i}\right)$ in $L^1(\mathbb{R}^{N-1})$. Replacing u with u_n in (3) and letting $n \rightarrow \infty$, we obtain

$$\int_{\mathbb{R}^{N-1}} \text{Tr}(u)(x') \frac{\partial \varphi}{\partial x_i}(x') dx = - \int_{\mathbb{R}^{N-1}} \text{Tr}\left(\frac{\partial u}{\partial x_i}\right)(x') \frac{\partial u}{\partial x_i}(x', 0) dx.$$

This shows that $\text{Tr}(u) \in W^{1,1}(\mathbb{R}^{N-1})$. The bound (1) now follows from (2). ■

Next we show that $\text{Tr}(W^{2,1}(\mathbb{R}_+^N)) \subset B^{1,1}(\mathbb{R}^{N-1})$.

Theorem 2 *Let $N \geq 2$. Then $\text{Tr}(W^{2,1}(\mathbb{R}_+^N)) \subset B^{1,1}(\mathbb{R}^{N-1})$ and there exists a constant $C = C(N) > 0$ such that for all $u \in W^{2,1}(\mathbb{R}_+^N)$,*

$$|\text{Tr}(u)|_{B^{1,1}(\mathbb{R}^{N-1})} \leq C \int_{\mathbb{R}_+^N} |\nabla^2 u(x)| dx. \quad (4)$$

Proof. Assume that $u \in C^2(\overline{\mathbb{R}_+^N})$. Then taking $\theta = p = 1$ in Step 2 of the proof of Theorem 14 in the file "Higher order Besov spaces", we get (4). A density argument shows that the same result continues to hold for an arbitrary $u \in W^{2,1}(\mathbb{R}_+^N)$. ■

We now prove that

$$\text{Tr}(W^{2,1}(\mathbb{R}_+^N)) = B^{1,1}(\mathbb{R}^{N-1}).$$

Theorem 3 *Let $N \geq 2$ and let $f \in B^{1,1}(\mathbb{R}^{N-1})$. Then there exists a function $u \in W^{2,1}(\mathbb{R}_+^N)$ such that $\text{Tr}(u) = f$ and*

$$\|u\|_{W^{2,1}(\mathbb{R}_+^N)} \leq C \|f\|_{B^{1,1}(\mathbb{R}^{N-1})}, \quad (5)$$

where $C = C(N) > 0$.

Proof. Consider a cut-off function $\varphi \in C_c^\infty(B(0,1))$ such that φ is even in each variable x_i , $i = 1, \dots, N-1$, and $\int_{\mathbb{R}^{N-1}} \varphi(x') dx' = 1$. For $x' \in \mathbb{R}^{N-1}$ and $x_N > 0$ define

$$\begin{aligned} v(x', x_N) &:= \frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \varphi\left(\frac{x' - y'}{x_N}\right) f(y') dy' \\ &= \int_{\mathbb{R}^{N-1}} \varphi(z') f(x' - x_N z') dy'. \end{aligned}$$

Since $B^{1,1}(\mathbb{R}^{N-1}) \subset W^{1,1}(\mathbb{R}^{N-1})$ by Corollary 15 in the file "Higher order Besov spaces" (taking $\theta = p = 1$), following the proof of Theorem 15.21 and using Theorems B.53 and C.20, for $x' \in \mathbb{R}^{N-1}$ and $x_N > 0$, we have

$$\begin{aligned} \frac{\partial v}{\partial x_i}(x) &= \frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \varphi\left(\frac{x' - y'}{x_N}\right) \frac{\partial f}{\partial x_i}(y') dy', \\ \frac{\partial v}{\partial x_N}(x) &= -\frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \varphi\left(\frac{x' - y'}{x_N}\right) \sum_{i=1}^{N-1} \left(\frac{x_i - y_i}{x_N}\right) \frac{\partial f}{\partial x_i}(y') dy', \end{aligned}$$

for every $i = 1, \dots, N-1$. Hence, by Theorem C.19 we have that for all $x_N > 0$,

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} |v(x', x_N)| dx' &\leq \int_{\mathbb{R}^{N-1}} |f(x')| dx', \\ \int_{\mathbb{R}^{N-1}} \left| \frac{\partial v}{\partial x_i}(x', x_N) \right| dx' &\leq \int_{\mathbb{R}^{N-1}} \left| \frac{\partial f}{\partial x_i}(x') \right| dx', \\ \int_{\mathbb{R}^{N-1}} \left| \frac{\partial v}{\partial x_N}(x', x_N) \right| dx' &\leq C \int_{\mathbb{R}^{N-1}} |\nabla_{x'} f(x')| dx', \end{aligned} \quad (6)$$

where in the last inequality we have used also the fact that $\text{supp } \varphi \subset B(0,1)$. Moreover, taking $\theta = p = 1$ in Theorem 14 in the file "Higher order Besov spaces", we get

$$\left\| \frac{\partial^2 v}{\partial x_i \partial x_j} \right\|_{L^1(\mathbb{R}_+^N)} \leq C \|f\|_{B^{1,1}(\mathbb{R}^{N-1})} \quad (7)$$

for all $i, j = 1, \dots, N$. Define

$$u(x) := e^{-x_N} v(x), \quad x \in \mathbb{R}_+^N.$$

Then

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= e^{-x_N} \frac{\partial v}{\partial x_i}, & \frac{\partial u}{\partial x_N} &= -e^{-x_N} v + e^{-x_N} \frac{\partial v}{\partial x_N}, \\ \frac{\partial^2 u}{\partial x_j \partial x_i} &= e^{-x_N} \frac{\partial^2 v}{\partial x_j \partial x_i}, & \frac{\partial^2 u}{\partial x_N \partial x_i} &= -e^{-x_N} \frac{\partial v}{\partial x_i} + e^{-x_N} \frac{\partial^2 v}{\partial x_N \partial x_i}, \\ \frac{\partial^2 u}{\partial x_N^2} &= e^{-x_N} v - 2e^{-x_N} \frac{\partial v}{\partial x_N} + e^{-x_N} \frac{\partial^2 v}{\partial x_N^2}, \end{aligned} \quad (8)$$

for all $i, j = 1, \dots, N-1$. Hence, as in the proof of Theorem 15.21 and by (6), we have that

$$\begin{aligned} \int_{\mathbb{R}_+^N} |u| dx &\leq \int_{\mathbb{R}^{N-1}} |f| dx', & \int_{\mathbb{R}_+^N} \left| \frac{\partial u}{\partial x_i} \right| dx &\leq \int_{\mathbb{R}^{N-1}} \left| \frac{\partial f}{\partial x_i} \right| dx', \\ \int_{\mathbb{R}_+^N} \left| \frac{\partial u}{\partial x_N} \right| dx &\leq \int_{\mathbb{R}^{N-1}} |f| dx' + C \int_{\mathbb{R}^{N-1}} |\nabla_{x'} f| dx', \end{aligned}$$

for $i = 1, \dots, N-1$, while by (7) and the estimates we just obtained for the partial derivatives of u ,

$$\begin{aligned} \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| dx &\leq C |f|_{B^{1,1}(\mathbb{R}^{N-1})}, \\ \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx &\leq \int_{\mathbb{R}^{N-1}} \left| \frac{\partial f}{\partial x_i} \right| dx' + C |f|_{B^{1,1}(\mathbb{R}^{N-1})} \\ \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N^2} \right| dx &\leq 2 \int_{\mathbb{R}^{N-1}} |f| dx' + C \int_{\mathbb{R}^{N-1}} |\nabla_{x'} f| dx' + C |f|_{B^{1,1}(\mathbb{R}^{N-1})}. \end{aligned}$$

This concludes the proof. \blacksquare

Next we study the normal derivative. In the proof of Theorem 1 we have seen that if $u \in W^{2,1}(\mathbb{R}_+^N)$, then $\text{Tr} \left(\frac{\partial u}{\partial x_N} \right) \in L^1(\mathbb{R}^{N-1})$ with

$$\int_{\mathbb{R}^{N-1}} \left| \text{Tr} \left(\frac{\partial u}{\partial x_N} \right) (x') \right| dx' \leq \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} (x) \right| dx. \quad (9)$$

Theorem 4 *Let $g \in L^1(\mathbb{R}^{N-1})$, $N \geq 2$. Then for every $0 < \varepsilon < 1$ there exists*

a function $u \in W^{2,1}(\mathbb{R}_+^N)$ such that $\text{Tr}(u) = 0$, $\text{Tr}\left(\frac{\partial u}{\partial x_N}\right) = g$, and

$$\begin{aligned} \int_{\mathbb{R}_+^N} |u| dx &\leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx', & \int_{\mathbb{R}_+^N} |\nabla u| dx &\leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx', \\ \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx &\leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx', & \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N^2} \right| dx &\leq (1 + \varepsilon) \int_{\mathbb{R}^{N-1}} |g| dx', \\ \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx &\leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx' \end{aligned}$$

for all $i, j = 1, \dots, N-1$.

Proof. If $g = 0$, it suffices to take $u = 0$. Thus, assume that $g \neq 0$. By Theorem C.23 there exists a sequence $\{g_n\} \subset C_c^\infty(\mathbb{R}^{N-1})$ such that $g_n \rightarrow g$ in $L^1(\mathbb{R}^{N-1})$. For each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that for all $n \geq N_k$,

$$\|g_n - g\|_{L^1(\mathbb{R}^{N-1})} \leq \frac{\varepsilon}{2^{k+1}} \|g\|_{L^1(\mathbb{R}^{N-1})}.$$

Let $n_k := \max\{N_k, N_{k-1} + 1\}$ and define $h_k := g_{n_k}$ and $h_0 \equiv 0$. The sequence $\{h_k\}$ satisfies the inequalities

$$\begin{aligned} \|h_{k+1} - h_k\|_{L^1(\mathbb{R}^{N-1})} &\leq \frac{\varepsilon}{2^k} \|g\|_{L^1(\mathbb{R}^{N-1})} \quad \text{for all } k \in \mathbb{N}, \\ \|h_k\|_{L^1(\mathbb{R}^{N-1})} &\leq (1 + \varepsilon) \|g\|_{L^1(\mathbb{R}^{N-1})} \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \quad (10)$$

Construct a strictly decreasing sequence $\{t_k\} \subset (0, 1)$, $k \in \mathbb{N}_0$, such that $t_k \rightarrow 0$ and

$$|t_{k+1} - t_k| \leq \frac{\varepsilon}{2^k} \frac{\|g\|_{L^1(\mathbb{R}^{N-1})}}{\|\nabla_{x'} h_{k+1}\|_{W^{1,1}} + \|\nabla_{x'} h_k\|_{W^{1,1}} + 1}, \quad t_0 \leq \frac{\varepsilon}{4}. \quad (11)$$

For $x \in \mathbb{R}_+^N$ define

$$w(x) := \begin{cases} 0 & \text{if } x_N \geq t_0, \\ \frac{t_k - x_N}{t_k - t_{k+1}} h_{k+1}(x') + \frac{x_N - t_{k+1}}{t_k - t_{k+1}} h_k(x') & \text{if } t_{k+1} \leq x_N \leq t_k. \end{cases}$$

As in the proof of Theorem 15.6, we have that $w \in W^{1,1}(\mathbb{R}_+^N)$, $\text{Tr}(w) = g$, and

$$\begin{aligned} \int_{\mathbb{R}_+^N} |w| dx &\leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx', & \int_{\mathbb{R}_+^N} \left| \frac{\partial w}{\partial x_i} \right| dx &\leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx' \\ \int_{\mathbb{R}_+^N} \left| \frac{\partial w}{\partial x_N} \right| dx &\leq (1 + 2\varepsilon) \int_{\mathbb{R}^{N-1}} |g| dx', \end{aligned} \quad (12)$$

$i = 1, \dots, N-1$. Moreover, for $t_{k+1} < x_N < t_k$,

$$\left| \frac{\partial^2 w}{\partial x_i \partial x_j}(x) \right| \leq \left| \frac{\partial^2 h_{k+1}}{\partial x_i \partial x_j}(x') \right| + \left| \frac{\partial^2 h_k}{\partial x_i \partial x_j}(x') \right| \quad \text{for all } i, j = 1, \dots, N-1. \quad (13)$$

For $x \in \mathbb{R}_+^N$ define

$$v(x', x_N) := e^{-x_N} \int_0^{x_N} w(x', s) ds, \quad u(x', x_N) := e^{-x_N} v(x', x_N).$$

By Theorem 10.35 and the fact that $w \in W^{1,1}(\mathbb{R}_+^N)$, we have that $u \in W_{\text{loc}}^{2,1}(\mathbb{R}_+^N)$. Since $w \in L^1(\mathbb{R}_+^N)$, we have that

$$\lim_{x_N \rightarrow 0^+} \int_{\mathbb{R}^{N-1}} |u(x', x_N)| dx' = \lim_{x_N \rightarrow 0^+} e^{-x_N} \int_0^{x_N} \int_{\mathbb{R}^{N-1}} |w(x', s)| dx' ds = 0,$$

which, reasoning as in (15.6), implies that $\text{Tr}(u) = 0$. Moreover, since

$$\frac{\partial u}{\partial x_N}(x) = -e^{-x_N} \int_0^{x_N} w(x', s) ds + e^{-x_N} w(x) = -u(x) + e^{-x_N} w(x) \quad (14)$$

for all $x \in \mathbb{R}_+^N$, it follows that $\text{Tr}\left(\frac{\partial u}{\partial x_N}\right) = \text{Tr}(w) = g$. By (8), and (12),

$$\begin{aligned} \int_{\mathbb{R}_+^N} |u(x)| dx &= \int_0^\infty \int_{\mathbb{R}^{N-1}} \left| e^{-x_N} \int_0^{x_N} w(x', s) ds \right| dx' dx_N \\ &\leq \int_0^\infty e^{-x_N} dx_N \int_{\mathbb{R}^{N-1}} \int_0^\infty |w(x', s)| ds dx' \leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx', \end{aligned}$$

while

$$\begin{aligned} \int_{\mathbb{R}_+^N} \left| \frac{\partial u}{\partial x_N}(x) \right| dx &\leq \int_{\mathbb{R}_+^N} |u(x)| dx + \int_{\mathbb{R}_+^N} |e^{-x_N} w(x)| dx \\ &\leq 2\varepsilon \int_{\mathbb{R}^{N-1}} |g| dx' \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^N} \left| \frac{\partial u}{\partial x_i}(x) \right| dx &\leq \int_{\mathbb{R}_+^N} e^{-x_N} \int_0^{x_N} \left| \frac{\partial w}{\partial x_i}(x', s) \right| ds dx \\ &\leq \int_0^\infty e^{-x_N} dx_N \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial w}{\partial x_i}(x', s) \right| ds dx' \\ &\leq \varepsilon \int_{\mathbb{R}^{N-1}} |g| dx'. \end{aligned}$$

Similarly, by (12) and (14),

$$\begin{aligned} \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N^2} \right| dx &\leq \int_{\mathbb{R}_+^N} \left| \frac{\partial w}{\partial x_N} \right| dx + 2 \int_{\mathbb{R}_+^N} |u| dx + \int_{\mathbb{R}_+^N} \left| \frac{\partial w}{\partial x_N} \right| dx \\ &\leq (1 + 5\varepsilon) \int_{\mathbb{R}^{N-1}} |g| dx' \\ \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u}{\partial x_N \partial x_i} \right| dx &\leq \int_{\mathbb{R}_+^N} \left| \frac{\partial w}{\partial x_i} \right| dx + \int_{\mathbb{R}_+^N} \left| \frac{\partial u}{\partial x_i} \right| dx \leq 2\varepsilon \int_{\mathbb{R}^{N-1}} |g| dx' \end{aligned}$$

$i = 1, \dots, N-1$, while for $i, j = 1, \dots, N-1$, by (13),

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2 u}{\partial x_j \partial x_i}(x', s) \right| dx' dx_N &= \int_{\mathbb{R}^{N-1}} \int_0^\infty e^{-x_N} \int_0^{x_N} \left| \frac{\partial^2 w}{\partial x_j \partial x_i}(x', s) \right| ds dx' dx_N \\
&\leq \int_{\mathbb{R}^{N-1}} \int_0^\infty e^{-x_N} \sum_{k=0}^\infty \int_{t_{k+1}}^{t_k} \left| \frac{\partial^2 h_{k+1}}{\partial x_j \partial x_i}(x') \right| + \left| \frac{\partial^2 h_k}{\partial x_j \partial x_i}(x') \right| ds dx_N dx' \\
&\leq \sum_{k=0}^\infty \left(\|\nabla_{x'}^2 h_{k+1}\|_{L^1(\mathbb{R}^{N-1})} + \|\nabla_{x'}^2 h_k\|_{L^1(\mathbb{R}^{N-1})} \right) |t_{k+1} - t_k| \\
&\leq \varepsilon \|g\|_{L^1(\mathbb{R}^{N-1})}.
\end{aligned}$$

This concludes the proof. \blacksquare

Corollary 5 *Let $N \geq 2$, let $g \in B^{1,1}(\mathbb{R}^{N-1})$, and let $f \in L^1(\mathbb{R}^{N-1})$. Then there exists $u \in W^{2,1}(\mathbb{R}_+^N)$ such that $\text{Tr}(u) = f$ and $\text{Tr}\left(\frac{\partial u}{\partial x_N}\right) = g$.*

Proof. By Theorem 2 there exists $w \in W^{2,1}(\mathbb{R}_+^N)$ such that $\text{Tr}(w) = f$ and

$$\|w\|_{W^{2,1}(\mathbb{R}_+^N)} \leq C \|f\|_{B^{1,1}(\mathbb{R}^{N-1})},$$

while by Theorem 4 there exists $v \in W^{2,1}(\mathbb{R}_+^N)$ such that $\text{Tr}(v) = 0$, $\text{Tr}\left(\frac{\partial v}{\partial x_N}\right) = g - \text{Tr}\left(\frac{\partial w}{\partial x_N}\right)$, and

$$\|v\|_{W^{2,1}(\mathbb{R}_+^N)} \leq C \left\| g - \text{Tr}\left(\frac{\partial w}{\partial x_N}\right) \right\|_{L^1(\mathbb{R}^{N-1})} \leq C \left(\|g\|_{L^1(\mathbb{R}^{N-1})} + \|f\|_{B^{1,1}(\mathbb{R}^{N-1})} \right),$$

where in the last inequality we have used (9). \blacksquare

2 Traces of Functions in $W^{k,p}(\mathbb{R}_+^N)$

Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $k \geq 2$. An induction argument shows that the linear mapping

$$u \in W^{k,p}(\mathbb{R}^N) \mapsto \gamma u := (\text{Tr}_0(u), \dots, \text{Tr}_{k-1}(u)),$$

where $\text{Tr}_0(u) := \text{Tr}(u)$ and $\text{Tr}_j(u) := \text{Tr}\left(\frac{\partial^j u}{\partial x_N^j}\right)$ for $j = 1, \dots, k-1$, is a well-defined continuous linear operator. Then we have the following result.

Theorem 6 *Let $1 \leq p \leq \infty$, $k \in \mathbb{N}$, $k \geq 2$. Then*

$$\gamma(W^{k,p}(\mathbb{R}_+^N)) = \prod_{j=0}^{k-1} B^{k-j-\frac{1}{p},p}(\mathbb{R}^{N-1})$$

if $1 < p \leq \infty$, while

$$\gamma(W^{k,1}(\mathbb{R}_+^N)) = \left(\prod_{j=0}^{k-2} B^{k-j-1,1}(\mathbb{R}^{N-1}) \right) \times L^1(\mathbb{R}^{N-1})$$

if $p = 1$.

Proof. The proof follows as in the previous section, using Theorem 15 in the file "Higher order Besov spaces". Alternatively, one could use an induction argument of k using Theorem 18 in the file "Higher order Besov spaces" and separating the cases $p = 1$ and $p > 1$. We omit the details. ■

3 Notes

For more information on the space $B^{1,p,\theta}(\mathbb{R}^{N-1})$, we refer to [1], [2], [3], and [5]. Theorems 2 and 3 are adapted from [5], see also [3] for a different proof. Theorem 4 is based on a paper of Demengel [4].

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