Linear and Quasi-linear Evolution Equations in Hilbert Spaces

Supplemental Chapter
Von Karman Equations

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Contents

Von Karman Equations

In this chapter we consider two types of evolution problems, one hyperbolic and the other parabolic, related to a highly nonlinear elliptic system of equations of von Karman type on $\mathbb{R}^{2m}$, $m \geq 2$. These equations are called "of von Karman type" because of a formal analogy with the well-known equations of the same name in the theory of elasticity (see, e.g., Ciarlet and Rabier [**9**]). While their physical significance may not be evident, their interest resides in a number of specific analytical features, which makes their study a rich subject of investigation. In fact, when $m = 1$ (which case we do not consider), the usual von Karman equations model the dynamics of the vertical oscillations (buckling) of an elastic two-dimensional thin plate, due to both internal and external stresses. In [**1**], Berger devised a remarkable variational method to investigate the elliptic model, corresponding to the stationary state of the usual von Karman system, in a bounded domain of $\mathbb{R}^2$; weak solutions of the corresponding hyperbolic evolution equation have been established in Lions [**12**, ch.1, sct.4] (see also Favini et alii [**10, 11**]). In Cherrier and Milani [**2**], we considered a formally similar elliptic system on a compact Kähler manifold, without boundary, and arbitrary complex dimension $m \geq 2$. This generalization involves a number of analytic difficulties, due to the rather drastic role played by the limit cases of the Sobolev imbeddings. We then considered the corresponding parabolic and hyperbolic evolution problems, respectively in Cherrier and Milani [**3**] and [**5**]. Systems of this type can also be studied in the context of Riemannian manifolds with boundary, with some extra difficulties due to the curvature of the metric of the manifold, and the presence of boundary conditions. For more precise modeling issues related to the von Karman equations, as well as their physical motivations, we refer, e.g., to Ciarlet [**7, 8**]; in addition, a recent exhaustive study of a large class of initial-boundary value problems of von Karman type on domains of $\mathbb{R}^2$, with a multitude of different boundary conditions, can be found in Chuesov and Lasiecka [**6**].

Our goal is to apply the methods of Chapter 3 to show the existence and uniqueness of a local strong solution to the Cauchy problem corresponding to each of the two systems, in a suitable class of function spaces. For both problems, we also establish an almost global existence result, in the spirit of Theorem 4.4.1 of Chapter 4, if the data are sufficiently small. As we mentioned in the introduction, these systems do not fit exactly in the framework of the evolution equations we have presented so far (that is, of type (0.0.2)); however, we believe this example to be of particular interest, as it shows that the unified methods we presented in Chapter 3 can be applied to a much wider class of systems. In addition, we explicitly point out that working on the whole space $\mathbb{R}^{2m}$ introduces a number of serious extra difficulties, which are not present in the case of a compact manifold.

0.1. The Equations

1: The operators. Let $m \in \mathbb{N}_{\geq 2}$, and $u_1, \ldots, u_m, u_{m+1}, u \in C^\infty(\mathbb{R}^{2m})$. We define

$$N(u_1, \ldots, u_m) := \delta^{i_1 \cdots i_m}_{j_1 \cdots j_m} \nabla^{j_1}_{i_1} u_1 \cdots \nabla^{j_m}_{i_m} u_m , \tag{0.1.1}$$

$$I(u_1, \ldots, u_m, u_{m+1}) := \langle N(u_1, \ldots, u_m), u_{m+1} \rangle , \tag{0.1.2}$$

$$M(u) := N(u, \ldots, u) = m! \, \sigma_m(\nabla^2 u) , \tag{0.1.3}$$

where, recalling the summation convention for repeated indices, we adopted the following notations: for $i_1, \ldots, i_m, j_1, \ldots, j_m \in \{1, \ldots 2m\}$, $\delta^{i_1 \cdots i_m}_{j_1 \cdots j_m}$ denotes the Kronecker tensor; for $1 \leq i, j \leq 2m$, $\nabla^j_i := \partial_i \partial_j$, and σ_m is the m-th elementary symmetric function of the eigenvalues $\lambda_j = \lambda_j(u)$, $1 \leq j \leq 2m$, of the Hessian matrix $H(u) := [\partial_i \partial_j u]$, that is,

$$\sigma_m(\nabla^2 u) := \sum_{1 \leq j_1 < j_2 < \ldots < j_m \leq 2m} \lambda_{j_1} \cdots \lambda_{j_m} . \tag{0.1.4}$$

We often abbreviate

$$U_m := (u_1, \ldots, u_m) , \qquad N(U_m) := N(u_1, \ldots, u_m) ; \tag{0.1.5}$$

in addition, we introduce the convention

$$N\left(u_1^{(k_1)}, \ldots, u_p^{(k_p)}\right) := N(\underbrace{u_1, \ldots, u_1}_{k_1 \text{ factors}}, \ldots \underbrace{u_p, \ldots, u_p}_{k_p \text{ factors}}) , \tag{0.1.6}$$

with $k_1 + \cdots + k_p = m$. Finally, we set $\Delta := -\nabla^j_j u$.

2: The Equations. In Lemma 0.1.5 below, we shall show that the elliptic equation

$$\Delta^m f = -M(u) \tag{0.1.7}$$

can be uniquely solved, in a suitable functional frame, for f in terms of u, thereby defining a map $u \mapsto f := f(u)$. Let $T > 0$ and set $\tilde{Q}_T := [0,T]\times\mathbb{R}^{2m}$. Given a source term φ defined on $\tilde{Q}_T$, we consider the following two Cauchy problems, one of hyperbolic type and the other of parabolic type. In the hyperbolic case we wish to determine a function u on $\tilde{Q}_T$, satisfying the equation

$$u_{tt} + \Delta^m u = N(f(u), u^{(m-1)}) + N(\varphi^{(m-1)}, u)\,, \tag{0.1.8}$$

and subject to the initial conditions

$$u(0) = u_0\,, \qquad u_t(0) = u_1\,, \tag{0.1.9}$$

where u_0 and u_1 are given functions defined on $\mathbb{R}^{2m}$. We refer to this Cauchy problem as "problem (H)". In the parabolic case we wish to determine a function u on $\tilde{Q}_T$, satisfying the equation

$$u_t + \Delta^m u = N(f(u), u^{(m-1)}) + N(\varphi^{(m-1)}, u)\,, \tag{0.1.10}$$

and subject to the initial condition

$$u(0) = u_0\,. \tag{0.1.11}$$

We refer to this Cauchy problem as "problem (P)". In the original von Karman system on $\mathbb{R}^2$ the equations under consideration are (7.2.7) and (7.2.9), with $f(u)$ given by (7.2.6), all written for $m = 2$ instead of $m = 1$. In this model the unknown function u represents the vertical displacement of the plate, and the corresponding term $f(u)$ represents the so-called Airy stress function, which is related to the internal elastic forces acting on the plate and depends on the deformation u of the plate (see, e.g., Ciarlet and Rabier [**9**]).

3: Basic Function Spaces. For $k \geq 0$, we define the space $\bar{H}^k$ as the completion of $C_0^\infty(\mathbb{R}^{2m})$ with respect to the norm

$$u \mapsto \|u\|_{\overline{k}} := \begin{cases} |\Delta^{k/2} u|_2 & \text{if} \quad k \text{ is even}\,, \\ |\nabla\, \Delta^{(k-1)/2} u|_2 & \text{if} \quad k \text{ is odd}\,, \end{cases} \tag{0.1.12}$$

with corresponding scalar product

(0.1.13)

$$\langle u, v\rangle_{\overline{k}} := \begin{cases} \langle \Delta^{k/2} u, \Delta^{k/2} v\rangle & \text{if} \quad k \text{ is even}\,, \\ \langle \nabla\Delta^{(k-1)/2} u, \nabla\Delta^{(k-1)/2} v\rangle & \text{if} \quad k \text{ is odd}\,. \end{cases}$$

In the sequel, to avoid unnecessary distinctions between the cases k even and odd, we formally rewrite (0.1.13) and (0.1.12) as

$$\langle u, v\rangle_{\overline{k}} =: \langle \nabla^k u, \nabla^k v\rangle , \qquad \|u\|^2_{\overline{k}} = \langle \nabla^k u, \nabla^k u\rangle . \tag{0.1.14}$$

We also consider in $H^k := H^k(\mathbb{R}^{2m})$ the norm

$$H^k \ni u \mapsto \Big(\|u\|^2_{\overline{k}} + |u|^2_2\Big)^{1/2} =: \|u\|_k , \tag{0.1.15}$$

which is equivalent to the standard norm of H^k, as can be seen by means of the Fourier transform. The main properties of the spaces $\bar{H}^k$ are described in

Proposition 0.1.1. 1) *Let $k \geq 0$. Then $H^k \hookrightarrow \bar{H}^k$; the equivalency*

$$u \in \bar{H}^k \iff \partial_x^k u \in L^2 \tag{0.1.16}$$

holds, and if $u \in \bar{H}^k$,

$$\|u\|_{\overline{k}} \leq \|\partial_x^k u\|_0 \leq C\, \|u\|_{\overline{k}} , \tag{0.1.17}$$

with C independent of u.
2) *Let $s_1 \geq s \geq s_2$, and $u \in \bar{H}^{s_1} \cap \bar{H}^{s_2}$. Then $u \in \bar{H}^s$, and the interpolation inequality*

$$\|u\|_{\overline{s}} \leq C\, \|u\|^{\theta}_{\overline{s_1}} \|u\|^{1-\theta}_{\overline{s_2}} , \qquad \theta := \tfrac{s-s_2}{s_1-s_2} , \tag{0.1.18}$$

holds, with C independent of u.
3) *If $u \in \bar{H}^{m+h}$, $h \geq 0$, then $\partial_x^{h+1} u \in L^{2m}$, $\partial_x^{h+2} u \in L^m$, and*

$$\max\{|\partial_x^{h+1} u|_{2m}, |\partial_x^{h+2} u|_m\} \leq C\, |\partial_x^{m+h} u|_2 \leq C\, \|u\|_{\overline{m+h}} . \tag{0.1.19}$$

Proof. The first assertion is obvious. To show (0.1.16), assume first that $u \in \bar{H}^k$, and let $\alpha = (\alpha_1, \ldots, \alpha_{2m})$ be such that $|\alpha| = k$. Then, since $|\xi_i| \leq |\xi|$ for $1 \leq i \leq 2m$,

(0.1.20)

$$\begin{aligned} \int |\xi_1|^{2\alpha_1} \cdots |\xi_{2m}|^{2\alpha_m} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi &\leq \int |\xi|^{2|\alpha|} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \\ = \int |\xi|^{2k} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi &= \int \Big|\mathcal{F}(\Delta^k u)(\xi)\Big|^2 \,\mathrm{d}\xi = \|\Delta^k u\|^2_0 = \|u\|^2_{\overline{k}} . \end{aligned}$$

This shows that $\mathcal{F}(\partial_x^\alpha u) \in L^2$; therefore, $\partial_x^\alpha u \in L^2$, as well. The converse in (0.1.16) is obvious. Inequality (0.1.18) is proven exactly in the same way as (1.5.48) of Proposition 1.5.3, with the factor $1+|\xi|^2$ replaced by $|\xi|^2$. Finally, (0.1.19) corresponds to the limit cases of the Sobolev imbeddings $H^{m-1} \hookrightarrow L^{2m}$ and $H^{m-2} \hookrightarrow L^m$ (see imbedding (1.5.54) of Theorem 1.5.3); indeed, the estimates follow from the Gagliardo-Nirenberg inequalities (1.5.42), with $\theta = 1$ and $C_1 = 0$ because $\Omega = \mathbb{R}^{2m}$. □

4: Properties of N and I. From (0.1.1), we deduce that the quantity $N(u_1, \dots, u_m)$ is completely symmetric in all its arguments. Using integration by parts, the antisymmetry of Kronecker's tensor, and Schwarz's theorem on the symmetry of third order partial derivatives, we see that the same is true for I; in fact,

(0.1.21)
$$\begin{aligned} & I(u_1, \dots, u_m, u_{m+1}) \\ & = -\int \delta^{i_1 \cdots i_m}_{j_1 \cdots j_m} \nabla^{j_1}_{i_1} u_1 \dots \nabla^{j_{m-1}}_{i_{m-1}} u_{m-1}, \nabla_{i_m} u_m, \nabla^{j_m} u_{m+1} \, \mathrm{d}x \\ & =: J\left(\nabla^2 u_1, \dots, \nabla^2 u_{m-1}, \nabla u_m, \nabla u_{m+1}\right) . \end{aligned}$$

For a proof of this identity, in which the particular structure of the operator N plays a crucial role, we refer to Lemma 1 of Cherrier and Milani [**2**].

We now establish various estimates on the function $N(u_1, \dots, u_m)$ in the Sobolev spaces H^h, $h \geq 0$. The cases $0 \leq h \leq 2$ and $h > 2$ require different kinds of assumptions on the functions u_j, due to the restrictions imposed by the limit cases of the Sobolev imbeddings. In any case, we note that, by the density of $C_0^\infty(\mathbb{R}^{2m})$ in H^h, it is sufficient to establish these estimates when $u_1, \dots, u_m \in C_0^\infty(\mathbb{R}^{2m})$. For future reference, we also note that for $h \geq 0$, $q = (q_1, \dots, q_m) \in \mathbb{N}^m$, and $\alpha \in \mathbb{N}^m$ such that $|\alpha| = h$, we can decompose $\partial_x^\alpha N(u_1, u_2, \dots, u_m)$ as a sum of the type

$$\partial_x^\alpha N(u_1, u_2, \cdots, u_m) = \sum_{|q|=h} C_q \, N(\nabla^{q_1} u_1, \dots, \nabla^{q_m} u_m) , \tag{0.1.22}$$

for suitable constants C_q. In the sequel, whenever a constant C appears in an estimate, unless explicitly stated otherwise, it is understood that C is independent of each of the functions that appear in the estimate.

Lemma 0.1.1. [*Case $h = 0$.*] *Let $U_m = (u_1, \dots, u_m) \in (\bar{H}^{m+1})^m$. Then $N(U_m) \in L^2$, and*

$$\|N(U_m)\|_0 \leq C \prod_{j=1}^m \|u_j\|_{\overline{m+1}} . \tag{0.1.23}$$

Proof. By Hölder's inequality,

$$\|N(U_m)\|_0 \leq C \prod_{j=1}^m |\nabla^2 u_j|_{2m} , \tag{0.1.24}$$

from which (0.1.23) follows, by (0.1.19) with $h = 1$. □

Remark. We explicitly remark that (0.1.23) does not require the special structure of N, as it is sufficient to use the fact that N is the sum of products of second order derivatives of the functions u_j's. On the other hand, from (0.1.21) and (0.1.19) we deduce that

(0.1.25)
$$|I(u_1, \ldots, u_m, u_{m+1})| \leq C \left(\prod_{j=1}^{m-1} |\nabla^2 u_j|_m \right) |\nabla u_m|_{2m} |\nabla u_{m+1}|_{2m}$$
$$\leq C \prod_{j=1}^{m+1} \|u_j\|_{\overline{m}} \leq C \prod_{j=1}^{m+1} \|u_j\|_m ,$$

and this estimate, which requires less regularity of the u_j's, does use the special structure of N. ◇

Lemma 0.1.2. [*Case* $h = 1$.] *Let* $U_m = (u_1, \ldots, u_m) \in (\bar{H}^{m+2} \cap \bar{H}^{m+1})^m$. *Then* $N(U_m) \in H^1$, *and*

$$\|\nabla N(U_m)\|_0 \leq C \sum_{j=1}^{m} \|u_j\|_{\overline{m+2}} \prod_{\substack{i=1 \\ i \neq j}}^{m} \|u_i\|_{\overline{m+1}} . \tag{0.1.26}$$

Proof. Since $u_1, \ldots, u_m \in \bar{H}^{m+1}$, Lemma 0.1.1 implies that $N(U_m) \in L^2$. When $h = 1$, we can write (0.1.22) as

$$\nabla N(U_m) = \sum_{j=1}^{m} C_j \, N(u_1, \ldots, \nabla u_j, \ldots, u_m) , \tag{0.1.27}$$

and, recalling (0.1.19), we estimate

(0.1.28)
$$|N(u_1, \ldots, \nabla u_j, \ldots, u_m)| \leq C \, |\partial_x^2 u_1|_{2m} \cdots |\partial_x^3 u_j|_{2m} \cdots |\partial_x^2 u_m|_{2m}$$
$$\leq C \, \|u_j\|_{\overline{m+2}} \prod_{\substack{i=1 \\ i \neq j}} \|u_i\|_{\overline{m+1}} .$$

Adding all estimates (0.1.28) for $j = 1, \ldots, m$ yields (0.1.26). □

Lemma 0.1.3. [*Case* $h \geq 2$.] 1) *If* $m > 2$ *and* $h \geq 2$, *or if* $m = 2$ *and* $h > 2$, *let* $U_m = (u_1, \ldots, u_m) \in (\bar{H}^{m+h} \cap \bar{H}^m)^m$. *Then* $N(U_m) \in H^h$, *and*

$$\|\nabla^h N(U_m)\|_0 \leq C \prod_{j=1}^{m} \max \left\{ \|u_j\|_{\overline{m}}, \|u_j\|_{\overline{m+h}} \right\} . \tag{0.1.29}$$

2) *If* $m = h = 2$, *let* $u_1 \in \bar{H}^5 \cap \bar{H}^2$, *and let* $u_2 \in \bar{H}^4 \cap \bar{H}^2$. *Then* $N(u_1, u_2) \in H^2$, *and*

$$\|\nabla^2 N(u_1, u_2)\|_0 \le C \max\{\|u_1\|_{\overline{2}}, \|u_1\|_{\overline{5}}\} \max\{\|u_2\|_{\overline{2}}, \|u_2\|_{\overline{4}}\} \ . \tag{0.1.30}$$

Proof. 1) Since each $u_j \in \bar{H}^{m+h} \cap \bar{H}^m$, $1 \le j \le m$, by interpolation (part (2) of Proposition 0.1.1) it follows that $u_j \in \bar{H}^{m+1}$. Thus, by Lemma 0.1.1, $N(U_m) \in L^2$.

2) We refer to the decomposition (0.1.22). If $q_j \le h - 1$ for all $j = 1, \dots, m$, recalling (0.1.19) and the interpolation inequality (0.1.18), with $\theta_j = \frac{q_j+1}{h}$,

$$\begin{aligned} |N(\nabla^{q_1} u_1, \dots, \nabla^{q_m} u_m)|_2 &\le C \prod_{j=1}^m |\nabla^{q_j+2} u_j|_{2m} \le C \prod_{j=1}^m \|u_j\|_{\overline{m+q_j+1}} \\ &\le C \prod_{j=1}^m \|u_j\|_{\overline{m}}^{1-\theta_j} \|u_j\|_{\overline{m+h}}^{\theta_j} \le C \prod_{j=1}^m \max\{\|u_j\|_{\overline{m}}, \|u_j\|_{\overline{m+h}}\} \ , \end{aligned} \tag{0.1.31}$$

as desired in (0.1.29). If $q_1 = h$, so that $q_j = 0$ for $2 \le j \le m$, we proceed as follows. If $m > 2$, we set $p := \frac{2m(m-1)}{m-2}$, so that $\frac{1}{m} + \frac{m-1}{p} = \frac{1}{2}$ and, by the Gagliardo-Nirenberg inequalities and (0.1.19), with $\lambda_h = \frac{m}{h(m-1)} < 1$, we estimate

$$\begin{aligned} |N(\nabla^h u_1, u_2, \dots, u_m)|_2 &\le C\, |\nabla^{h+2} u_1|_m \prod_{j=2}^m |\partial_x^2 u_j|_p \\ &\le C\, \|u_1\|_{\overline{m+h}} \prod_{j=2}^m |\partial_x^{m+h} u_j|_2^{\lambda_h} \, |\partial_x^2 u_j|_m^{1-\lambda_h} \\ &\le C\, \|u_1\|_{\overline{m+h}} \prod_{j=2}^m \|u_j\|_{\overline{m+h}}^{\lambda_h} \|u_j\|_{\overline{m}}^{1-\lambda_h} \\ &\le C\, \|u_1\|_{\overline{m+h}} \prod_{j=2}^m \max\{\|u_j\|_{\overline{m}}, \|u_j\|_{\overline{m+h}}\} \ , \end{aligned} \tag{0.1.32}$$

again as desired in (0.1.29). An analogous estimate holds if $q_j = h$ for $2 \le j \le m$ and $q_i = 0$ for $i \ne j$; adding all these estimates to (0.1.31) yields (0.1.29). Next, suppose that $m = 2$ and $h > 2$. If $q_1 = h$ and $q_2 = 0$, using once more the Gagliardo-Nirenberg inequalities we obtain

$$
\begin{aligned}
|N(\nabla^h u_1, u_2)|_2 &\leq C\,|\partial_x^{h+2} u_1|_2\,|\partial_x^2 u_2|_\infty \\
&\leq C\,\|u_1\|_{\overline{2+h}}\,|\partial_x^{h+2} u_2|_2^{2/h}\,|\partial_x^2 u_2|_2^{1-2/h} \\
&\leq C\,\|u_1\|_{\overline{2+h}}\,\|u_2\|_{\overline{2+h}}^{2/h}\,\|u_2\|_{\overline{2}}^{1-2/h} \\
&\leq C\,\|u_1\|_{\overline{2+h}}\,\max\left\{\|u_2\|_{\overline{2}}, \|u_2\|_{\overline{2+h}}\right\}\,.
\end{aligned} \tag{0.1.33}
$$

In the same way, if $q_1 = 0$ and $q_2 = h$,

$$
\begin{aligned}
|N(u_1, \nabla^h u_2)|_2 &\leq C\,|\partial_x^2 u_1|_\infty\,|\partial_x^{h+2} u_2|_2 \\
&\leq C\,\max\left\{\|u_1\|_{\overline{2}}, \|u_1\|_{\overline{2+h}}\right\}\,\|u_2\|_{\overline{2+h}}\,.
\end{aligned} \tag{0.1.34}
$$

Finally, if $1 \leq q_1,\ q_2 \leq h-1$, estimate (0.1.31) still holds; namely,

$$
|N(\nabla^{q_1} u_1, \nabla^{q_2} u_2)|_2 \leq C \prod_{j=1}^{2} \max\left\{\|u_j\|_{\overline{2}}, \|u_j\|_{\overline{2+h}}\right\}\,. \tag{0.1.35}
$$

Adding (0.1.33), (0.1.34) and (0.1.35) yields (0.1.29) for $m = 2$, $h > 2$.

3) Let now $m = h = 2$, and refer again to (0.1.22). Then, recalling (0.1.19) and the interpolation inequality (0.1.18),

$$
\begin{aligned}
|N(\nabla^2 u_1, u_2)|_2 &\leq C\,|\partial_x^4 u_1|_4\,|\partial_x^2 u_2|_4 \leq C\,\|u_1\|_{\overline{5}}\,\|u_2\|_{\overline{3}} \\
&\leq C\,\|u_1\|_{\overline{5}}\,\|u_2\|_{\overline{4}}^{1/2}\,\|u_2\|_{\overline{2}}^{1/2} \\
&\leq C\,\|u_1\|_{\overline{5}}\,\max\{\|u_2\|_{\overline{2}}, \|u_2\|_{\overline{4}}\}\,;
\end{aligned} \tag{0.1.36}
$$

in the same way, by (0.1.18),

$$
\begin{aligned}
|N(\nabla u_1, \nabla u_2)|_2 &\leq C\,|\partial_x^3 u_1|_4\,|\partial_x^3 u_2|_4 \leq C\,\|u_1\|_{\overline{4}}\,\|u_2\|_{\overline{4}} \\
&\leq C\,\|u_1\|_{\overline{5}}^{2/3}\,\|u_1\|_{\overline{2}}^{1/3}\,\|u_2\|_{\overline{4}} \\
&\leq C\,\max\{\|u_1\|_{\overline{2}}, \|u_1\|_{\overline{5}}\}\,\|u_2\|_{\overline{4}}\,;
\end{aligned} \tag{0.1.37}
$$

and finally, by the Gagliardo-Nirenberg inequalities,

(0.1.38)

$$
\begin{aligned}
|N(u_1, \nabla^2 u_2)|_2 &\leq C\,|\partial_x^2 u_1|_\infty\,|\partial_x^4 u_2|_2 \leq C\,|\partial_x^5 u_1|_2^{2/3}\,|\partial_x^2 u_1|_2^{1/3}\,|\partial_x^4 u_2|_2 \\
&\leq C\,\|u_1\|_{\overline{5}}^{2/3}\,\|u_1\|_{\overline{2}}^{1/3}\,\|u_2\|_{\overline{4}} \\
&\leq C\,\max\{\|u_1\|_{\overline{2}}, \|u_1\|_{\overline{5}}\}\,\|u_2\|_{\overline{4}}\,.
\end{aligned}
$$

Adding (0.1.36), (0.1.37) and (0.1.38) yields (0.1.30). This concludes the proof of Lemma 0.1.3. □

If $h > m > 2$, the results of Lemma 0.1.3 can be somewhat improved; this is essentially due to the fact that H^h is an algebra if $h > m$.

Lemma 0.1.4. *[Case $h > m > 2$.] Let $U_m = (u_1, \ldots, u_m) \in (\bar{H}^{m+h-1} \cap \bar{H}^m)^m$. Then $N(U_m) \in H^h$, and*

$$\|\nabla^h N(U_m)\|_0 \leq C \prod_{j=1}^{m} \max\left\{\|u_j\|_{\overline{m}}, \|u_j\|_{\overline{m+h-1}}\right\} . \tag{0.1.39}$$

Proof. Since $m+h-1 > m+1$, we first deduce that $N(U_m) \in L^2$, as in part (1) of the proof of Lemma 0.1.3. Then, referring again to the decomposition (0.1.22), we need to consider three different cases; namely:

1) There is j such that $q_j = h$, and $q_i = 0$ for $i \neq j$;

2) There are j and k such that $q_j = h-1$, $q_k = 1$ and $q_i = 0$ for $i \neq j$ and $i \neq k$;

3) For all $j = 1, \ldots, m$, $q_j \leq h-2$.

Case 1. Assume, e.g., that $q_1 = h$ and $q_j = 0$ for $2 \leq j \leq m$. Then, with $\theta_h := \frac{2}{h-1} \in]0,1[$ and $\lambda_h := \frac{m-3}{h-1} \in [0,1[$, by the Gagliardo-Nirenberg inequalities and the interpolation inequality (0.1.18),

$$\begin{aligned} |N(\nabla^h u_1, u_2, \ldots, u_m)|_2 &\leq C\, |\partial_x^{h+2} u_1|_2 \prod_{j=2}^{m} |\partial_x^2 u_j|_\infty \\ &\leq\; C\, \|u_1\|_{\overline{h+2}} \prod_{j=2}^{m} |\partial_x^{m+h-1} u_j|_2^{\theta_h}\, |\partial_x^2 u_j|_m^{1-\theta_h} \\ &\leq\; C\, \|u_1\|_{\overline{m+h-1}}^{1-\lambda_h} \|u_1\|_{\overline{m}}^{\lambda_h} \prod_{j=2}^{m} \|u_j\|_{\overline{m}}^{1-\theta_h} \|u_j\|_{\overline{m+h-1}}^{\theta_h} \\ &\leq\; C \prod_{j=1}^{m} \max\left\{\|u_j\|_{\overline{m}}, \|u_j\|_{\overline{m+h-1}}\right\} , \end{aligned} \tag{0.1.40}$$

as desired in (0.1.39). An analogous estimate clearly holds for the other indices, that is, for $2 \leq j \leq m$.

Case 2. Assume, e.g., that $q_1 = h-1$, $q_2 = 1$, and $q_j = 0$ for $3 \leq j \leq m$. Define p, $r \geq 2$ by $\frac{1}{p} = \frac{m-2}{m(m-1)}$, $\frac{1}{r} = \frac{m-3}{2m(m-1)}$ (if $m = 3$, we take $r = +\infty$), so that $\frac{1}{m}+\frac{1}{p}+\frac{m-2}{r}=\frac{1}{2}$. Then, with $\eta_h := \frac{m+1}{(m-1)(h-1)} \in]0,1[$,

(0.1.41)

$$
\begin{aligned}
|N(\nabla^{h-1}u_1, \nabla u_2, u_3, \ldots, u_m)|_2 &\le C\,|\partial_x^{h+1}u_1|_m\,|\partial_x^3 u_2|_p \prod_{j=3}^m |\partial_x^2 u_j|_r \\
&\le C\,|\partial_x^{m+h-1}u_1|_2 \prod_{j=2}^m |\partial_x^{m+h-1}u_j|_2^{\eta_h}\,|\partial_x^2 u_j|_m^{1-\eta_h} \\
&\le C\,\|u_1\|_{\overline{m+h-1}} \prod_{j=2}^m \|u_j\|_{\overline{m}}^{1-\eta_h}\,\|u_j\|_{\overline{m+h-1}}^{\eta_h} \\
&\le C \prod_{j=1}^m \max\left\{\|u_j\|_{\overline{m}}, \|u_j\|_{\overline{m+h-1}}\right\} ,
\end{aligned}
$$

again as desired in (0.1.39). An analogous estimate clearly holds for the other similar choices of indices.

Case 3. Assume now that $q_j \le 2$ for all indices j. Then, with $\gamma_{jh} := \frac{q_j+1}{h-1} \in]0,1[$,

(0.1.42)

$$
\begin{aligned}
|N(\nabla^{q_1}u_1, \ldots, \nabla^{q_m}u_m)|_2 &\le C \prod_{j=1}^m |\partial_x^{q_j+2}u_j|_{2m} \\
&\le C \prod_{j=1}^m |\partial_x^{m+q_j+1}u_j|_2 \le C \prod_{j=1}^m |\partial_x^{m+h-1}u_j|_2^{\gamma_{jh}}\,|\partial_x^2 u_j|_m^{1-\gamma_{jh}} \\
&\le C \prod_{j=1}^m \|u_j\|_{\overline{m}}^{1-\gamma_{jh}}\,\|u_j\|_{\overline{m+h-1}}^{\gamma_{jh}} \le C \prod_{j=1}^m \max\left\{\|u_j\|_{\overline{m}}, \|u_j\|_{\overline{m+h-1}}\right\} ,
\end{aligned}
$$

again as desired in (0.1.39). Adding all estimates (0.1.40), (0.1.41), and (0.1.42) yields (0.1.39). This ends the proof of Lemma 0.1.4. □

5: Elliptic Type Estimates on f. We now turn to equation (0.1.7), which defines $f(u)$ in both problems (H) and (P), and show that estimate (0.1.29) can be somewhat improved if u_1 is replaced by $f(u_1)$.

Lemma 0.1.5. *Let $u \in \bar{H}^m$. There exists a unique $f \in \bar{H}^m$, which is a weak solution of* (0.1.7), *in the sense that for all $\varphi \in \bar{H}^m$,*

(0.1.43)
$$\langle f, \varphi\rangle_{\overline{m}} = -\langle M(u), \varphi\rangle_0 .$$

The function f satisfies the estimate

$$\|f\|_{\overline{m}} \leq C\,\|u\|_{\overline{m}}^{m}\,, \tag{0.1.44}$$

with C independent of u.

Proof. We first note that if $u \in \bar{H}^m$, then $\partial_x^2 u \in \bar{H}^{m-2} \hookrightarrow L^m$; thus, by (0.1.19) and Hölder's inequality, $M(u)$ is at least in L^1, as we see from the estimate

$$|M(u)|_1 \leq C\,|\partial_x^2 u|_m^m \leq C\,\|u\|_{\overline{m}}^{m}\,. \tag{0.1.45}$$

Consequently, the right side of (0.1.43) makes sense if $\varphi \in L^\infty$. Note that since the space dimension is $N = 2\,m$, the imbedding $H^m \hookrightarrow L^\infty$ does not hold; hence, neither does the imbedding $\bar{H}^m \hookrightarrow L^\infty$. On the other hand, by (0.1.25), if $\varphi \in C_0^\infty(\mathbb{R}^{2m}) \hookrightarrow L^\infty \cap \bar{H}^m$ the estimate

$$|\langle M(u), \varphi\rangle| = |I(u, \ldots, u, \varphi)| \leq C\,\|u\|_{\overline{m}}^{m}\,\|\varphi\|_{\overline{m}} \tag{0.1.46}$$

holds, with C independent of u and φ. By the density of $C_0^\infty(\mathbb{R}^{2m})$ in $\bar{H}^m$, (0.1.46) also hold for all $\varphi \in \bar{H}^m$; hence, by Riesz' representation theorem, there is a unique $f \in \bar{H}^m$, solution of (0.1.43). Taking $\varphi = f$ in (0.1.43), which is admissible because $f \in \bar{H}^m$, and using (0.1.46), we obtain (0.1.44). □

We now establish further regularity results for f.

Lemma 0.1.6. 1) *Let $u \in \bar{H}^{m+1} \cap \bar{H}^m$, and let $f \in \bar{H}^m$ be the weak solution of* (0.1.7)*, given by Lemma* 0.1.5*. Then $f \in \bar{H}^{m+h}$ for $0 < h \leq m$, and*

$$\|f\|_{\overline{m+h}} \leq C\,\|u\|_{\overline{m}}^{m-h}\,\|u\|_{\overline{m+1}}^{h}\,. \tag{0.1.47}$$

2) *Let $h > m$, and $u \in \bar{H}^{h+1} \cap \bar{H}^m$. Then $f \in \bar{H}^{m+h}$, and*

$$\|f\|_{\overline{m+h}} \leq C\left(\max\{\|u\|_{\overline{m}}\,, \|u\|_{\overline{h}}\,, \|u\|_{\overline{h+1}}\}\right)^m\,. \tag{0.1.48}$$

3) *If $u \in \bar{H}^{m+h} \cap \bar{H}^m$, $h \geq 0$, then*

$$\|f\|_{\overline{m+h}} \leq C\,\|u\|_{\overline{m}}^{m-1}\,\|u\|_{\overline{m+h}}\,. \tag{0.1.49}$$

In (0.1.47)*,* (0.1.48)*, and* (0.1.49)*, C is independent of u.*

Proof. 1) If $u \in \bar{H}^{m+1}$, Lemma 0.1.1 implies that $\Delta^m f = -\,M(u) \in L^2$; in addition, by (0.1.24),

$$\|f\|_{\overline{2m}} = \|\Delta^m f\|_0 = \|M(u)\|_0 \leq C\,\|u\|_{\overline{m+1}}^{m}\,. \tag{0.1.50}$$

This implies (0.1.47) for $h = m$. If $0 < h < m$ and $u \in \bar{H}^{m+1} \cap \bar{H}^m$, we obtain (0.1.47) from the case $h = m$ and (0.1.44), by means of the interpolation inequality (0.1.18), that is,

$$\|f\|_{\overline{m+h}} \leq C\,\|f\|_{\overline{2m}}^{h/m}\,\|f\|_{\overline{m}}^{1-h/m}\,. \tag{0.1.51}$$

2) If $h > m$ and $u \in \bar{H}^{h+1} \cap \bar{H}^m$, we write $h = m + r$, $r > 0$. Then $u \in \bar{H}^{m+r+1} \cap \bar{H}^m$, which implies, by interpolation (part (2) of Proposition 0.1.1), that $u \in \bar{H}^{m+r+1} \cap \bar{H}^{m+r} \cap \bar{H}^{m+1} \cap \bar{H}^m$. Thus, we can apply Lemma 0.1.2 if $r = 1$, or Lemma 0.1.3 if $r > 1$, to deduce that $M(u) \in H^r$. Consequently, $\nabla^r \Delta^m f = -\nabla^r M(u) \in L^2$, which means that $f \in \bar{H}^{2m+r} = \bar{H}^{m+h}$. To estimate $\|f\|_{\overline{m+h}}$, we (formally) multiply (0.1.7) in L^2 by $\Delta^h f$, to obtain

(0.1.52)
$$\begin{aligned} \|f\|^2_{\overline{m+h}} &= \|\nabla^{m+h} f\|_0^2 = \langle \Delta^m f, \Delta^h f \rangle \\ &= -\langle M(u), \Delta^h f \rangle = -\langle \nabla^{h-m} M(u), \nabla^{h+m} f \rangle . \end{aligned}$$

If $r = h - m = 1$, (0.1.52) and (0.1.26) imply that

(0.1.53)
$$\begin{aligned} \|f\|_{\overline{m+h}} &= \|f\|_{\overline{2m+1}} \le C \|\nabla M(u)\|_0 \\ &\le C \|u\|_{\overline{m+2}} \|u\|_{\overline{m}}^{m-1} = C \|u\|_{\overline{m}}^{m-1} \|u\|_{\overline{h+1}} , \end{aligned}$$

from which (0.1.48) follows. If $r \ge 2$ and $m > 2$, or if $r > 2$ and $m = 2$, by (0.1.52) and (0.1.29),

(0.1.54)
$$\begin{aligned} \|f\|_{\overline{m+h}} \le C \|\nabla^r M(u)\|_0 &\le C \left(\max\{\|u\|_{\overline{m}}, \|u\|_{\overline{m+r}}\}\right)^m \\ &= C \left(\max\{\|u\|_{\overline{m}}, \|u\|_{\overline{h}}\}\right)^m , \end{aligned}$$

which again yields (0.1.48). Finally, if $r = m = 2$, so that $h = 4$, we use (0.1.30) to obtain

(0.1.55)
$$\begin{aligned} \|f\|_{\overline{m+h}} &= \|f\|_{\overline{5}} \\ &\le C \left(\max\{\|u\|_{\overline{2}}, \|u\|_{\overline{5}}\}\right) \left(\max\{\|u\|_{\overline{2}}, \|u\|_{\overline{4}}\}\right) \\ &\le C \left(\max\{\|u\|_{\overline{2}}, \|u\|_{\overline{4}}, \|u\|_{\overline{5}}\}\right)^2 \\ &= C \left(\max\{\|u\|_{\overline{m}}, \|u\|_{\overline{h}}, \|u\|_{\overline{h+1}}\}\right)^2 , \end{aligned}$$

which is (0.1.48).

3) If $h = 0$, (0.1.49) coincides with (0.1.44). If $0 < h \le m$ and $u \in \bar{H}^{m+h} \cap \bar{H}^m$, (0.1.49) follows from (0.1.50), (0.1.44) and the interpolation inequality

(0.1.56)
$$\|u\|_{\overline{m+1}} \le C \|u\|_{\overline{m}}^{1-1/h} \|u\|_{\overline{m+h}}^{1/h} .$$

If $h \geq m$, we use interpolation to modify estimate (0.1.31) (with $u_j = u$ for all j) into

(0.1.57)
$$\begin{aligned} \|\nabla^{h-m} M(u)\|_0 &\leq C \sum_{|q|=h-m} \prod_{j=1}^{m} \|u\|_{\overline{m+1+q_j}} \\ &\leq C \sum_{|q|=h-m} \prod_{j=1}^{m} \|u\|_{\overline{m}}^{1-\theta_j} \, \|u\|_{\overline{m+h}}^{\theta_j} \,, \end{aligned}$$

where $\theta_j = \frac{q_j+1}{h} \in]0,1[$. Thus, noting that $\sum_{j=1}^{m} \theta_j = 1$,

(0.1.58)
$$\|\nabla^{h-m} M(u)\|_0 \leq C \sum_{|q|=h-m} \|u\|_{\overline{m}}^{\sum(1-\theta_j)} \, \|u\|_{\overline{m+h}}^{\sum \theta_j} = C \, \|u\|_{\overline{m}}^{m-1} \, \|u\|_{\overline{m+h}} \,,$$

as claimed in (0.1.49). □

Using (0.1.19), we deduce the following consequence of part (1) of the proof of Lemma 0.1.6.

Corollary 0.1.1. *Let $u \in \bar{H}^m$, and let $f = f(u)$ be the corresponding solution of (0.1.7). Then $\partial_x^2 f \in L^m$. If $u \in \bar{H}^{m+1} \cap \bar{H}^m$, then $\partial_x^2 f \in L^{2m}$. In addition, the estimates*

(0.1.59)
$$|\partial_x^2 f|_m \leq C \, \|f\|_{\overline{m}} \,, \qquad |\partial_x^2 f|_{2m} \leq C \, \|f\|_{\overline{m+1}}$$

hold, respectively, with C independent of u and f.

Proof. If $u \in \bar{H}^m$, the claim follows by Lemma 0.1.5, which implies that $f \in \bar{H}^m$, so that $\partial_x^2 f \in \bar{H}^{m-2} \hookrightarrow L^m$, and the first of (0.1.59) holds. Likewise, if $u \in \bar{H}^{m+1} \cap \bar{H}^m$, part (1) of Lemma 0.1.6, with $h = 1$, implies that $f \in \bar{H}^{m+1}$, and the second of (0.1.59) is a consequence of (0.1.19). □

0.2. The Hyperbolic System

0.2.1. Local Existence.

In this section we give a local existence result for problem (H). For h and $k \in \mathbb{N}$ with $k \leq h$, and given $T > 0$, we consider the space

(0.2.1)
$$\mathcal{H}^{h,k}(T) := C([0,T]; H^h) \cap C^1([0,T]; H^k) \,,$$

endowed with its natural norm

(0.2.2)
$$\|u\|_{\mathcal{H}^{h,k}(T)}^2 := \max_{0 \leq t \leq T} \left(\|u_t(t)\|_k^2 + \|u(t)\|_h^2 \right) \,.$$

We claim:

Theorem 0.2.1. *Assume that* $u_0 \in H^{2m}$, $u_1 \in H^m$, *and* $\varphi \in C([0,T];H^{2m})$ *if* $m > 2$, $\varphi \in C([0,T];H^{4+\varepsilon})$, $\varepsilon > 0$, *if* $m = 2$. *There exist* $\tau_* \in]0,T]$ *and a unique* $u \in \mathcal{H}^{2m,m}(\tau_*)$, *with* $f(u) \in C([0,\tau_*];\bar{H}^{2m})$, *solution of problem* (H). *This solution depends continuously on the data* $\{u_0, u_1, \varphi\}$.

Proof. We loosely follow Cherrier and Milani [**5**], and proceed along the lines of the linearization and fixed point method described in section 3.2 of Chapter 3. For simplicity of exposition, whenever we write $\varphi \in C([0,T];H^{2m})$, we tacitly understand that, if $m = 2$, we mean to write $\varphi \in C([0,T];H^{4+\varepsilon})$, $\varepsilon > 0$. We proceed in five steps: Linearization, Contractivity, Picard's Iterations, Continuity, and Well-Posedness.

1: Linearization. Given $\tau \in]0,T]$ and $R > 0$, we define

(0.2.3)
$$B_m(\tau,R) := \{u \in \mathcal{H}^{2m,m}(\tau) \ \mid \ \|u\|_{\mathcal{H}^{2m,m}(\tau)} \le R\,,\ u(0) = u_0\,,\ u_t(0) = u_1\}\,.$$

As we remarked after Proposition 3.2.1, for any $R > \left(\|u_1\|_m^2 + \|u_0\|_{2m}^2\right)^{1/2}$ there exists $\tau = \tau(R) \in]0,T]$ such that the ball $B_m(\tau,R)$ is not empty. Choose such a pair (R,τ). For fixed $w \in B_m(\tau,R)$, we consider the linearized equation

$$u_{tt} + \Delta^m u = N(f(w), w^{(m-2)}, u) + N(\varphi^{(m-1)}, u)\,, \tag{0.2.4}$$

with initial data (0.1.9). We first show that the function $t \mapsto M(w(t))$ is continuous from $[0,\tau]$ into L^2. For t and $t_0 \in [0,\tau]$, we decompose

(0.2.5)
$$\begin{aligned} M(w(t)) - M(w(t_0)) &= \sum_{k=1}^{m} N(w(t) - w(t_0), (w(t))^{(m-k)}, (w(t_0))^{(k-1)}) \\ &=: \sum_{k=1}^{m} \Psi_k(t,t_0)\,, \end{aligned}$$

and, recalling (0.1.23), we estimate

$$\begin{aligned} |\Psi_k(t,t_0)|_2 &\le C\,\|w(t) - w(t_0)\|_{m+1}\,\|w(t)\|_{m+1}^{m-k}\,\|w(t_0)\|_{m+1}^{k-1} \\ &\le C\,R^{m-1}\,\|w(t) - w(t_0)\|_{m+1}\,. \end{aligned} \tag{0.2.6}$$

Since $w \in C([0,T];H^{2m}) \hookrightarrow C([0,T];H^{m+1})$, (0.2.6) yields the asserted continuity. By Lemma 0.1.6, $f(t) := f(w(t)) \in \bar{H}^{2m}$ for all $t \in [0,\tau]$, and $\partial_x^2 f(t) \in L^{2m}$, because $w(t) \in H^{2m}$. In fact, by (0.1.47), for $0 \le h \le m$,

$$\|f(t)\|_{\overline{m+h}} \le C\,\|w(t)\|_{m+1}^m \le C\,R^m\,, \tag{0.2.7}$$

and (0.2.6) implies that $f \in C([0,\tau];\bar{H}^{2m})$, because the function $t \mapsto \Delta^m f(t) = -M(w(t))$ is continuous from $[0,\tau]$ into L^2. From this observation it follows that for all $t \in [0,\tau]$, the map $u \mapsto N(f(t),(w(t))^{(m-2)},u)$ is well defined and continuous from H^{2m} into L^2. In fact, by (0.1.23) and (0.1.47) with $h=1$, omitting the variable t,

$$
\begin{aligned}
|N(f,w^{(m-2)},u)|_2 &\leq C\,\|f\|_{\overline{m+1}}\,\|w\|_{m+1}^{m-2}\,\|u\|_{m+1} \\
&\leq C\,\|w\|_m^{m-1}\,\|w\|_{m+1}^{m-1}\,\|u\|_{m+1} \qquad (0.2.8)\\
&\leq C\,R^{2(m-1)}\,\|u\|_{2m}\,.
\end{aligned}
$$

Likewise, the map $u \mapsto N(\varphi^{(m-1)},u)$ is well defined and continuous from H^{2m} into L^2, with

$$
|N(\varphi^{(m-1)},u)|_2 \leq C\,\|\varphi\|_{2m}^{m-1}\,\|u\|_{2m}\,. \qquad (0.2.9)
$$

As a consequence, for each $w \in B_m(\tau,R)$ the existence and uniqueness of a solution $u \in \mathcal{H}^{2m,m}(\tau)$ to problem (0.2.4)+(0.1.9), with $f \in C([0,\tau];\bar{H}^{2m})$, can be established by methods analogous to those of Chapter 2. As in section 3.3.1, this allows us to define the map $w \mapsto u =: \Gamma(w)$ from $B_m(\tau,R)$ into $\mathcal{H}^{2m,m}(\tau)$.

Proposition 0.2.1. *There exist $\tau_0 \in]0,T]$ and $R_* > 0$ such that for all $\tau \in]0,\tau_0]$, Γ maps the ball $B_m(\tau,R_*)$ into itself.*

Proof. 1) Multiplying (formally) equation (0.2.4) in L^2 by $2\,\Delta^m u_t$ and integrating by parts, we obtain, abbreviating again $f := f(w)$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|_{\overline{m}}^2 + \|u\|_{\overline{2m}}^2\right) \\
&\quad = 2\,\langle N(f,w^{(m-2)},u) + N(\varphi^{(m-1)},u),\ \Delta^m u_t\rangle \qquad (0.2.10)\\
&\quad = 2\,\langle \underbrace{\nabla^m N(f,w^{(m-2)},u)}_{=:\,A_m} + \underbrace{\nabla^m N(\varphi^{(m-1)},u)}_{=:\,B_m}, \nabla^m u_t\rangle\,.
\end{aligned}
$$

We estimate A_m and B_m by means of Lemma 0.1.3. If $m > 2$, we use (0.1.29), with $h = m > 2$, as well as (0.2.7) with $h = 0$ and $h = m$, to obtain

$$
\begin{aligned}
|A_m|_2 &\leq \|\nabla^m N(f,w^{(m-2)},u)\|_0 \\
&\leq C\,\max\{\|f\|_{\overline{m}}, \|f\|_{\overline{2m}}\}\,\|w\|_{2m}^{m-2}\,\|u\|_{2m} \qquad (0.2.11)\\
&\leq C\,R^{2(m-1)}\,\|u\|_{2m}\,.
\end{aligned}
$$

Similarly,

$$
|B_m|_2 \leq C\,\|\varphi\|_{2m}^{m-1}\,\|u\|_{2m}\,. \qquad (0.2.12)
$$

If $m = 2$, by (0.1.30), (0.2.7) with $h = 2$, and (0.1.48) with $h = 3$,

$$\begin{aligned} |A_2|_2 &\leq \|\nabla^2 N(f,u)\|_0 \\ &\leq C\,(\max\{\|f\|_{\overline{4}}\,,\|f\|_{\overline{5}}\})\,(\max\{\|u\|_{\overline{2}}\,,\|u\|_{\overline{4}}\}) \qquad (0.2.13) \\ &\leq C\,R^2\,\|u\|_4\,, \end{aligned}$$

in accord with (0.2.11). To estimate B_2, we make use of the imbedding $H^{2+\varepsilon} \cdot H^2 \hookrightarrow H^2$ and estimate

$$|B_2|_2 \leq \|N(\varphi,u)\|_2 \leq C\,\|\partial_x^2\varphi\|_{2+\varepsilon}\,\|\partial_x^2 u\|_2 \leq C\,\|\varphi\|_{4+\varepsilon}\,\|u\|_4\,, \qquad (0.2.14)$$

in accord with (0.2.12).

2) Inserting (0.2.11) and (0.2.12) into (0.2.10), we obtain

(0.2.15)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2_{\overline{m}} + \|u\|^2_{\overline{2m}}\right) \leq C\,\left(R^{2(m-1)} + \|\varphi\|^{m-1}_{2m}\right)\left(\|u_t\|^2_m + \|u\|^2_{2m}\right)\,.$$

Multiplying (0.2.4) in L^2 by $2\,u_t$, and recalling (0.2.8) and (0.2.9),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2_0 + \|u\|^2_{\overline{m}}\right) &= 2\,\langle N(f,w^{(m-2)},u) + N(\varphi^{(m-1)},u),u_t\rangle \qquad (0.2.16) \\ &\leq C\,\left(R^{2(m-1)} + \|\varphi\|^{m-1}_{2m}\right)\left(\|u_t\|^2_0 + \|u\|^2_{2m}\right)\,; \end{aligned}$$

Finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|^2_0 = 2\,\langle u,u_t\rangle \leq \|u\|^2_0 + \|u_t\|^2_0\,. \qquad (0.2.17)$$

Adding (0.2.17) and (0.2.16) to (0.2.15), recalling (0.1.15), and assuming that $R \geq 1$, we obtain

(0.2.18)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2_m + \|u\|^2_{2m}\right) \leq C\,\left(R^{2(m-1)} + \|\varphi\|^{m-1}_{2m}\right)\left(\|u_t\|^2_m + \|u\|^2_{2m}\right)\,.$$

Integrating (0.2.18) yields, by Gronwall's inequality, that for all $t \in [0,\tau]$,

(0.2.19)
$$\|u_t(t)\|^2_m + \|u(t)\|^2_{2m} \leq \left(\|u_1\|^2_m + \|u_0\|^2_{2m}\right)\exp\left(\left(C\,R^{2(m-1)} + C_\varphi\right)\,t\right)\,,$$

where

$$C_\varphi := \begin{cases} C\,\max\limits_{0\leq t\leq T}\|\varphi(t)\|^{m-1}_{2m} & \text{if} \quad m > 2\,, \\ C\,\max\limits_{0\leq t\leq T}\|\varphi(t)\|_{4+\varepsilon} & \text{if} \quad m = 2\,. \end{cases} \qquad (0.2.20)$$

Thus, if we define (for example) R_* by

$$R_*^2 := \max\left\{1, 4\left(\|u_1\|^2_m + \|u_0\|^2_{2m}\right)\right\}\,, \qquad (0.2.21)$$

and then $\tau_0 \in]0,T]$ by

$$\tau_0 := \min \left\{ \tau(R_*), \frac{\ln 4}{C\, R_*^{2(m-1)} + C_\varphi} \right\} , \tag{0.2.22}$$

we deduce from (0.2.19) that for all $\tau \in]0, \tau_0]$, Γ maps $B_m(\tau, R_*)$ into itself. This ends the proof of Proposition 0.2.1. □

2: Contractivity. We now prove that a further restriction on τ makes the map Γ a contraction with respect to the lower order norm of $\mathcal{H}^{m,0}(\tau)$.

Proposition 0.2.2. *There exists $\tau_* \in]0, \tau_0]$ with the property that for all w, $\tilde{w} \in B_m(\tau_*, R_*)$,*

$$\|\Gamma(w) - \Gamma(\tilde{w})\|_{\mathcal{H}^{m,0}(\tau_*)} \le \tfrac{1}{2} \|w - \tilde{w}\|_{\mathcal{H}^{m,0}(\tau_*)} \, . \tag{0.2.23}$$

Proof. Let $\tau \in]0, \tau_0]$, and w, $\tilde{w} \in B_m(\tau, R_*)$; set $u := \Gamma(w)$, $\tilde{u} := \Gamma(\tilde{w})$, $v := w - \tilde{w}$, $z := u - \tilde{u}$, $f := f(w)$, $\tilde{f} := f(\tilde{w})$, and $g := f - \tilde{f}$. Taking the difference between the equations satisfied by u and $\tilde{u}$, and recalling the symmetry of N, we see that z and g solve the system

$$z_{tt} + \Delta^m z \;=\; F + N(\varphi^{(m-1)}, z) \, , \tag{0.2.24}$$

$$\Delta^m g \;=\; -G \, , \tag{0.2.25}$$

where $F = F(g, z, u, \tilde{u}, w, \tilde{w}, f, \tilde{f})$ and $G = G(v, w, \tilde{w})$ are defined by

$$\begin{aligned} F \;:=\; & N(g, w^{(m-2)}, u) + \sum_{k=2}^{m-1} N(\tilde{f}, v, w^{(m-k-1)}, \tilde{w}^{(k-2)}, u) \\ & + N(\tilde{f}, \tilde{w}^{(m-2)}, z) \;=: \sum_{k=1}^{m} F_k \, , \end{aligned} \tag{0.2.26}$$

and

$$G := \sum_{k=1}^{m} N(v, w^{(m-k)}, \tilde{w}^{(k-1)}) =: \sum_{k=1}^{m} G_k \, . \tag{0.2.27}$$

We multiply (0.2.24) in L^2 by $2\, z_t$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|z_t\|_0^2 + \|z\|_{\overline{m}}^2 \right) = 2 \, \langle F + N(\varphi^{m-1}, z), z_t \rangle \, . \tag{0.2.28}$$

To simplify notations in the estimates that follow, we denote by $\hat{w}$ either one of the functions w or $\tilde{w}$, and by $\hat{w}^k$ a generic product (in N) of k_1 factors w and k_2 factors $\tilde{w}$, with $k_1 + k_2 = k$; we use a similar notation for $\hat{u}$.

1) We first consider the term at the right side of (0.2.26) with F_1. By (0.1.23),

$$\|F_1\|_0 \;\le\; C \, \|g\|_{\overline{m+1}} \, \|w\|_{m+1}^{m-2} \, \|u\|_{m+1} \le C \, R_*^{m-1} \, \|g\|_{\overline{m+1}} \, . \tag{0.2.29}$$

We can estimate g in $\bar{H}^{m+1}$ as in (0.1.47), obtaining

$$\|g\|_{\overline{m+1}} \le C\,\|\hat{w}\|_m^{m-2}\,\|\hat{w}\|_{m+1}\,\|v\|_m \le C\,R_*^{m-1}\,\|v\|_m\,; \tag{0.2.30}$$

more precisely, multiplying (0.2.25) in L^2 by $\Delta\, g$ and recalling (0.2.27), we obtain

$$\|g\|^2_{\overline{m+1}} = -\,\langle G, \Delta g\rangle = -\sum_{k=1}^{m}\langle G_k, \Delta g\rangle\,. \tag{0.2.31}$$

Each of the terms at the right side of (0.2.31) has the form $I(v, \hat{w}^{m-1}, \Delta g)$; thus, recalling (0.1.25),

$$\begin{aligned} |I(v,\hat{w}^{m-1},\Delta g)| &\le C\,|\nabla^2\hat{w}|_m^{m-2}\,|\nabla^2\hat{w}|_{2m}\,|\nabla v|_{2m}\,|\nabla\Delta g|_m \\ &\le C\,\|\hat{w}\|_m^{m-2}\,\|\hat{w}\|_{m+1}\,\|v\|_m\,|\partial_x^{m-2}\nabla\Delta g|_2\,, \end{aligned} \tag{0.2.32}$$

from which (0.2.30) follows, recalling (0.1.16). Replacing (0.2.30) into (0.2.29) we conclude that

$$\|F_1\|_0 \;\le\; C\,R_*^{2(m-1)}\,\|v\|_m\,. \tag{0.2.33}$$

2) Next, for $2 \le k \le m-1$ we see that each term F_k of (0.2.26) has the form $F_k = N(\tilde{f}, v, \hat{w}^{(m-3)}, u)$. To estimate these terms, we proceed as follows. If $m > 2$, so that $H^{2m-2} \hookrightarrow L^p$ for all $p \in [2, +\infty]$, we let $p = \frac{2m(m-2)}{m-3}$ (so that $\frac{1}{2m} + \frac{1}{m} + \frac{m-2}{p} = \frac{1}{2}$; if $m = 3$, we take $p = +\infty$), and, by (0.1.47) with $h = 1$,

$$\begin{aligned} \|F_k\|_0 &\le C\,|\nabla^2\tilde{f}|_{2m}\,|\nabla^2 v|_m\,|\nabla^2\hat{w}|_p^{m-3}\,|\nabla^2 u|_p \\ &\le C\,\|\tilde{f}\|_{\overline{m+1}}\,\|v\|_m\,\|\hat{w}\|_{2m}^{m-3}\,\|u\|_{2m} \le C\,R_*^{2(m-1)}\,\|v\|_m\,. \end{aligned} \tag{0.2.34}$$

Analogously,

$$\begin{aligned} \|F_m\|_0 &\le C\,|\nabla^2\tilde{f}|_{2m}\,|\nabla^2\tilde{w}|_p^{m-2}\,|\nabla^2 z|_m \\ &\le C\,\|\tilde{f}\|_{\overline{m+1}}\,\|\tilde{w}\|_{2m}^{m-2}\,\|z\|_m \le C\,R_*^{2(m-1)}\,\|z\|_m\,. \end{aligned} \tag{0.2.35}$$

If instead $m = 2$, (0.2.26) and (0.2.27) reduce to

$$F = N(g,u) + N(\tilde{f}, z)\,, \qquad\qquad G = N(v, w+\tilde{w})\,; \tag{0.2.36}$$

thus, at first, recalling (0.2.30),

$$\begin{aligned} \|N(g,u)\|_0 &\le C\,|\nabla^2 g|_4\,|\nabla^2 u|_4 \le C\,\|g\|_{\overline{3}}\,\|u\|_3 \\ &\le C\,\|\hat{w}\|_3\,\|v\|_2\,\|u\|_3 \le C\,R_*^2\,\|v\|_2\,, \end{aligned} \tag{0.2.37}$$

in accord with (0.2.33). Next, by (0.1.48) with $h = 3 > m = 2$, and (0.1.44),

$$\begin{aligned} \|N(\tilde f, z)\|_0 &\le C\,|\nabla^2 \tilde f|_\infty\,|\nabla^2 z|_2 \le C\,|\nabla^5 \tilde f|_2^{2/3}\,|\nabla^2 \tilde f|_2^{1/3}\,\|\nabla^2 z\|_0 \\ &\le C\,\|\tilde f\|_5^{2/3}\,\|\tilde f\|_2^{1/3}\,\|z\|_2 \le C\,\|\tilde u\|_4^{4/3}\,\|\tilde u\|_2^{2/3}\,\|z\|_2 \\ &\le C\,R_*^2\,\|z\|_2\,, \end{aligned} \tag{0.2.38}$$

in accord with (0.2.35). In conclusion, from (0.2.33), ..., (0.2.38), we obtain that

$$\begin{aligned} 2\,|\langle F, z_t\rangle| &\le C\,R_*^{2(m-1)}\,(\|v\|_m + \|z\|_m)\,\|z_t\|_0 \\ &\le C\,R_*^{2(m-1)}\left(\|v\|_m^2 + \|z\|_m^2 + \|z_t\|_0^2\right)\,. \end{aligned} \tag{0.2.39}$$

3) We estimate the last term of (0.2.28) in a similar way. If $m > 2$, with $p = \frac{2m(m-1)}{m-2}$ (so that $\frac{m-1}{p} + \frac{1}{m} = \frac{1}{2}$),

$$\begin{aligned} 2\,|\langle N(\varphi^{(m-1)}, z), z_t\rangle| &\le 2\,|N(\varphi^{(m-1)}, z)|_2\,|z_t|_2 \\ &\le 2\,C\,|\partial_x^2\varphi|_p^{m-1}\,|\partial_x^2 z|_m\,|z_t|_2 \\ &\le 2\,C\,\|\partial_x^2\varphi\|_{2m-2}^{m-1}\,\|\partial_x^2 z\|_{m-2}\,\|z_t\|_0 \\ &\le 2\,C\,\|\varphi\|_{2m}^{m-1}\,\|z\|_m\,\|z_t\|_0 \\ &\le C_\varphi\,\left(\|z\|_m^2 + \|z_t\|_0^2\right)\,, \end{aligned} \tag{0.2.40}$$

where C_φ is as in (0.2.20). If instead $m = 2$,

$$\begin{aligned} 2\,|\langle N(\varphi, z), z_t\rangle| &\le 2\,C\,|\partial_x^2\varphi|_\infty\,|\partial_x^2 z|_2\,|z_t|_2 \\ &\le 2\,C\,\|\varphi\|_{4+\varepsilon}\,\|z\|_2\,\|z_t\|_0 \\ &\le C_\varphi\,\left(\|z_t\|_0^2 + \|z\|_2^2\right)\,, \end{aligned} \tag{0.2.41}$$

in accord with (0.2.40). Inserting (0.2.39) and (0.2.40) into (0.2.28), adding the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|z\|_0^2 = 2\,\langle z, z_t\rangle \le \|z\|_0^2 + \|z_t\|_0^2\,, \tag{0.2.42}$$

and recalling that $R_* \ge 1$, we obtain

(0.2.43)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\left(\|z_t\|_0^2 + \|z\|_m^2\right) \le \left(C\,R_*^{2(m-1)} + C_\varphi\right)\left(\|v\|_m^2 + \|z\|_m^2 + \|z_t\|_0^2\right)\,.$$

From this, recalling that $z(0) = u(0) - \tilde{u}(0) = u_0 - u_0 = 0$ and, similarly, $z_t(0) = 0$, by Gronwall's inequality we obtain that for all $t \in [0, \tau]$,

(0.2.44)
$$\|z_t(t)\|_0^2 + \|z(t)\|_m^2 \leq \gamma(R_*, \varphi) \left(\max_{0 \leq t \leq \tau} \|v(t)\|_m^2 \right) t \, \exp(\gamma(R_*, \varphi)\, t) \,,$$

where $\gamma(R_*, \varphi) := C\, R_*^{2(m-1)} + C_\varphi$. Thus, choosing $\tau_* \in]0, \tau_0]$ such that

$$\gamma(R_*, \varphi)\, \tau_* \, \exp(\gamma(R_*, \varphi)\, \tau_*) \leq \tfrac{1}{4} \,, \tag{0.2.45}$$

we deduce from (0.2.44) that for all $t \in [0, \tau_*]$,

$$\|u_t(t) - \tilde{u}_t(t)\|_0^2 + \|u(t) - \tilde{u}(t)\|_m^2 \leq \tfrac{1}{4} \max_{0 \leq t \leq \tau} \|v(t)\|_m^2 \,. \tag{0.2.46}$$

Recalling that $u = \Gamma(v)$ and $\tilde{u} = \Gamma(\tilde{u})$, and that $v = w - \tilde{w}$, we see that (0.2.46) implies (0.2.23). Thus, Γ is a contraction on $B_m(\tau_*, R_*)$, with respect to the weaker norm of $\mathcal{H}^{m,0}(\tau_*)$, as claimed. □

3: Picard's Iterations. We now consider the Picard's iterations of Γ, that is, the sequence $(u^n)_{n \geq 0}$ defined recursively by $u^{n+1} = \Gamma(u^n)$, starting from an arbitrary $u^0 \in B_m(\tau_*, R_*)$. Explicitly, the functions u^{n+1} are defined in terms of u^n, by the equation

$$u_{tt}^{n+1} + \Delta^m u^{n+1} = N(f^n, (u^n)^{(m-2)}, u^{n+1}) + N(\varphi^{(m-1)}, u^{n+1}) \,, \tag{0.2.47}$$

where $f^n := f(u^n)$, and by the initial conditions

$$u^{n+1}(0) = u_0 \,, \qquad u_t^{n+1}(0) = u_1 \,. \tag{0.2.48}$$

1) By Proposition 0.2.1 the sequence $(u^n)_{n \geq 0}$ is bounded in $\mathcal{H}^{2m,m}(\tau_*)$, because $u^n \in B_m(\tau_*, R_*)$ for all $n \geq 0$. Thus, there is a subsequence, still denoted $(u^n)_{n \geq 0}$, and a function $u \in L^\infty(0, \tau_*; H^{2m})$, with $u_t \in L^\infty(0, \tau_*; H^m)$, such that

$$u^n \to u \quad \text{in} \quad L^\infty(0, \tau_*; H^{2m}) \quad \text{weak}^* \,, \tag{0.2.49}$$
$$u_t^n \to u_t \quad \text{in} \quad L^\infty(0, \tau_*; H^m) \quad \text{weak}^* \,. \tag{0.2.50}$$

By Proposition 0.2.2, $u^n \to u$ in $\mathcal{H}^{m,0}(\tau_*)$; that is,

$$u^n \to u \quad \text{in} \quad C([0, \tau_*]; H^m) \,, \tag{0.2.51}$$
$$u_t^n \to u_t \quad \text{in} \quad C([0, \tau_*]; L^2) \,. \tag{0.2.52}$$

By Proposition 1.7.1, (0.2.49) and (0.2.51) imply that $u \in C_{\mathrm{w}}([0, \tau_*]; H^{2m})$, and (0.2.50) and (0.2.52) imply that $u_t \in C_{\mathrm{w}}([0, \tau_*]; H^m)$; moreover, the maps $t \mapsto \|u(t)\|_{2m}$ and $t \mapsto \|u_t(t)\|_m$ are bounded on $[0, \tau_*]$. We set

$$R_0^2 := \sup_{0 \leq t \leq \tau_*} \left(\|u_t(t)\|_m^2 + \|u(t)\|_{2m}^2 \right) \,. \tag{0.2.53}$$

From (0.2.49) and (0.2.51) we deduce that for $0 \le r \le m-1$,

$$u^n \to u \qquad \text{in} \quad C([0,\tau_*]; H^{m+r}) \ : \tag{0.2.54}$$

indeed, by interpolation, for all $t \in [0,\tau_*]$,

$$\begin{aligned} \|u^n(t) - u(t)\|_{m+r} &\le C\, \|u^n(t) - u(t)\|_{2m}^{r/m}\, \|u^n(t) - u(t)\|_m^{1-r/m} \\ &\le C\,(R_* + R_0)^{r/m}\, \|u^n(t) - u(t)\|_m^{1-r/m} , \end{aligned} \tag{0.2.55}$$

so that (0.2.54) follows from (0.2.51). In the same way, (0.2.50) and (0.2.52) imply that for $0 \le r \le m-1$,

$$u_t^n \to u_t \qquad \text{in} \quad C([0,\tau_*]; H^r) \,. \tag{0.2.56}$$

2) As from (0.2.5) and (0.2.6), with w replaced by u, we know that $M(u) \in C([0,\tau_*]; L^2)$; we now show that

$$M(u^n) \to M(u) \qquad \text{in} \quad C([0,\tau_*]; L^2) \,. \tag{0.2.57}$$

Indeed, setting $z^n := u^n - u$ we decompose, as in (0.2.5),

$$M(u^n) - M(u) = \sum_{k=1}^{m} N(z^n, (u^n)^{(m-k)}, u^{(k-1)}) \,; \tag{0.2.58}$$

thus, acting as in (0.2.6),

$$\begin{aligned} &\|M(u^n) - M(u)\|_{C([0,\tau_*];L^2)} \\ &\quad \le \ C \sum_{k=1}^{m} R_*^{m-k}\, R_0^{k-1}\, \|u^n - u\|_{C([0,\tau_*];H^{m+1})} , \end{aligned} \tag{0.2.59}$$

and (0.2.57) follows from (0.2.54) with $r = 1$. Let $f := f(u)$. Recalling that $f^n = f(u^n)$, (0.2.57) implies that

$$f^n \to f \qquad \text{in} \quad C([0,\tau_*]; \bar{H}^{2m}) \,. \tag{0.2.60}$$

3) We proceed to show that

$$N(f^n, (u^n)^{(m-2)}, u^{n+1}) \to N(f, u^{(m-1)}) \qquad \text{in} \quad C([0,\tau_*]; L^2) \,. \tag{0.2.61}$$

To this end, if $m > 2$ we compute that

$$\begin{aligned} &N(f^n, (u^n)^{(m-2)}, u^{n+1}) - N(f, u^{(m-1)}) \\ &\quad = \ N(f^n - f, (u^n)^{(m-2)}, u^{n+1}) \\ &\quad\quad + \sum_{k=2}^{m-1} N(f, u^{(k-2)}, u^n - u, (u^n)^{(m-k-1)}, u^{n+1}) \\ &\quad\quad + N(f, u^{(m-2)}, u^{n+1} - u) \ =: N_n^{(1)} + N_n^{(2)} + N_n^{(3)} \,. \end{aligned} \tag{0.2.62}$$

By (0.1.23) and interpolation, since u^n and $u^{n+1} \in B_m(\tau_*, R_*)$,

$$\begin{aligned} \|N_n^{(1)}\|_0 &\leq C\,\|f^n - f\|_{\overline{m+1}} \left(\prod_{k=2}^{m-1} \|u^n\|_{m+1} \right) \|u^{n+1}\|_{m+1} \\ &\leq C\,R_*^{m-1}\,\|f^n - f\|_{\overline{m+1}} \\ &\leq C\,R_*^{m-1}\,\|f^n - f\|_{\overline{m}}^{1-1/m}\,\|f^n - f\|_{\overline{2m}}^{1/m}\,. \end{aligned} \tag{0.2.63}$$

By (0.1.44),

$$\|f^n - f\|_{\overline{m}} \leq \|f^n\|_{\overline{m}} + \|f\|_{\overline{m}} \leq C\,R_*^m\,; \tag{0.2.64}$$

consequently, (0.2.63) and (0.2.60) imply that $N_n^{(1)} \to 0$ in $C([0,\tau_*]; L^2)$. Acting likewise, and setting

$$R_f := \max_{0 \leq t \leq \tau_*} \|f(t)\|_{\overline{2m}} \tag{0.2.65}$$

(which makes sense, by (0.2.60)), we see that

$$\|N_n^{(2)}\|_0 \leq C\,R_f \sum_{k=2}^{m-1} R_0^{k-2}\,R_*^{m-k}\,\|u^n - u\|_{m+1}\,, \tag{0.2.66}$$

$$\|N_n^{(3)}\|_0 \leq C\,R_f\,R_0^{m-2}\,\|u^{n+1} - u\|_{m+1}\,; \tag{0.2.67}$$

thus, by (0.2.54) with $r = 1$, also $N_n^{(2)}$ and $N_n^{(3)} \to 0$ in $C([0,\tau_*]; L^2)$, and (0.2.61) follows. If instead $m = 2$,

$$N(f^n, u^{n+1}) - N(f, u) = N(f^n - f, u^{n+1}) + N(f, u^{n+1} - u)\,, \tag{0.2.68}$$

and by (0.1.23), since $f^n \to f$ in $\bar{H}^3$ by (0.2.60) and (0.2.64),

$$|N(f^n - f, u^{n+1})|_2 \leq C\,\|f^n - f\|_{\overline{3}}\,\|u^{n+1}\|_3 \leq C\,R_*\,\|f^n - f\|_{\overline{3}} \to 0\,, \tag{0.2.69}$$

as well as, by (0.2.54) with $r = 1$,

$$|N(f, u^{n+1} - u)|_2 \leq C\,\|f\|_{\overline{3}}\,\|u^{n+1} - u\|_3 \to 0\,. \tag{0.2.70}$$

4) From (0.2.61) and (0.2.49), it also follows that the sequence $(u_{tt}^n)_{n \geq 0}$ is bounded in $L^\infty(0,\tau_*; L^2)$; consequently, we can suppose that

$$u_{tt}^n \to u_{tt} \qquad \text{in } L^\infty(0,\tau_*; L^2) \qquad \text{weak}^*\,. \tag{0.2.71}$$

We can then let $n \to +\infty$ in (0.2.47); more precisely, by (0.2.49) and (0.2.61),

$$\begin{aligned} u_{tt}^{n+1} &= -\Delta^m u^{n+1} + N(f^n, (u^n)^{(m-2)}, u^{n+1}) + N(\varphi^{(m-1)}, u^{n+1}) \\ &\to -\Delta^m u + N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u) \end{aligned} \tag{0.2.72}$$

in $L^\infty(0,\tau_*;L^2)$ weak*. Comparing (0.2.72) with (0.2.71) we deduce that u satisfies equation (0.1.8) in L^2, at least for a.a. $t \in [0,\tau_*]$. Moreover, u satisfies the initial conditions (0.1.9), because by (0.2.48), (0.2.51) and (0.2.52), $u^n(0)=u_0 \to u(0)$ in H^m and $u^n_t(0)=u_1 \to u_t(0)$ in L^2.

4: Continuity. We now proceed as in the proof of Lemma 2.3.2, to show that $u \in C([0,\tau_*];H^{2m})$ and $u_t \in C([0,\tau_*];H^m)$. Setting

$$E(t) := \|u_t(t)\|_m^2 + \|u(t)\|_{2m}^2 \,, \tag{0.2.73}$$

we deduce from (0.1.8) that for all t, $t_0 \in [0,\tau_*]$,

$$E(t)-E(t_0) = 2\int_{t_0}^t \langle \nabla^m \left(N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u)\right), \nabla^m u_t\rangle \,\mathrm{d}\theta \,. \tag{0.2.74}$$

As in the proof of Proposition 0.2.1 (replacing w with u in the estimate of A_m in (0.2.11)), we see that $\nabla^m \left(N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u)\right) \in L^\infty(0,\tau_*;L^2)$; since also $\nabla^m u_t \in L^\infty(0,\tau_*;L^2)$, the integrand at the right side of (0.2.74) is in $L^1(0,\tau_*)$. Thus, E is continuous on $[0,\tau_*]$. We set further

$$F(t,t_0) := \|u_t(t)-u_t(t_0)\|_m^2 + \|u(t)-u(t_0)\|_{2m}^2 \,. \tag{0.2.75}$$

Recalling that $u_t \in C_{\mathrm{w}}([0,\tau_*];H^m)$ and $u \in C_{\mathrm{w}}([0,\tau_*];H^{2m})$, it follows that as $t \to t_0$,

$$\begin{aligned} F(t,t_0) &= E(t)+E(t_0) - 2\,\langle u_t(t),u_t(t_0)\rangle_m - 2\,\langle u(t),u(t_0)\rangle_{2m} \\ &\to E(t_0)+E(t_0) - 2\,\|u_t(t_0)\|_m^2 - 2\,\|u(t_0)\|_{2m}^2 \\ &= 0\,. \end{aligned} \tag{0.2.76}$$

Thus,

$$0 \;\le\; \|u_t(t)-u_t(t_0)\|_m^2 \le F(t,t_0) \;\to 0\,, \tag{0.2.77}$$

$$0 \;\le\; \|u(t)-u(t_0)\|_m^2 \;\le F(t,t_0) \;\to 0\,, \tag{0.2.78}$$

from which it follows that, indeed, $u \in \mathcal{H}^{2m,m}(\tau_*)$.

5: Well-Posedness. We conclude the proof of Theorem 0.2.1 by showing that the solutions of problem (H) depend continuously on their data.

Proposition 0.2.3. *Let* u_0, $\tilde u_0 \in H^{2m}$, *let* u_1, $\tilde u_1 \in H^m$, *and let* φ, $\tilde\varphi \in C([0,T];H^{2m})$ *(if* $m=2$, *let* φ, $\tilde\varphi \in C([0,T];H^{4+\varepsilon})$*). Assume that problem* (H) *has corresponding solutions* u, $\tilde u \in \mathcal{H}^{2m,m}(\tau)$, *for some* $\tau \in\,]0,T]$. *Then,* *the difference* $u - \tilde{u}$ *satisfies the estimate*

(0.2.79)

$$\|u-\tilde u\|_{\mathcal{H}^{2m,m}(\tau)} \le h(\rho,\delta,T)\Big(\|u_0-\tilde u_0\|_{2m} + \|u_1-\tilde u_1\|_m + \|\varphi-\tilde\varphi\|_{C([0,T];H^{2m})}\Big)\,,$$

where $h \in \mathcal{K}$ *and*

$$\rho := \max\left\{1, \|u\|_{\mathcal{H}^{2m,m}(\tau)}, \|\tilde u\|_{\mathcal{H}^{2m,m}(\tau)}\right\}\,, \tag{0.2.80}$$

$$\delta := \max\left\{\|\varphi\|_{C([0,T];H^{2m})}, \|\tilde\varphi\|_{C([0,T];H^{2m})}\right\}\,. \tag{0.2.81}$$

In particular, there is at most one solution of problem (H) *in* $\mathcal{H}^{2m,m}(\tau)$.

Sketch of Proof. 1) We consider first the case $m > 2$. With notations analogous to those of the proof of Proposition 0.2.2, let $z := u-\tilde u$, $f := f(u)$, $\tilde f := f(\tilde u)$, $g := f - \tilde f$, $\psi := \varphi - \tilde\varphi$. By the symmetry of N, z and g solve the system

$$z_{tt} + \Delta^m z = F + \Phi\,, \tag{0.2.82}$$

$$\Delta^m g = -G\,, \tag{0.2.83}$$

where now (compare to (0.2.24) and (0.2.25))

$$F := N(g, u^{(m-1)}) + \sum_{k=2}^{m} N(\tilde f, z, u^{(m-k)}, \tilde u^{(k-2)}) =: \sum_{k=1}^{m} F_k\,, \tag{0.2.84}$$

$$\Phi := N(z, \varphi^{(m-1)}) + \sum_{k=2}^{m} N(\tilde u, \psi, \varphi^{(m-k)}, \tilde\varphi^{(k-2)})\,, \tag{0.2.85}$$

$$G := \sum_{k=1}^{m} N(z, u^{(m-k)}, \tilde u^{(k-1)})\,. \tag{0.2.86}$$

We multiply (0.2.82) in L^2 by $2\,\Delta^m z_t$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z_t\|^2_{\overline{m}} + \|z\|^2_{\overline{2m}}\right) = 2\,\langle \nabla^m(F+\Phi), \nabla^m z_t\rangle\,. \tag{0.2.87}$$

By (0.1.29) with $h = m$,

$$\|F_1\|_{\overline{m}} \le C\,\max\{\|g\|_{\overline{m}}, \|g\|_{\overline{2m}}\}\,\|u\|^{m-1}_{2m}\,. \tag{0.2.88}$$

From (0.2.83) and (0.2.86), as in (0.1.44), and recalling our notation for $\hat u$,

$$\|g\|_{\overline{m}} \le C\,\|z\|_m\,\|\hat u\|^{m-1}_m \le C\,\rho^{m-1}\,\|z\|_m\,; \tag{0.2.89}$$

and as in (0.1.50), by (0.1.23),

$$
\begin{aligned}
\|g\|_{\overline{2m}} &\le C\,\|N(z,\hat u^{m-1})\|_0 \\
&\le C\,\|z\|_{m+1}\,\|\hat u\|_{m+1}^{m-1} \le C\,\rho^{m-1}\,\|z\|_{m+1}\,.
\end{aligned} \tag{0.2.90}
$$

Thus, from (0.2.88),

$$
\|F_1\|_{\overline{m}} \le C\,\rho^{2(m-1)}\,\|z\|_{2m}\,. \tag{0.2.91}
$$

Likewise, for $k \ge 2$, by (0.1.44) and (0.1.50), with f and w replaced by $\tilde f$ and $\tilde u$,

$$
\begin{aligned}
\|F_k\|_{\overline{m}} &\le C\,\max\left\{\|\tilde f\|_{\overline{m}},\|\tilde f\|_{\overline{2m}}\right\}\,\|z\|_{2m}\,\|\hat u\|_{2m}^{m-2} \\
&\le C\,\rho^m\,\|z\|_{2m}\,\rho^{m-2} \le C\,\rho^{2(m-1)}\,\|z\|_{2m}\,.
\end{aligned} \tag{0.2.92}
$$

In conclusion, from (0.2.91) and (0.2.92) we deduce that

$$
\|F\|_{\overline{m}} \le C\,\rho^{2(m-1)}\,\|z\|_{2m}\,. \tag{0.2.93}
$$

2) The estimate of Φ is simpler, since $H^{2m-2} \hookrightarrow H^m$ and H^{2m-2} is an algebra if $m > 2$. Recalling (0.2.85), (0.2.80) and (0.2.81),

$$
\begin{aligned}
\|\Phi\|_m &\le \|N(z,\varphi^{(m-1)})\|_{2m-2} + \sum_{k=2}^{m}\|N(\tilde u,\psi,\hat\varphi^{(m-2)})\|_{2m-2} \\
&\le C\,\|\partial_x^2 z\|_{2m-2}\,\|\partial_x^2\varphi\|_{2m-2}^{m-1} \\
&\qquad + C\,\|\partial_x^2\tilde u\|_{2m-2}\,\|\partial_x^2\psi\|_{2m-2}\,\|\partial_x^2\hat\varphi\|_{2m-2}^{m-2} \\
&\le C\,\delta^{m-1}\,\|z\|_{2m} + C\,\rho\,\delta^{m-2}\,\|\psi\|_{2m}\,.
\end{aligned} \tag{0.2.94}
$$

Replacing (0.2.93) and (0.2.94) into (0.2.87) yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z_t\|_{\overline{m}}^2 + \|z\|_{\overline{2m}}^2\right) \le C_1\,\left(\|\psi\|_{2m} + \|z\|_{2m}\right)\,\|z_t\|_m\,, \tag{0.2.95}
$$

where, here and in the sequel, C_1, C_2, … denote suitable constants depending on ρ and δ. We proceed as in the proof of Proposition 0.2.2. Starting from identities analogous to (0.2.28) and (0.2.42), we can obtain as in (0.2.43), that also

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z_t\|_0^2 + \|z\|_m^2\right) \le C_2\,\left(\|\psi\|_m + \|z\|_m\right)\,\|z_t\|_m\,. \tag{0.2.96}
$$

Adding (0.2.95) and (0.2.96) we deduce that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z_t\|_m^2 + \|z\|_{2m}^2\right) &\le C_3\,\left(\|\psi\|_{2m} + \|z\|_{2m}\right)\,\|z_t\|_m \\
&\le C_3\|\psi\|_{2m}^2 + 2\,C_3\,\left(\|z_t\|_m^2 + \|z\|_{2m}^2\right)\,.
\end{aligned} \tag{0.2.97}
$$

Integrating this, we obtain that for all $t \in [0, \tau]$,

$$\begin{aligned} &\|z_t(t)\|_m^2 + \|z(t)\|_{2m}^2 \\ \text{(0.2.98)} \qquad &\leq \left(\|z_t(0)\|_m^2 + \|z(0)\|_{2m}^2 + C_3\, T\, \|\psi\|_{C([0,T];H^{2m})}^2\right) \mathrm{e}^{2\, C_3\, T}\,, \end{aligned}$$

from which (0.2.79) follows, with

$$\text{(0.2.99)} \qquad h(\rho, \delta, T) := \max\{1, \sqrt{C_3(\rho, \delta)\, T}\}\, \mathrm{e}^{C_3(\rho,\delta)\, T}\,,$$

if $m > 2$. If $m = 2$, we rewrite

$$\text{(0.2.100)} \qquad F = N(g, u) + N(\tilde{f}, z) =: F_1 + F_2\,,$$

$$\text{(0.2.101)} \qquad \Phi = N(\varphi, z) + N(\psi, \tilde{u}) =: \Phi_1 + \Phi_2\,.$$

By (0.1.30),

$$\text{(0.2.102)} \qquad \|F_1\|_{\overline{2}} \leq C\, \max\{\|g\|_{\overline{2}}, \|g\|_{\overline{5}}\}\, \|u\|_4\,.$$

When $m = 2$, (0.2.83) and (0.2.86) yield that

$$\text{(0.2.103)} \qquad \Delta^2 g = -\,N(z, u + \tilde{u})\,;$$

thus, by (0.1.26),

$$\begin{aligned} \|g\|_{\overline{5}} &\leq C\, \|\nabla N(z, u + \tilde{u})\|_0 \\ \text{(0.2.104)} \qquad &\leq C\, (\|z\|_4\, \|u + \tilde{u}\|_3 + \|z\|_3\, \|u + \tilde{u}\|_4) \\ &\leq C\, \|\hat{u}\|_4\, \|z\|_4\,. \end{aligned}$$

Recalling also (0.2.89), it follows from (0.2.102) and (0.2.104) that

$$\text{(0.2.105)} \qquad \|F_1\|_{\overline{2}} \leq C\, \rho^2\, \|z\|_4\,.$$

Similarly,

$$\text{(0.2.106)} \qquad \|F_2\|_{\overline{2}} \leq C\, \max\left\{\|\tilde{f}\|_{\overline{2}}, \|\tilde{f}\|_{\overline{5}}\right\}\, \|z\|_4 \leq C\, \rho^2\, \|z\|_4\,.$$

The estimate of Φ is simpler, using the imbedding $H^{2+\varepsilon} \cdot H^2 \hookrightarrow H^2$. As in (0.2.94),

$$\text{(0.2.107)} \qquad \|\Phi_1\|_{\overline{2}} \leq \|N(\varphi, z)\|_2 \leq C\, \|\partial_x^2 \varphi\|_{2+\varepsilon}\, \|\partial_x^2 z\|_2 \leq C_\varphi\, \|z\|_4\,,$$

$$\text{(0.2.108)} \qquad \|\Phi_2\|_{\overline{2}} \leq \|N(\psi, \tilde{u})\|_2 \leq C\, \|\partial_x^2 \psi\|_{2+\varepsilon}\, \|\partial_x^2 \tilde{u}\|_2 \leq \rho\, \|\psi\|_{4+\varepsilon}\,.$$

Estimates (0.2.105), (0.2.106), (0.2.107), and (0.2.108) allow us to deduce the analogous of (0.2.95) for $m = 2$; we can then proceed in the same way, and obtain (0.2.79) for $m = 2$ as well. This concludes the proof of Proposition 0.2.3 and, therefore, that of Theorem 0.2.1. □

0.2.2. Higher Regularity.

Higher regularity results for problem (H) can be established by a suitable generalization of Theorem 0.2.1.

Theorem 0.2.2. *Let $k \geq 0$, and assume that $u_0 \in H^{2m+k}$, $u_1 \in H^{m+k}$, $\varphi \in C([0,T]; H^{2m+k})$. There is $\tau_k \in]0,T]$, such that problem* (H) *admits a unique solution $u \in \mathcal{H}^{2m+k,m+k}(\tau_k)$.*

The proof of this theorem follows the same arguments of the proof of Theorem 0.2.1, based on a priori estimates similar to the ones we establish in the proof of Proposition 0.2.4 below. Note that Theorem 0.2.1 corresponds to Theorem 0.2.2 when $k = 0$, with $\tau_0 = \tau_*$ (and the additional assumption $\varphi \in C([0,T]; H^{4+\varepsilon})$ if $m = 2$). In this section we show that the regularity result of Theorem 0.2.2 is uniform in k, in the sense that $\inf_{k>0} \tau_k \geq \tau_*$. Roughly speaking, this means that increasing the regularity of the data does not decrease the life-span of the solution. This is a consequence of the following time-independent a priori estimate:

Proposition 0.2.4. *Let $k > 0$, and u_0, u_1, φ satisfy the assumptions of Theorem* 0.2.2. *Assume that problem* (H) *has a corresponding solution $u \in \mathcal{H}^{2m,m}(\tau) \cap \mathcal{H}^{2m+k,m+k}(\tau')$, with $0 < \tau' < \tau \leq T$. There exists Λ_k, depending on τ but not on τ', such that*

$$\sup_{0 \leq t \leq \tau'} \left(\|u_t(t)\|^2_{m+k} + \|u(t)\|^2_{2m+k} \right) \leq \Lambda_k^2 \,. \tag{0.2.109}$$

Consequently, $u \in \mathcal{H}^{2m+k,m+k}(\tau)$.

Sketch of Proof. It is sufficient to estimate the map

$$[0, \tau'] \ni t \ \mapsto \ \|u_t(t)\|^2_{\overline{m+k}} + \|u(t)\|^2_{\overline{2m+k}} \tag{0.2.110}$$

independently of τ'. We proceed by induction on $k \geq 0$, and assume, for simplicity, that $\varphi \equiv 0$. If $k = 0$, we have already remarked that we can take $\tau' = \tau$, and (0.2.109) holds, with

$$\Lambda_0^2 := \sup_{0 \leq t \leq \tau} \left(\|u_t(t)\|^2_m + \|u(t)\|^2_{2m} \right) \,. \tag{0.2.111}$$

Thus, we can assume that $u_0 \in H^{2m+k+1}$, $u_1 \in H^{m+k+1}$, and that, correspondingly, problem (H) has a solution

$$u \in \mathcal{H}^{2m+k,m+k}(\tau) \cap \mathcal{H}^{2m+k+1,m+k+1}(\tau') \,, \tag{0.2.112}$$

with $0 < \tau' < \tau$, satisfying (0.2.109), with Λ_k independent of τ'; in particular,

$$\sup_{0 \leq t \leq \tau} \left(\|u_t(t)\|^2_{m+k} + \|u(t)\|^2_{2m+k} \right) \leq \Lambda_k^2 \,. \tag{0.2.113}$$

We wish to show the existence of Λ_{k+1}, independent of τ', such that (0.2.109) holds with k replaced by $k+1$; that is, explicitly,

$$\sup_{0 \le t \le \tau'} \left(\|u_t(t)\|^2_{m+k+1} + \|u(t)\|^2_{2m+k+1} \right) \le \Lambda^2_{k+1} \, . \tag{0.2.114}$$

To this end, we multiply equation (0.1.8) in L^2 by $2\,\Delta^{m+k+1} u_t$, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(\|u_t\|^2_{\overline{m+k+1}} \; + \; \|u\|^2_{\overline{2m+k+1}} \right) \\ &\qquad = \; 2\,\langle \nabla^{m+k+1} N(f, u^{(m-1)}), \nabla^{m+k+1} u_t \rangle \\ &\qquad \le \; 2\,\|N(f, u^{(m-1)})\|_{\overline{m+k+1}} \, \|u_t\|_{\overline{m+k+1}} \, , \end{aligned} \tag{0.2.115}$$

where $f := f(u)$. If $m > 2$, we estimate $N(f, u^{(m-1)})$ by means of (0.1.39) of Lemma 0.1.4, with $h = m+k+1 > m$, which yields

$$\|N(f, u^{(m-1)})\|_{\overline{m+k+1}} \le C \, \max\left\{ \|f\|_{\overline{m}}, \|f\|_{\overline{2m+k}} \right\} \, \|u\|^{m-1}_{2m+k} \, . \tag{0.2.116}$$

By (0.1.48) with $h = m+k > m$,

$$\|f\|_{\overline{2m+k}} \le C \, \|u\|^m_{m+k+1} \le C \, \|u\|^m_{2m+k} \, ; \tag{0.2.117}$$

thus, using also (0.1.44), by (0.2.113) we obtain from (0.2.116) and (0.2.117) that

$$\|N(f, u^{(m-1)})\|_{\overline{m+k+1}} \le C \, \Lambda_k^{2m-1} \, . \tag{0.2.118}$$

Replacing this into (0.2.115) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|u_t\|^2_{\overline{m+k+1}} \; + \; \|u\|^2_{\overline{2m+k+1}} \right) &\le C \, \Lambda_k^{2m-1} \, \|u_t\|_{\overline{m+k+1}} \\ &\le C \, \Lambda_k^{2(2m-1)} + \|u_t\|^2_{\overline{m+k+1}} \, ; \end{aligned} \tag{0.2.119}$$

thus, by Gronwall's inequality, for all $t \in [0, \tau']$,

$$\begin{aligned} &\|u_t(t)\|^2_{\overline{m+k+1}} \; + \; \|u(t)\|^2_{\overline{2m+k+1}} \\ &\qquad \le \; \left(\|u_1\|^2_{\overline{m+k+1}} \; + \; \|u_0\|^2_{\overline{2m+k+1}} + C \, \Lambda_k^{2(2m-1)} \, \tau \right) \, \mathrm{e}^{\tau} =: \Lambda^2_{k+1} \, , \end{aligned} \tag{0.2.120}$$

from which (0.2.114) follows. If $m = 2$, we use instead (0.1.29) of Lemma 0.1.3 with $h = 3 + k > 2$, to obtain

$$\|N(f, u)\|_{\overline{3+k}} \le C \, \max\left\{ \|f\|_{\overline{2}}, \|f\|_{\overline{5+k}} \right\} \, \|u\|_{5+k} \, . \tag{0.2.121}$$

By (0.1.48),

$$\|f\|_{\overline{5+k}} \le C \, \|u\|^2_{4+k} \le C \, \Lambda_k^2 \, ; \tag{0.2.122}$$

hence, we deduce from (0.2.121), via (0.1.44) as above, that

$$\|N(f,u)\|_{\overline{3+k}} \le C\,\Lambda_k^2\,\|u\|_{5+k}\,. \tag{0.2.123}$$

Replacing this into (0.2.115) with $m=2$, yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2_{\overline{3+k}} + \|u\|^2_{\overline{5+k}}\right) &\le 2\,C\,\Lambda_k^2\,\|u\|_{\overline{5+k}}\,\|u_t\|_{\overline{3+k}} \\ &\le C\,\Lambda_k^2\left(\|u_t\|^2_{\overline{3+k}} + \|u\|^2_{\overline{5+k}}\right)\,, \end{aligned} \tag{0.2.124}$$

and we can conclude by means of Gronwall's inequality. This ends the proof of Proposition 0.2.4. □

0.2.3. Almost Global Existence.

The estimates we established in the proof of Theorem 0.2.1 allow us to give an almost global existence result for problem (H), in the spirit of Theorem 4.4.1 of Chapter 4. More precisely, with $E(t)$ as in (0.2.73), we claim:

Theorem 0.2.3. *In the same assumptions of Theorem* 0.2.1*, given arbitrary* $T>0$ *there is* $\delta>0$ *such that if* $E(0)\le\delta$*, problem* (H) *admits a unique solution* $u\in\mathcal{H}^{2m,m}(T)$*, with* $f(u)\in C([0,T];\bar{H}^{2m})$*.*

Sketch of Proof. Replacing w by u in the procedure that led to estimate (0.2.18), we deduce that on any interval $[0,\tau]\subseteq[0,T]$ on which u is defined, E satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E \le C\,\|u\|_{2m}^{2m-1}\,\|u_t\|_m + C_\varphi\,\|u\|_{2m}\,\|u_t\|_m \le C\,E^m + C_\varphi\,E\,, \tag{0.2.125}$$

with C_φ as in (0.2.20). The Bernoulli-type inequality (0.2.125) implies the exponential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{1-m} + (m-1)\,C_\varphi\,E^{1-m} \ge -\,C\,(m-1)\,, \tag{0.2.126}$$

the integration of which leads to

$$(E(t))^{1-m} \ge \left((E(0))^{1-m} + \tfrac{C}{C_\varphi}\right)\mathrm{e}^{-(m-1)C_\varphi\,t} - \tfrac{C}{C_\varphi}\,. \tag{0.2.127}$$

Assuming that $E(0)\le\delta$, we deduce from (0.2.127) that for all $t\in[0,\tau]$,

$$(E(t))^{m-1} \le \frac{C_\varphi\,\mathrm{e}^{(m-1)C_\varphi\,t}\,\delta^{m-1}}{C_\varphi + C\,\delta^{m-1} - C\,\delta^{m-1}\,\mathrm{e}^{(m-1)C_\varphi\,t}}\,. \tag{0.2.128}$$

Thus, the life-span T_c of u satisfies the estimate $T_c\ge T_\delta$, where T_δ is the blow-up time of the right side of (0.2.128); that is,

$$T_c \ge T_\delta := \tfrac{1}{(m-1)\,C_\varphi}\,\ln\left(1 + \tfrac{C_\varphi}{C\,\delta^{m-1}}\right)\,. \tag{0.2.129}$$

In particular, since $T_\delta \to +\infty$ as $\delta \to 0$, given $T > 0$ it is possible to have $T_c > T$ by choosing δ, and therefore $E(0)$, sufficiently small, so that $T_c \geq T_\delta > T$. □

0.3. The Parabolic System

0.3.1. Local Existence.

In this section we give a local existence result for problem (P). For h and $k \in \mathbb{N}$, with $k \leq h$, and given $T > 0$, we consider the spaces

$$\mathcal{P}^{h,k}(T) := \{u \in L^2(0,T;H^h) \mid u_t \in L^2(0,T;H^k)\}\,, \tag{0.3.1}$$

$$\mathcal{Y}^{h,k}(T) := L^2(0,T;H^h) \cap C([0,T];H^{(h+k)/2})\,, \tag{0.3.2}$$

$$\overline{\mathcal{Y}}^{h,k}(T) := L^2(0,T;\bar{H}^h) \cap C([0,T];\bar{H}^{(h+k)/2})\,, \tag{0.3.3}$$

endowed with their natural norms

$$\|u\|^2_{\mathcal{P}^{h,k}(T)} := \int_0^T \left(\|u_t\|_k^2 + \|u\|_h^2\right) \,\mathrm{d}t\,, \tag{0.3.4}$$

$$\|u\|^2_{\mathcal{Y}^{h,k}(T)} := \int_0^T \|u\|_h^2 \,\mathrm{d}t + \max_{0\leq t\leq T} \|u(t)\|^2_{(h+k)/2}\,, \tag{0.3.5}$$

$$\|u\|^2_{\overline{\mathcal{Y}}^{h,k}(T)} := \int_0^T \|u\|_{\overline{h}}^2 \,\mathrm{d}t + \max_{0\leq t\leq T} \|u(t)\|^2_{\overline{(h+k)/2}}\,. \tag{0.3.6}$$

Note that by the trace Theorem 1.7.4, $u \in C([0,T];H^{(h+k)/2})$ if $u \in \mathcal{P}^{h,k}(T)$, and, by (1.7.61),

$$\|u\|_{C([0,T];H^{(h+k)/2})} \leq C\, \|u\|_{\mathcal{P}^{h,k}(T)}\,. \tag{0.3.7}$$

In particular, (0.3.7) implies that $\mathcal{P}^{2m,0}(T) \hookrightarrow \mathcal{Y}^{2m,0}(T)$. We claim:

Theorem 0.3.1. *Assume that $u_0 \in H^m$ and $\varphi \in \mathcal{Y}^{2m,0}(T)$. There exist $\tau_* \in]0,T]$ and a unique $u \in \mathcal{P}^{2m,0}(\tau_*)$, with $f(u) \in \overline{\mathcal{Y}}^{2m,0}(\tau_*)$, solution of problem* (P). *This solution depends continuously on the data $\{u_0, \varphi\}$.*

Proof. We loosely follow Cherrier and Milani [**3, 4**]. In contrast to the hyperbolic problem (H), we cannot prove Theorem 0.3.1 directly; rather, we first prove a higher regularity result for problem (P), and then use this result to prove Theorem 0.3.1 by means of an approximation argument. This roundabout procedure, which we do not know to what extent is necessary, is due to a rather drastic role played by the limit case of the Sobolev imbedding theorem $H^{m-1} \hookrightarrow L^{2m}$, which, as we saw in (0.1.23), allows us to estimate $N(u_1, \ldots, u_m)$ in L^2 only in terms of the u_j's in H^{m+1} (as opposed to H^m; see (0.3.31) and (0.3.33) below). Thus, we postpone the proof of Theorem 0.3.1 to section 0.3.2, and first prove

Theorem 0.3.2. *Assume that $u_0 \in H^{m+1}$ and $\varphi \in \mathcal{Y}^{2m+1,1}(T)$. There exist $\tau_1 \in]0,T]$ and a unique $u \in \mathcal{P}^{2m+1,1}(\tau_1)$, with $f(u) \in \overline{\mathcal{Y}}^{2m+1,1}(\tau_1)$, solution of problem* (P)*.*

Proof. The proof of Theorem 0.3.2 proceeds along the lines of the linearization and fixed point method described in section 3.4 of Chapter 3, following essentially the same steps of the proof of Theorem 0.2.1 for problem (H). Since the methods are similar, and the required estimates are by now familiar, we will occasionally omit a detailed proof of the most straightforward steps of the proof.

1: Linearization. Given $\tau \in [0,T]$ and $R > 0$, we define

$$B_m(\tau, R) := \left\{ u \in \mathcal{P}^{2m+1,1}(\tau) \;\middle|\; \|u\|_{\mathcal{P}^{2m+1,1}(\tau)} \le R\,, \quad u(0) = u_0 \right\}\,. \tag{0.3.8}$$

Note that the condition on $u(0)$ makes sense because $u \in C([0,T]; H^{m+1})$, by the trace theorem; in fact, by (0.3.7), if $u \in B_m(\tau, R)$,

$$\max_{0\le t\le \tau} \|u(t)\|_{m+1} \le C\,R\,. \tag{0.3.9}$$

For fixed $w \in B_m(\tau, R)$, we consider the linearized equation

$$u_t + \Delta^m u = N(f(w), w^{(m-2)}, u) + N(\varphi^{(m-1)}, u)\,, \tag{0.3.10}$$

with initial data (0.1.11). By (0.1.23) and (0.1.26) of lemmas 0.1.1 and 0.1.2, $M(w) \in L^2(0,\tau; H^1)$, with

$$\begin{aligned} \int_0^\tau \|M(w)\|_1^2 \,\mathrm{d}t &\le C \int_0^\tau \|w\|_{m+1}^{2(m-1)} \left(\|w\|_{m+1}^2 + \|w\|_{m+2}^2 \right) \mathrm{d}t \\ &\le C \left(\max_{0\le t\le\tau} \|w(t)\|_{m+1}^{2(m-1)} \right) \int_0^\tau \|w\|_{2m}^2 \,\mathrm{d}t \\ &\le C\,R^{2m}\,. \end{aligned} \tag{0.3.11}$$

Thus, $f := f(w) \in L^2(0,\tau; \bar{H}^{2m+1})$. In the same way, we see that the map $u \mapsto N(f, w^{(m-2)}, u)$ is continuous from $\mathcal{P}^{2m+1,1}(\tau)$ into $L^2(0,\tau; H^1)$; indeed, by means of (0.1.26) and (0.1.47) we can show that

$$\begin{aligned} &\int_0^\tau \|N(f, w^{(m-2)}, u)\|_1^2 \,\mathrm{d}t \\ &\quad \le C\,\|w\|_{C([0,\tau];H^{m+1})}^{2(2m-3)} \left(\int_0^\tau \|w\|_{2m}^2 \,\mathrm{d}t \right) \|u\|_{C([0,\tau];H^{m+1})}^2 \\ &\qquad + C\,\|w\|_{C([0,\tau];H^{m+1})}^{4(m-1)} \left(\int_0^\tau \|u\|_{2m}^2 \,\mathrm{d}t \right) \\ &\quad \le C\,R^{4(m-1)}\,\|u\|_{\mathcal{P}^{2m+1,1}(\tau)}^2\,, \end{aligned} \tag{0.3.12}$$

having used (0.3.7) in the last step. Likewise, the map $u \mapsto N(\varphi^{(m-1)}, u)$ is continuous from $\mathcal{P}^{2m+1,1}(\tau)$ into $L^2(0,\tau;H^1)$, with

$$\begin{aligned}
&\int_0^\tau \|\nabla N(\varphi^{(m-1)}, u)\|^2 \,\mathrm{d}t \\
&\quad \leq \; C\,\|\varphi\|^{2(m-2)}_{C([0,T];H^{m+1})} \left(\int_0^T \|\varphi\|^2_{2m}\,\mathrm{d}t\right) \|u\|^2_{C([0,\tau];H^{m+1})} \\
&\qquad + C\,\|\varphi\|^{2(m-1)}_{C([0,T];H^{m+1})} \left(\int_0^\tau \|u\|^2_{2m}\,\mathrm{d}t\right) \\
&\quad \leq: \; C_1(\varphi)\,\|u\|^2_{\mathcal{P}^{2m+1,1}(\tau)} \,.
\end{aligned} \tag{0.3.13}$$

Thus, the existence and uniqueness of a solution $u \in \mathcal{P}^{2m+1,1}(\tau)$ of problem (0.3.10)+ (0.1.11), with $f \in \overline{\mathcal{Y}}^{2m+1,1}(\tau)$ can be established by methods analogous to those of Chapter 2. As in section 3.4, this allows us to define the map $w \mapsto u := \Gamma(w)$, from $B_m(\tau, R)$ into $\mathcal{P}^{2m+1,1}(\tau)$.

Proposition 0.3.1. *There exist $\tau_0 \in]0,T]$ and $R_1 \geq 1$ such that for all $\tau \in]0,\tau_0]$, Γ maps the ball $B_m(\tau, R_1)$ into itself.*

Proof. Multiplying (formally) equation (0.3.10) in L^2 by $\Delta^{m+1}u + \Delta u_t$, and integrating by parts, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|^2_{\overline{m+1}} + \|u\|^2_{\overline{2m+1}} + \|\nabla u_t\|^2 \\
&\quad = \; \langle N(f, w^{(m-2)}, u) + N(\varphi^{(m-1)}, u), \Delta^{m+1}u + \Delta u_t\rangle \\
&\quad = \; \langle \underbrace{\nabla N(f, w^{(m-2)}, u)}_{=:\,A'_m} + \underbrace{\nabla N(\varphi^{(m-1)}, u)}_{=:\,B'_m}, \nabla(\Delta^m u + u_t)\rangle \,.
\end{aligned} \tag{0.3.14}$$

We estimate A'_m and B'_m by means of lemmas 0.1.2 and 0.1.6; recalling that for $v \in H^r$, $r \geq 0$, $\|v\|_{\overline{r}} \leq \|v\|_r$, we obtain

$$\begin{aligned}
\|A'_m\|_0 \;\leq\; & C\,\|w\|^{m-2}_m\,\|w\|^m_{m+1}\,\|u\|_{m+1} \\
& + C\,\|w\|^{m-1}_m\,\|w\|^{m-2}_{m+1}\,\|w\|_{m+2}\,\|u\|_{m+1} \\
& + C\,\|w\|^{m-1}_m\,\|w\|^{m-1}_{m+1}\,\|u\|_{\overline{m+2}} \,.
\end{aligned} \tag{0.3.15}$$

Using interpolation, recalling (0.3.9) and assuming that $R \geq 1$, we obtain from (0.3.15)

$$\begin{aligned} &|\langle A'_m, \nabla(\Delta^m u + u_t)\rangle| \\ &\quad \leq C\Big(\|w\|_{m+1}^{2m-2}\,\|u\|_{m+1} + \|w\|_{m+1}^{2m-2-1/m}\,\|w\|_{2m}^{1/m}\,\|u\|_{m+1} \\ &\qquad + \|w\|_{m+1}^{2m-2}\,\|u\|_{m+1}^{1-1/m}\,\|u\|_{\overline{2m+1}}^{1/m}\Big)\,\left(\|u\|_{\overline{2m+1}} + \|\nabla u_t\|\right) \\ &\quad \leq C\,R^{2m-2}\left(1 + \|w\|_{2m}^{1/m}\right)\|u\|_{m+1}\left(\|u\|_{\overline{2m+1}} + \|\nabla u_t\|\right) \\ &\qquad + C\,R^{2m-2}\,\|u\|_{m+1}^{1-1/m}\left(\|u\|_{\overline{2m+1}}^{1+1/m} + \|u\|_{\overline{2m+1}}^{1/m}\,\|\nabla u_t\|\right). \end{aligned} \tag{0.3.16}$$

From this, by the weighted Minkowski's inequality ((1.1.12) of Proposition 1.1.1),

$$\begin{aligned} &|\langle A'_m, \nabla(\Delta^m u + u_t)\rangle| \\ &\quad \leq C\,R^{4m}\left(1 + \|w\|_{2m}^{2/m}\right)\|u\|_{m+1}^2 + \tfrac{1}{4}\left(\|u\|_{\overline{2m+1}}^2 + \|\nabla u_t\|^2\right). \end{aligned} \tag{0.3.17}$$

Next,

$$|\langle B'_m, \nabla(\Delta^m u + u_t)\rangle| \leq \|B'_m\|_0\,\left(\|u\|_{\overline{2m+1}} + \|\nabla u_t\|_0\right)\ ; \tag{0.3.18}$$

by (0.1.26) and interpolation,

$$\begin{aligned} \|B'_m\|_0 &\leq C\,\|\varphi\|_{m+2}\,\|\varphi\|_{m+1}^{m-2}\,\|u\|_{m+1} + C\,\|\varphi\|_{m+1}^{m-1}\,\|u\|_{m+2} \\ &\leq C\,\|\varphi\|_{m+1}^{m-1-\frac{1}{m}}\,\|\varphi\|_{2m+1}^{\frac{1}{m}}\,\|u\|_{m+1} \\ &\quad +\ C\,\|\varphi\|_{m+1}^{m-1}\,\|u\|_{m+1}^{1-\frac{1}{m}}\,\|u\|_{\overline{2m+1}}^{\frac{1}{m}}. \end{aligned} \tag{0.3.19}$$

Inserting (0.3.19) into (0.3.18) we arrive at

$$\begin{aligned} &|\langle B'_m, \nabla(\Delta^m u + u_t)\rangle| \\ &\quad \leq C\left(\|\varphi\|_{m+1}^{2m} + \|\varphi\|_{2m}^{\frac{2m}{m+1}}\right)\|u\|_{m+1}^2 + \tfrac{1}{4}\left(\|u\|_{\overline{2m+1}}^2 + \|\nabla u_t\|^2\right). \end{aligned} \tag{0.3.20}$$

Inserting (0.3.17) and (0.3.20) into (0.3.14), adding (0.2.17) and integrating, we obtain

$$\begin{aligned} &\|u(t)\|_{m+1}^2 + \tfrac{1}{2}\int_0^t \left(\|u\|_{2m+1}^2 + \|\nabla u_t\|^2\right)\,\mathrm{d}t \le \|u_0\|_{m+1}^2 + \int_0^t \|u\|_0^2\,\mathrm{d}\theta \\ &\quad + \int_0^t \|u_t\|_0^2\,\mathrm{d}\theta + C\,R^{4m}\int_0^t \left(1 + \|w\|_{2m}^{\frac{2}{m}}\right)\|u\|_{m+1}^2\,\mathrm{d}\theta \\ &\quad + C\int_0^t \left(\|\varphi\|_{m+1}^{2m} + \|\varphi\|_{2m}^{\frac{2m}{m+1}}\right)\|u\|_{m+1}^2\,\mathrm{d}\theta\,. \end{aligned} \tag{0.3.21}$$

Integrating (0.2.17) we deduce that if $\tau \le \frac{1}{2}$,

$$\int_0^t \|u\|_0^2\,\mathrm{d}\theta \le \|u_0\|_0^2 + \int_0^t \|u_t\|_0^2\,\mathrm{d}\theta\,; \tag{0.3.22}$$

adding this to (0.3.21), as well as the term $\frac{1}{2}\int_0^t \|u_t\|_0^2\,\mathrm{d}\theta$ to both sides, we obtain

$$\begin{aligned} &2\,\|u(t)\|_{m+1}^2 + \int_0^t \left(\|u\|_{2m+1}^2 + \|u_t\|_1^2\right)\,\mathrm{d}\theta \le\; 4\,\|u_0\|_{m+1}^2 + 5\int_0^t \|u_t\|_0^2\,\mathrm{d}\theta \\ &\quad + C\,R^{4m}\int_0^t \left(1 + \|w\|_{2m}^{\frac{2}{m}}\right)\|u\|_{m+1}^2\,\mathrm{d}\theta \\ &\quad + C\int_0^t \left(\|\varphi\|_{m+1}^{2m} + \|\varphi\|_{2m}^{\frac{2m}{m+1}}\right)\|u\|_{m+1}^2\,\mathrm{d}\theta\,. \end{aligned} \tag{0.3.23}$$

We estimate u_t in $L^2(0,t;L^2)$ directly from equation (0.3.10). By (0.1.23) of Lemma 0.1.1 and (0.1.47),

$$\begin{aligned} \|u_t\|_0 &\le \|\Delta^m u\|_0 + \|N(f, w^{(m-2)}, u)\|_0 + \|N(\varphi^{(m-1)}, u)\|_0 \\ &\le \|u\|_{\overline{m}} + C\,\|w\|_m^{m-1}\,\|w\|_{m+1}\,\|w\|_{m+1}^{m-2}\,\|u\|_{m+1} \\ &\quad +\; C\,\|\varphi\|_{m+1}^{m-1}\,\|u\|_{m+1}\,, \end{aligned} \tag{0.3.24}$$

from which

$$\|u_t\|_0^2 \le C\left(\|u\|_m^2 + R^{2(m-1)}\,\|u\|_{m+1}^2 + \|\varphi\|_{m+1}^{2(m-1)}\,\|u\|_{m+1}^2\right)\,. \tag{0.3.25}$$

This shows that the term with u_t at the right side of (0.3.23) can be absorbed by the other terms. Thus, by the Gronwall and Hölder's inequalities, we deduce from (0.3.23) that for all $t \in [0,\tau]$, $\tau \le \frac{1}{2}$,

$$\begin{aligned} &2\,\|u(t)\|_{m+1}^2 + \int_0^t \left(\|u\|_{2m+1}^2 + \|u_t\|_1^2\right)\,\mathrm{d}\theta \\ &\qquad \le\; 4\,\|u_0\|_{m+1}^2\,\exp\left(C\,\tau^{1-\frac{1}{m}}\,\psi(R,T,w,\varphi)\right)\,, \end{aligned} \tag{0.3.26}$$

where

(0.3.27)

$$\psi(R,T,w,\varphi) := R^{4m}\,T^{\frac{1}{m}} + R^{4m}\,\left(\int_0^\tau \|w\|_{2m}^2\,\mathrm{d}t\right)^{1/m}$$

$$+\;C_2(\varphi,T)\;\leq R^{4m}\,(R+T^{1/m}) + C_2(\varphi,T)\,,$$

and

$$C_2(\varphi,T) := T^m \max_{0\leq t\leq\tau} \|\varphi(t)\|_{m+1}^{2m} + \left(\int_0^T \|\varphi\|_{2m}^2\,\mathrm{d}t\right)^{1/m}\,. \tag{0.3.28}$$

Choosing then (e.g.) $R_1 := 2\,\|u_0\|_{m+1}$, and then $\tau_0 \in \left]0, \min\left\{\frac{1}{2}, T\right\}\right]$ such that

$$\exp\left(C\,\tau_0^{1-1/m}\,\psi(R,T,w,\varphi)\right) \leq \tfrac{4}{3}\,, \tag{0.3.29}$$

we deduce from (0.3.26) that for all $\tau \in]0,\tau_0]$, Γ maps $B_m(\tau,R_1)$ into itself, as claimed. □

Remark. As we have mentioned in the beginning of the proof of Theorem 0.3.1, the proof of Proposition 0.3.1 cannot be adapted to show the existence of an invariant ball for Γ in $\mathcal{P}^{2m,0}(\tau)$. To see this, it is sufficient to consider the case $\varphi \equiv 0$ and $m = 2$ (it is straightforward to see that this argument can be carried over to the general case). Then, (0.3.14) would be replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_2^2 + \|u\|_4^2 + \|u_t\|_0^2 = \langle N(f,u), \Delta^2 u + u_t\rangle\,; \tag{0.3.30}$$

in turn, by (0.1.23) and (0.1.47), (0.3.15) would be replaced by

$$\begin{aligned}\|N(f,u)\|_0 &\leq C\,\|f\|_{\overline{3}}\,\|u\|_{\overline{3}} \leq C\,\|w\|_2\,\|w\|_3\,\|u\|_{\overline{3}}\\ &\leq C\,\|w\|_2^{3/2}\,\|w\|_4^{1/2}\,\|u\|_2^{1/2}\,\|u\|_{\overline{4}}^{1/2}\,.\end{aligned} \tag{0.3.31}$$

Thus, (0.3.17) would be replaced by

$$|\langle N(f,u), \Delta^2 u + u_t\rangle| \leq C\,R^6\,\|w\|_4^2\,\|u\|_2^2 + \tfrac{1}{4}\,\left(\|u\|_4^2 + \|u_t\|_0^2\right)\,, \tag{0.3.32}$$

and, ultimately, (0.3.26) would become

$$\int_0^t \left(\|u\|_4^2 + \|u_t\|_0^2\right)\,\mathrm{d}\theta \leq 4\,\|u_0\|_2^2\,\exp\left(C\,R^6 \int_0^\tau \|w\|_4^2\,\mathrm{d}t\right)\,. \tag{0.3.33}$$

This last estimate is not sufficient to allow us to choose small τ, uniformly with respect to w, so that the right side of (0.3.33) be not larger than $4\,\|u_0\|_2^2$, as done in (0.3.29). ◇

2: Contractivity. We now prove that a further restriction on τ makes Γ a contraction on $B_m(\tau,R_1)$, with respect to the weaker norm of $\mathcal{P}^{2m,0}(\tau)$.

Proposition 0.3.2. *There exists $\tau_1 \in]0, \tau_0]$ such that for all w and $\tilde{w} \in B_m(\tau_1, R_1)$,*

$$\|\Gamma(w) - \Gamma(\tilde{w})\|_{\mathcal{P}^{2m,0}(\tau_1)} \leq \tfrac{1}{2}\,\|w - \tilde{w}\|_{\mathcal{P}^{2m,0}(\tau_1)}\,. \tag{0.3.34}$$

Proof. Let $\tau \in]0, \tau_0]$, and w, $\tilde{w} \in B_m(\tau_1, R_1)$; as in the proof of Proposition 0.2.2, set $u := \Gamma(w)$, $\tilde{u} := \Gamma(\tilde{w})$, $v := w - \tilde{w}$, $z := u - \tilde{u}$, $f := f(u)$, $\tilde{f} := f(\tilde{u})$, and $g := f - \tilde{f}$. By difference between the equations satisfied by u and $\tilde{u}$, and by the symmetry of N, we see that z and g solve the system

$$z_t + \Delta^m z = F + N(\varphi^{(m-1)}, z)\,, \tag{0.3.35}$$

$$\Delta^m g = -G\,, \tag{0.3.36}$$

where F and G are defined in (0.2.26) and (0.2.27). Multiplying (0.3.35) in L^2 by $\Delta^m z + z_t$, we obtain

(0.3.37)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\|z\|^2_{\overline{m}} + \|z\|^2_{\overline{2m}} + \|z_t\|^2_0 &= \langle F + N(\varphi^{(m-1)}, z), \Delta^m z + z_t\rangle \\ &\leq 2\,\|F + N(\varphi^{(m-1)}, z)\|^2_0 + \frac{1}{4}\,\left(\|z\|^2_{\overline{2m}} + \|z_t\|^2_0\right)\,.\end{aligned}$$

We estimate F as in the proof of Proposition 0.2.2, obtaining

$$\|F\|^2_0 \leq C\,R_1^{4(m-1)}\left(\|v\|^2_m + \|z\|^2_{\overline{m}}\right)\,. \tag{0.3.38}$$

We then modify the estimate of $N(\varphi^{(m-1)}, z)$ given in (0.2.40), proceeding instead with

$$\begin{aligned}\|N(\varphi^{(m-1)}, z)\|_0 &\leq C\,|\partial_x^2\varphi|^{m-2}_{2m}\,|\partial_x^2\varphi|_\infty\,|\partial_x^2 z|_m \\ &\leq C\,\|\partial_x^2\varphi\|^{m-1}_{m-1}\,\|\partial_x^2\varphi\|_{2m-1}\,|\partial_x^2 z|_m \\ &\leq C\,\|\varphi\|^{m-1}_{m+1}\,\|\varphi\|_{2m+1}\,\|z\|_{\overline{m}}\,.\end{aligned} \tag{0.3.39}$$

Inserting (0.3.38) and (0.3.39) into (0.3.37), and adding the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|z\|^2_0 \leq 8\,\|z\|^2_0 + \tfrac{1}{8}\,\|z_t\|^2_0 \tag{0.3.40}$$

(compare to (0.2.42)), we obtain

(0.3.41)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\,\|z\|^2_m + \frac{3}{4}\,\|z\|^2_{\overline{2m}} + \frac{5}{8}\,\|z_t\|^2_0 \\ &\quad\leq C\,R_1^{4(m-1)}\left(\|v\|^2_m + \|z\|^2_m\right) + C\,\|\varphi\|^{2(m-1)}_{m+1}\,\|\varphi\|^2_{2m+1}\,\|z\|^2_m\,.\end{aligned}$$

Similarly as in (0.3.22), and recalling that $z(0) = z_t(0) = 0$,

$$\int_0^t \|z\|_0^2 \,\mathrm{d}\theta \le t \int_0^t \|z_t\|_0^2 \,\mathrm{d}\theta \,. \tag{0.3.42}$$

Thus, integrating (0.3.41) and adding (0.3.42), if we impose further that $\tau \le \frac{1}{8}$, recalling (0.3.7) with $h = 2m$ and $k = 0$, we deduce that for all $t \in [0, \tau]$,

$$\begin{aligned} &2\,\|z(t)\|_m^2 + \int_0^t \left(\|z\|_{2m}^2 + \|z_t\|_0^2\right) \,\mathrm{d}\theta \\ &\le\; C\,R_1^{4(m-1)} \left(\max_{0\le t\le \tau} \|v(t)\|_m^2\right) t \,\exp\left(C\,R_1^{4(m-1)}\,t + C_3(\varphi)\right) \\ &\le\; C\,R_1^{4(m-1)}\,\|v\|_{\mathcal{P}^{2m,0}(\tau)}^2 \,t\,\exp\left(C\,R_1^{4(m-1)}\,t + C_3(\varphi)\right)\,, \end{aligned} \tag{0.3.43}$$

with

$$C_3(\varphi) := C_1(\varphi) \int_0^T \|\varphi\|_{2m+1}^2 \,\mathrm{d}t \,, \tag{0.3.44}$$

$C_1(\varphi)$ as in (0.3.13). Thus, if we choose $\tau_1 \in \left]0, \frac{1}{4}\,\tau_0\right]$ such that

$$C\,R^{4(m-1)}\,\tau_1 \,\exp\left(C\,R_1^{4(m-1)}\,\tau_1 + C_3(\varphi)\right) \le \tfrac{1}{4} \,, \tag{0.3.45}$$

recalling that $z = u - \tilde{u} = \Gamma(w) - \Gamma(\tilde{w})$ and $v = w - \tilde{w}$, we see that (0.3.34) follows from (0.3.43). This ends the proof of Proposition 0.3.2. □

3: Picard's Iterations. We now consider the Picard's iterations of Γ, that is, the sequence $(u^n)_{n\ge 0}$, defined recursively by $u^{n+1} = \Gamma(u^n)$, starting from an arbitrary $u^0 \in B_m(\tau_1, R_1)$. Explicitly, the functions u^{n+1} are defined in terms of u^n, by the equation

$$u_t^{n+1} + \Delta^m u^{n+1} = N(f^n, (u^n)^{(m-2)}, u^{n+1}) + N(\varphi^{(m-1)}, u^{n+1}) \,, \tag{0.3.46}$$

where $f^n := f(u^n)$, and by the initial condition $u^{n+1}(0) = u_0$. By Proposition 0.3.1, the sequence $(u^n)_{n\ge 0}$ is bounded in $\mathcal{P}^{2m+1,1}(\tau_1)$; by Proposition 0.3.2, $(u^n)_{n\ge 0}$ is a Cauchy sequence in $\mathcal{P}^{2m,0}(\tau_1)$. Thus, there is a subsequence, still denoted $(u^n)_{n\ge 0}$, and a function $u \in \mathcal{P}^{2m+1,1}(\tau_1)$, such that

$$u^n \to u \quad \text{in } L^2(0, \tau_1; H^{2m+1}) \text{ weak}^* \text{ and } L^2(0, \tau_1; H^{2m}) \,, \tag{0.3.47}$$

$$u_t^n \to u_t \quad \text{in } L^2(0, \tau_1; H^1) \text{ weak}^* \text{ and } L^2(0, \tau_1; L^2) \,. \tag{0.3.48}$$

By the trace theorem, (0.3.47) and (0.3.48) imply that also

$$u^n \to u \qquad \text{in} \quad C([0, \tau_1]; H^m) \,. \tag{0.3.49}$$

We now show, with methods similar to those used in the proof of (0.2.57), that

$$M(u^n) \to M(u) \quad \text{in} \quad L^2(0,\tau_1;L^2)\,. \tag{0.3.50}$$

Indeed, recalling (0.2.58) and (0.1.23), at first

$$\|M(u^n)-M(u)\|_0 \le C\sum_{k=1}^{m} \|u^n-u\|_{m+1}\,\|u^n\|_{m+1}^{m-k}\,\|u\|_{m+1}^{k-1}\,. \tag{0.3.51}$$

By interpolation and (0.3.49),

$$\|u^n-u\|_{m+1} \le C\,\|u^n-u\|_{2m}^{1/m}\,\|u^n-u\|_{m}^{1-1/m} \le C_n\,\|u^n-u\|_{2m}^{1/m}\,, \tag{0.3.52}$$

with $C_n \to 0$; likewise, with C now depending on R_1,

$$\|u^n\|_{m+1}^{m-k} \le \|u^n\|_{2m}^{(m-k)/m}\,\|u^n\|_{m}^{(m-k)(m-1)/m} \le C\,\|u^n\|_{2m}^{(m-k)/m}\,, \tag{0.3.53}$$

$$\|u\|_{m+1}^{k-1} \le \|u\|_{2m}^{(k-1)/m}\,\|u\|_{m}^{(k-1)(m-1)/m} \le C\,\|u\|_{2m}^{(k-1)/m}\,. \tag{0.3.54}$$

By Hölder's inequality,

$$\begin{aligned} &\int_0^{\tau_1} \|u^n-u\|_{2m}^{\frac{2}{m}}\,\|u^n\|_{2m}^{\frac{2(m-k)}{m}}\,\|u\|_{2m}^{\frac{2(k-1)}{m}}\,\mathrm{d}\theta \\ &\le \left(\int_0^{\tau_1}\|u^n-u\|_{2m}^2\right)^{\frac{1}{m}}\left(\int_0^{\tau_1}\|u^n\|_{2m}^2\right)^{\frac{m-k}{m}}\left(\int_0^{\tau_1}\|u\|_{2m}^2\right)^{\frac{k-1}{m}} \\ &\le C\left(\int_0^{\tau_1}\|u^n-u\|_{2m}^2\right)^{\frac{1}{m}}\,, \end{aligned} \tag{0.3.55}$$

so that (0.3.50) follows from the second of (0.3.47). Thus, (0.3.50) implies that

$$f^n \to f(u) =: f \quad \text{in} \quad L^2(0,\tau_1;\bar{H}^{2m})\,. \tag{0.3.56}$$

In an analogous way, with methods similar to those used to prove (0.2.61) and (0.3.50) we can also show that

$$N(f^n,(u^n)^{(m-2)},u^{n+1}) \to N(f,u^{(m-1)}) \quad \text{in} \quad L^2(0,\tau_1;L^2)\,. \tag{0.3.57}$$

We can then proceed as in part (3) of the proof of Theorem 0.2.1, and conclude that u solves problem (P). Finally, since $u \in C([0,\tau_1];H^{m+1})$, $\partial_x^2 u \in C([0,\tau_1];H^{m-1}) \hookrightarrow C([0,\tau_1];L^{2m})$; thus, $M(u) \in C([0,\tau_1];L^2)$ and, therefore, $f(u) \in C([0,\tau_1];\bar{H}^{2m})$. Using interpolation (see (0.1.18)), we can then deduce that $f(u) \in C([0,\tau_1];\bar{H}^{m+1})$. To see this, let t_1, $t_2 \in [0,\tau_1]$. Then, by (0.1.44),

$$\|f(u(t_1))\|_{\overline{m}} + \|f(u(t_2))\|_{\overline{m}} \le C\,(\|u(t_1)\|_m^m + \|u(t_2)\|_m^m) \le C\,R_1^m\,. \tag{0.3.58}$$

From this, it follows that

(0.3.59)

$$
\begin{aligned}
&\|f(u(t_1)) - f(u(t_2))\|_{\overline{m+1}} \\
&\qquad \le\; C\,\|f(u(t_1)) - f(u(t_2))\|_{\overline{2m}}^{\frac{1}{m}}\,\|f(u(t_1)) - f(u(t_2))\|_{\overline{m}}^{1-\frac{1}{m}} \\
&\qquad \le\; C\,\|f(u(t_1)) - f(u(t_2))\|_{\overline{2m}}^{\frac{1}{m}}\left(\|f(u(t_1))\|_{\overline{m}} + \|f(u(t_2))\|_{\overline{m}}\right)^{1-\frac{1}{m}} \\
&\qquad \le\; C\,R_1^{m-1}\,\|f(u(t_1)) - f(u(t_2))\|_{\overline{2m}}^{\frac{1}{m}}\,.
\end{aligned}
$$

Since $f(u) \in C([0,\tau_1];\bar{H}^{2m})$, (0.3.59) shows that $f(u) \in C([0,\tau_1];\bar{H}^{m+1})$ as well. Since also, by (0.1.48) of Lemma 0.1.6,

$$
\begin{aligned}
\int_0^{\tau_1} \|f(u)\|_{\overline{2m+1}}^2\,\mathrm{d}t &\le C\int_0^{\tau_1} \|u\|_{m+2}^{2m}\,\mathrm{d}t \\
&\le C\int_0^{\tau_1} \|u\|_{m+1}^{2(m-1)}\,\|u\|_{2m+1}^2\,\mathrm{d}t \qquad (0.3.60)\\
&\le C\,R_1^{2(m-1)}\int_0^{\tau_1} \|u\|_{2m+1}^2\,\mathrm{d}t\,,
\end{aligned}
$$

we conclude that $f(u) \in \overline{\mathcal{Y}}^{2m+1,1}(\tau_1)$. This ends the proof of Theorem 0.3.2. □

0.3.2. Proof of Theorem 0.3.1.

We now go back to the proof of Theorem 0.3.1. We proceed in two steps, first proving the continuous dependence of solutions in $\mathcal{P}^{2m,0}(\tau)$ on their data, and then the existence of such solutions, by means of an approximation argument, which uses the more regular solutions given by Theorem 0.3.2.

1: Well-Posedness. We show that if problem (P) has solutions in $\mathcal{P}^{2m,0}(\tau)$, they are unique, and depend continuously on their data.

Proposition 0.3.3. *Let* u_0, $\tilde{u}_0 \in H^m$, *and* φ, $\tilde{\varphi} \in \mathcal{Y}^{2m,0}(T)$. *Assume that problem* (P) *has corresponding solutions* u, $\tilde{u} \in \mathcal{P}^{2m,0}(\tau)$, *with* $f := f(u)$ *and* $\tilde{f} := f(\tilde{u}) \in \overline{\mathcal{Y}}^{2m,0}(\tau)$, *for some* $\tau \in\,]0,T]$. *Then, the difference* $u - \tilde{u}$ *satisfies the estimate*

(0.3.61) $$\|u-\tilde{u}\|_{\mathcal{P}^{2m,0}(\tau)} \le h_1(\rho_1,\delta_1,T)\,\left(\|u_0-\tilde{u}_0\|_m + \|\varphi-\tilde{\varphi}\|_{\mathcal{Y}^{2m,0}(T)}\right)\,,$$

where $h_1 \in \mathcal{K}$ *and*

$$\rho_1 := \max\left\{1, \|u\|_{\mathcal{P}^{2m,0}(\tau)}, \|\tilde{u}\|_{\mathcal{P}^{2m,0}(\tau)}\right\}, \tag{0.3.62}$$

$$\delta_1 := \max\left\{1, \|\varphi\|_{\mathcal{Y}^{2m,0}(\tau)}, \|\tilde{\varphi}\|_{\mathcal{Y}^{2m,0}(\tau)}\right\}. \tag{0.3.63}$$

In particular, there is at most one solution of problem (P) *in* $\mathcal{P}^{2m,0}(\tau)$.

Sketch of Proof. With notation analogous to those of the proof of Propositions 0.2.3 and 0.3.2, let $z := u - \tilde{u}$, $g := f - \tilde{f}$, $\psi := \varphi - \tilde{\varphi}$. By the symmetry of N, z and g solve the system

$$z_t + \Delta^m z = F + \Phi, \tag{0.3.64}$$

$$\Delta^m g = -G, \tag{0.3.65}$$

where F, Φ and G are as in (0.2.84), (0.2.85), and (0.2.86). Multiplying (0.3.64) in L^2 by $\Delta^m z + z_t$, we obtain, as in (0.3.37),

(0.3.66)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2_{\overline{m}} + \|z\|^2_{\overline{2m}} + \|z_t\|^2_0 = \langle F + \Phi, \Delta^m z + z_t\rangle$$

$$\leq 4\|F+\Phi\|_0^2 + \frac{1}{8}\left(\|z\|^2_{\overline{2m}} + \|z_t\|_0^2\right).$$

We estimate F and Φ as in the proof of Proposition 0.2.3. Using (0.1.23) and (0.1.49) and acting as in (0.2.30) to estimate g, we obtain

$$\begin{aligned}\|F\|_0 &\leq C\|g\|_{\overline{m+1}}\|u\|^{m-1}_{m+1} + C\|\tilde{f}\|_{m+1}\|z\|_{\overline{m+1}}\|\hat{u}\|^{m-2}_{m+1}\\ &\leq C\|\hat{u}\|^{m-2}_m\|\hat{u}\|_{m+1}\|u\|^{m-1}_{m+1}\|z\|_{\overline{m}} \qquad (0.3.67)\\ &\quad + C\|\tilde{u}\|^{m-1}_m\|\hat{u}\|^{m-1}_{m+1}\|z\|_{\overline{m+1}}.\end{aligned}$$

Using interpolation and the weighted Minkowski's inequality (1.1.12), recalling (0.3.62) and that $\rho_1 \geq 1$, we obtain from (0.3.67) that for small $\eta > 0$,

$$\|F\|_0 \leq C\rho_1^{2m-1}\|\hat{u}\|_{2m}\|z\|_{\overline{m}} + \eta\|z\|_{\overline{2m}}. \tag{0.3.68}$$

Likewise, from (0.2.85), recalling also (0.3.63) and using (0.3.7), we obtain that

$$\begin{aligned}\|\Phi\|_0 &\leq C\|\varphi\|^{m-1}_{m+1}\|z\|_{\overline{m+1}} + C\|\hat{u}\|_{m+1}\|\hat{\varphi}\|^{m-2}_{m+1}\|\psi\|_{m+1}\\ &\leq C\delta_1^{m-1}\|\varphi\|_{2m}\|z\|_{\overline{m}} + \eta\|z\|_{\overline{2m}} \qquad (0.3.69)\\ &\quad + C\rho_1^{\frac{m-1}{m}}\delta_1^{\frac{(m-1)(m-2)}{m}}\|\psi\|^{\frac{m-1}{m}}_{\overline{m}}\|\psi\|^{1/m}_{2m}\|\hat{\varphi}\|^{\frac{m-2}{m}}_{2m}\|\hat{u}\|^{\frac{1}{m}}_{2m}.\end{aligned}$$

Inserting this into (0.3.66), taking small enough η, and adding the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|z\|_0^2 \le \sigma\,\|z\|_0^2 + \tfrac{1}{\sigma}\,\|z_t\|_0^2 \tag{0.3.70}$$

with $\sigma = 8$, we deduce that

(0.3.71)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\|z\|_m^2 + \tfrac{3}{4}\,\left(\|z\|_{2m}^2 + \|z_t\|_0^2\right) \\ &\qquad \le\; C\,\rho_1^{2(2m-1)}\,\|\hat{u}\|_{2m}^2\,\|z\|_m^2 + C\,\delta_1^{2(m-1)}\,\|\varphi\|_{2m}^2\,\|z\|_m^2 + 8\,\|z\|_0^2 \\ &\qquad +\; \underbrace{C\,\rho_1^{\frac{2(m-1)}{m}}\,\delta_1^{\frac{2(m-1)(m-2)}{m}}}_{=:\,C_1(\rho_1,\delta_1)}\,\|\psi\|_m^{\frac{2(m-1)}{m}}\,\|\psi\|_{2m}^{\frac{2}{m}}\,\|\hat{\varphi}\|_{2m}^{\frac{2(m-2)}{m}}\,\|\hat{u}\|_{2m}^{\frac{2}{m}}\,. \end{aligned}$$

Integrating (0.3.70) with $\sigma = 4\tau$ twice yields

$$\int_0^t \|z\|_0^2\,\mathrm{d}\theta \le \tau\,\|z(0)\|_0^2 + 4\,\tau^2\int_0^t \|z\|_0^2\,\mathrm{d}\theta + \tfrac{1}{4}\int_0^t \|z_t\|_0^2\,\mathrm{d}\theta\,; \tag{0.3.72}$$

thus, integrating (0.3.71) and adding (0.3.72),

$$\begin{aligned} &\|z(t)\|_m^2 + \tfrac{1}{2}\int_0^t \left(\|z\|_{2m}^2 + \|z_t\|_0^2\right)\,\mathrm{d}\theta \le (1+T)\,\|z(0)\|_m^2 \\ &\qquad + C_1(\rho_1,\delta_1)\,\Psi_\tau^{\frac{2(m-1)}{m}}\int_0^\tau \|\hat{u}\|_{2m}^{\frac{2}{m}}\,\|\hat{\varphi}\|_{2m}^{\frac{2(m-2)}{m}}\,\|\psi\|_{2m}^{\frac{2}{m}}\,\mathrm{d}\theta \\ &\qquad + C_2(\rho_1,\delta_1)\int_0^t \left(\|\hat{u}\|_{2m}^2 + \|\varphi\|_{2m}^2\right)\,\|z\|_m^2\,\mathrm{d}\theta\,, \end{aligned} \tag{0.3.73}$$

where

$$C_2(\rho_1,\delta_1) := C\,\max\left\{4(2+T^2), \rho_1^{2(2m-1)}, \delta_1^{2(m-1)}\right\}\,, \tag{0.3.74}$$

and, by (0.3.7),

$$\Psi_\tau := \max_{0\le t\le\tau}\|\psi(t)\|_m \le C\,\|\psi\|_{\mathcal{P}^{2m,0}(\tau)}\,. \tag{0.3.75}$$

By Hölder's inequality, as in (0.3.55) with $k = 2$,

$$\begin{aligned} &\int_0^\tau \|\hat{u}\|_{2m}^{\frac{2}{m}}\,\|\hat{\varphi}\|_{2m}^{\frac{2(m-2)}{m}}\,\|\psi\|_{2m}^{\frac{2}{m}}\,\mathrm{d}\theta \\ &\qquad \le \left(\int_0^\tau \|\hat{u}\|_{2m}^2\right)^{\frac{1}{m}}\left(\int_0^\tau \|\hat{\varphi}\|_{2m}^2\right)^{1-\frac{2}{m}}\left(\int_0^\tau \|\psi\|_{2m}^2\right)^{\frac{1}{m}} \\ &\qquad \le \rho_1^{2/m}\,\delta_1^{2(1-2/m)}\,\|\psi\|_{\mathcal{P}^{2m,0}(\tau)}^{2/m}\,. \end{aligned} \tag{0.3.76}$$

Inserting (0.3.75) and (0.3.76) into (0.3.73) and recalling (0.3.75), we obtain, via Gronwall's inequality, that for all $t \in [0, \tau]$,

$$\|z(t)\|_m^2 + \tfrac{1}{2} \int_0^t \left(\|z\|_{2m}^2 + \|z_t\|_0^2 \right) \,\mathrm{d}\theta$$
$$\leq \left((1+T)\, \|z(0)\|_m^2 + C_1(\rho_1, \delta_1)\, \rho_1^{\frac{2}{m}}\, \delta_1^{2\left(1-\frac{2}{m}\right)}\, \|\psi\|_{\mathcal{P}^{2m,0}(\tau)}^2 \right) \tag{0.3.77}$$
$$\cdot\ \exp\left(C_2(\rho_1, \delta_1) \int_0^\tau \left(\|\hat{u}\|_{2m}^2 + \|\varphi\|_{2m}^2 \right) \mathrm{d}\theta \right) .$$

Recalling that $z = u - \tilde{u}$ and $\psi = \varphi - \tilde{\varphi}$, and that, by (0.3.62) and (0.3.63),

$$\int_0^\tau \left(\|\hat{u}\|_{2m}^2 + \|\varphi\|_{2m}^2 \right) \mathrm{d}\theta \leq \rho_1^2 + \delta_1^2 \,, \tag{0.3.78}$$

we see that (0.3.61) follows from (0.3.77). This ends the proof of Proposition 0.3.3. □

2: Regular Approximations. In this step we resort to Theorem 0.3.2 to construct a sequence $(u^n)_{n\geq 0} \subset \mathcal{P}^{2m+1,1}(\tau)$, for some $\tau \in]0, T]$, of approximate solutions to problem (P). We will then show that this sequence has a limit $u \in \mathcal{P}^{2m,0}(\tau)$, which is the desired solution of problem (P). Before doing this, we note that, if $u \in \mathcal{P}^{2m,0}(\tau)$ is a solution of problem (P), then, as in (0.3.24), we can use equation (0.1.10), together with (0.1.23) of Lemma 0.1.1, to estimate u_t in $L^2(0, \tau; L^2)$ and thus deduce an inequality of the form

$$\|u\|_{\mathcal{P}^{2m,0}(\tau)} \leq F\left(\|u\|_{\mathcal{Y}^{2m,0}(\tau)}, \|\varphi\|_{\mathcal{Y}^{2m,0}(\tau)} \right) \,, \tag{0.3.79}$$

with $F \in \mathcal{K}$. This implies that in order to show that a solution of problem (P) is in $\mathcal{P}^{2m,0}(\tau)$ for some $\tau \in]0, T]$, it is sufficient to establish an a priori bound on the norm of u in $\mathcal{Y}^{2m,0}(\tau)$. In fact, (0.3.79) implies that if u is a solution of problem (P), then

$$u \in \mathcal{P}^{2m,0}(\tau) \iff u \in \mathcal{Y}^{2m,0}(\tau) \,. \tag{0.3.80}$$

1) If $u_0 = 0$, the function $u \equiv 0$ is the only solution of problem (P), with $f(0) \equiv 0$, and there is nothing more to prove. Thus, we can assume that $u_0 \neq 0$, so that $R := 4\,\|u_0\|_m > 0$. Denoting by C_* the norm of the imbedding $\mathcal{P}^{2m,0}(\tau) \hookrightarrow \mathcal{Y}^{2m,0}(\tau)$ (recall (0.3.7)), and with h_1 as in (0.3.61), we define

$$\kappa(R) := C_*\, h_1\left(1 + 4\,R,\ 1 + 2\,\|\varphi\|_{\mathcal{Y}^{2m,0}(T)},\ T\right) . \tag{0.3.81}$$

If $\varphi \not\equiv 0$, we fix $\gamma \in]0, 1[$ such that

$$0 < \gamma \leq \min\left\{ \|u_0\|_m, \|\varphi\|_{\mathcal{Y}^{2m,0}(T)} \right\} , \qquad \frac{4\,\gamma}{1-\gamma} \leq \frac{R}{\kappa(R)} \,, \tag{0.3.82}$$

and choose sequences $(u_0^n)_{n\geq 0} \subset H^{m+1}$ and $(\varphi^n)_{n\geq 0} \subset \mathcal{Y}^{2m+1,1}(T)$ which approximate the data u_0 and φ, in the sense that for all $n \geq 0$,

$$\|u_0^n - u_0\|_m \leq \gamma^{n+1}\,, \qquad \|\varphi^n - \varphi\|_{\mathcal{Y}^{2m,0}(T)} \leq \gamma^{n+1}\,. \tag{0.3.83}$$

Note that the first of (0.3.82) and (0.3.83), together with the fact that $\gamma \in]0,1[$, imply that $u_0^n \not\equiv 0$ for all n. If instead $\varphi \equiv 0$, we replace the first of (0.3.82) with $0 < \gamma \leq \|u_0\|_m$, and take $\varphi^n \equiv 0$ for all n. In the sequel, we assume $\varphi \not\equiv 0$. With u_0^n and φ^n as data, we resort to Theorem 0.3.2 to determine local solutions $u^n \in \mathcal{P}^{2m+1,1}(\tau_n)$ of problem (P), for some $\tau_n \in]0,T]$.

2) We now claim that there is $\tau \in]0,T]$ such that for all $n \geq 0$,

$$\tau_n \geq \tau\,, \qquad \|u^n\|_{\mathcal{Y}^{2m,0}(\tau)} \leq 2\,R\,. \tag{0.3.84}$$

To see this, consider the first approximation u^0, corresponding to the data u_0^0 and φ^0. Since the function $t \mapsto \|u^0\|_{\mathcal{Y}^{2m,0}(t)}$ is continuous and nondecreasing, there is $\tau \in]0,\tau_0]$ such that

$$\|u^0\|_{\mathcal{Y}^{2m,0}(\tau)} \leq 2\,\|u^0\|_{\mathcal{Y}^{2m,0}(0)}\,. \tag{0.3.85}$$

Recalling (0.3.5), by the first of (0.3.82) and (0.3.81) we see that

$$\begin{aligned} \|u^0\|_{\mathcal{Y}^{2m,0}(0)} &= \|u_0^0\|_m \leq \|u_0^0 - u_0\|_m + \|u_0\|_m \\ &\leq \gamma + \|u_0\|_m \leq 2\,\|u_0\|_m\,; \end{aligned} \tag{0.3.86}$$

thus, from (0.3.85),

$$\|u^0\|_{\mathcal{Y}^{2m,0}(\tau)} \leq 4\,\|u_0\|_m = R\,. \tag{0.3.87}$$

Note that τ depends on $\|u_0^0\|_{m+1}$ and $\|\varphi^0\|_{\mathcal{Y}^{2m+1,1}(T)}$; however, τ will remain fixed throughout the rest of our argument.

3) We now show that if $\tau_n < \tau$ the function u^n can be extended to $[0,\tau]$, with $u^n \in \mathcal{Y}^{2m,0}(\tau)$ (and, therefore, by (0.3.80), $u^n \in \mathcal{P}^{2m,0}(\tau)$). We achieve this by showing that any extension (see section 4.2 of Chapter 4) of u^n (still denoted by u^n) to larger intervals $[0,{\tau_n}']$, $\tau_n < {\tau_n}' \leq \tau$, satisfies the uniform bound

$$\|u^n\|_{\mathcal{Y}^{2m,0}({\tau_n}')} \leq 2\,R \tag{0.3.88}$$

(compare to (0.3.84)). We prove (0.3.88) by induction on n. At first, we note that (0.3.83) and (0.3.82) yield that for all $n \geq 0$,

$$\begin{aligned} \|u^n(0)\|_m &= \|u_0^n\|_m \leq \|u_0^n - u_0\|_m + \|u_0\|_m \\ &\leq \gamma^{n+1} + \|u_0\|_m \leq \gamma + \|u_0\|_m \leq 2\,\|u_0\|_m \leq R\,. \end{aligned} \tag{0.3.89}$$

Likewise,

$$\|\varphi^n\|_{\mathcal{Y}^{2m,0}(T)} \leq \|\varphi^n - \varphi\|_{\mathcal{Y}^{2m,0}(T)} + \|\varphi\|_{\mathcal{Y}^{2m,0}(T)} \leq 2\,\|\varphi\|_{\mathcal{Y}^{2m,0}(T)}\,. \tag{0.3.90}$$

For $n = 0$, we know that $\tau \le \tau_0 \le T$, and (0.3.88) follows from (0.3.87). Fix then $\nu \ge 0$, and assume that (0.3.88) holds for $0 \le n \le \nu$, and any ${\tau_n}' \in]\tau_n, \tau]$. If (0.3.88) did not hold for $n = \nu + 1$, there would exist ${\tau_{\nu+1}}' =: \theta_\nu \in]\tau_{\nu+1}, \tau]$ such that

$$4\,R \ge \|u^{\nu+1}\|_{\mathcal{Y}^{2m,0}(\theta_\nu)} > 2\,R\,. \tag{0.3.91}$$

On the other hand, the induction assumption (0.3.88) implies that for $0 \le n \le \nu$,

$$\|u^n\|_{\mathcal{Y}^{2m,0}(\theta_\nu)} \le \|u^n\|_{\mathcal{Y}^{2m,0}(\tau)} \le 2\,R < 4\,R\,. \tag{0.3.92}$$

We now refer to estimate (0.3.61), on the interval $[0, \theta_\nu]$, with u, $\tilde{u}$, φ, $\tilde{\varphi}$, u_0 and $\tilde{u}_0$ replaced by, respectively, u^n, u^{n-1}, φ^n, φ^{n-1}, u_0^n and u_0^{n-1}, $0 \le n \le \nu + 1$. By (0.3.91) and (0.3.92), recalling (0.3.62),

$$\begin{aligned} \rho_1^n &:= \max\left\{1, \|u^n\|_{\mathcal{P}^{2m,0}(\theta_\nu)}, \|u^{n-1}\|_{\mathcal{P}^{2m,0}(\theta_\nu)}\right\} \\ &\le \max\{1, 4\,R\} \le 1 + 4\,R\,. \end{aligned} \tag{0.3.93}$$

Likewise, by (0.3.90), recalling (0.3.63),

$$\begin{aligned} \delta_1^n &:= \max\left\{1, \|\varphi^n\|_{\mathcal{Y}^{2m,0}(T)}, \|\varphi^{n-1}\|_{\mathcal{Y}^{2m,0}(T)}\right\} \\ &\le 1 + 2\,\|\varphi\|_{\mathcal{Y}^{2m,0}(T)}\,. \end{aligned} \tag{0.3.94}$$

Thus, recalling also (0.3.81), and that $h_1 \in \mathcal{K}$,

$$h_1(\rho_1^n, \delta_1^n, T) \le \tfrac{1}{C_*}\,\kappa(R)\,. \tag{0.3.95}$$

From this, by (0.3.61) it follows that for $1 \le n \le \nu + 1$,

$$\begin{aligned} \|u^n - u^{n-1}\|_{\mathcal{Y}^{2m,0}(\theta_\nu)} &\le C_*\,\|u^n - u^{n-1}\|_{\mathcal{P}^{2m,0}(\theta_\nu)} \\ &\le \kappa(R)\,\left(\|u_0^n - u_0^{n-1}\|_m + \|\varphi^n - \varphi^{n-1}\|_{\mathcal{Y}^{2m,0}(T)}\right) \\ &\le \kappa(R)\,\Big(\|u_0^n - u_0\|_m + \|u_0^{n-1} - u_0\|_m \\ &\quad + \|\varphi^n - \varphi\|_{\mathcal{Y}^{2m,0}(T)} + \|\varphi^{n-1} - \varphi\|_{\mathcal{Y}^{2m,0}(T)}\Big) \\ &\le 2\,\kappa(R)\,(\gamma^{n+1} + \gamma^n) \le 4\,\kappa(R)\,\gamma^n\,. \end{aligned} \tag{0.3.96}$$

Since $\theta_\nu \le \tau$, (0.3.87) yields that

$$\|u^0\|_{\mathcal{Y}^{2m,0}(\theta_\nu)} \le \|u^0\|_{\mathcal{Y}^{2m,0}(\tau)} \le R\,; \tag{0.3.97}$$

thus, by (0.3.96) and the second of (0.3.82),

$$\begin{aligned}\|u^{\nu+1}\|_{\mathcal{Y}^{2m,0}(\theta_\nu)} &\le \sum_{n=1}^{\nu+1}\|u^n-u^{n-1}\|_{\mathcal{Y}^{2m,0}(\theta_\nu)}+\|u^0\|_{\mathcal{Y}^{2m,0}(\theta_\nu)}\\ &\le 4\,\kappa(R)\sum_{n=1}^{\nu+1}\gamma^n+\|u^0\|_{\mathcal{Y}^{2m,0}(\tau)} \qquad (0.3.98)\\ &\le 4\,\kappa(R)\,\frac{\gamma}{1-\gamma}+R\le 2\,R\,.\end{aligned}$$

Since (0.3.98) contradicts (0.3.91), we deduce that (0.3.88) holds for $n=\nu+1$ as well. Thus, (0.3.88) holds for all $n\ge 0$. As already stated, this means that each u^n can be extended to $[0,\tau]$, with $u^n\in\mathcal{P}^{2m,0}(\tau)$; in addition, since (0.3.88) is independent of ${\tau_n}'$, we deduce that for all n,

$$\|u^n\|_{\mathcal{P}^{2m,0}(\tau)}\le 2\,R\,. \qquad (0.3.99)$$

In conclusion, all the approximations u^n are defined on the common interval $[0,\tau]$, with $u^n\in\mathcal{P}^{2m,0}(\tau)$, and, by (0.3.99), $(u^n)_{n\ge0}$ is bounded in $\mathcal{P}^{2m,0}(\tau)$.

3: Convergence. The rest of the argument proceeds similarly to part (3) of the proof of Theorem 0.3.2. In fact, the essential steps of that proof are the convergence of the nonlinear terms M and N in (0.3.50) and (0.3.57), which follow from the strong convergence $u^n\to u$ in $\mathcal{Y}^{2m,0}(\tau_1)$, as per the second claim of (0.3.47), and (0.3.49). Here, acting as in (0.3.98), we see that

$$\sum_{n=1}^{\infty}\|u^n-u^{n-1}\|_{\mathcal{Y}^{2m,0}(\tau)}\le 4\,\kappa(R)\sum_{n=1}^{\infty}\gamma^n=4\,\kappa(R)\,\frac{\gamma}{1-\gamma}\,, \qquad (0.3.100)$$

and this does imply that the sequence $(u^n)_{n\ge0}$ converges strongly to a limit $u\in\mathcal{Y}^{2m,0}(\tau)$. We can then proceed exactly as in part (3) of the proof of Theorem 0.3.2, with τ_1 replaced by τ and (0.3.57) replaced by

$$N(f^n,(u^n)^{(m-1)})\to N(f,u^{(m-1)}) \quad \text{in} \quad L^2(0,\tau;L^2)\,, \qquad (0.3.101)$$

and deduce that u is the desired solution of problem (P) in $\mathcal{P}^{2m,0}(\tau)$. This concludes the proof of Theorem 0.3.1. □

0.3.3. Higher Regularity.

Just as for problem (H), higher regularity results for problem (P) can be established by a suitable generalization of Theorem 0.3.2.

Theorem 0.3.3. *Let $k\ge0$, and assume that $u_0\in H^{m+k}$, $\varphi\in\mathcal{Y}^{2m+k,k}(T)$. There is $\tau_k\in\,]0,T]$ such that problem* (P) *admits a unique solution $u\in\mathcal{P}^{2m+k,k}(\tau_k)$.*

The proof of this theorem follows the same arguments of the proof of Theorem 0.3.2, based on a priori estimates similar to the ones we establish in the proof of Proposition 0.3.4 below. Note that Theorems 0.3.1 and 0.3.2 correspond to Theorem 0.3.3 when $k = 0$ and $k = 1$, with $\tau_0 = \tau$. In this section we show that the regularity result of Theorem 0.3.3 is uniform in k, in the sense that $\inf_{k\geq 0} \tau_k \geq \tau$. Roughly speaking, this means that increasing the regularity of the data does not decrease the life-span of the solution. This is a consequence of the following time-independent a priori estimate:

Proposition 0.3.4. *Let $k \geq 0$, and $u_0 \in H^{m+k}$, $\varphi \in \mathcal{Y}^{2m+k,k}(T)$. Assume that problem* (P) *has a corresponding solution $u \in \mathcal{P}^{2m,0}(\tau) \cap \mathcal{P}^{2m+k,k}(\tau')$, with $0 < \tau' < \tau \leq T$. There exists Λ_k, depending on τ but not on τ', such that*

$$\|u\|_{\mathcal{P}^{2m+k,k}(\tau')} \leq \Lambda_k \,. \tag{0.3.102}$$

Consequently, $u \in \mathcal{P}^{2m+k,k}(\tau)$.

Sketch of Proof. By a natural extension of (0.3.80), it is sufficient to estimate $\|u\|_{\mathcal{Y}^{2m+k,k}(\tau')}$ independently of τ'. We proceed by induction on $k \geq 0$ and assume, for simplicity, that $\varphi \equiv 0$. If $k = 0$, Theorem 0.3.1 implies that we can take $\tau' = \tau$, and (0.3.102) holds, with (obviously) $\Lambda_0 = \|u\|_{\mathcal{P}^{2m,0}(\tau)}$. Thus, we can assume that $u_0 \in H^{m+k+1}$ and that, correspondingly, problem (P) has a solution $u \in \mathcal{P}^{2m+k,k}(\tau) \cap \mathcal{P}^{2m+k+1,k+1}(\tau')$, $0 < \tau' < \tau$, satisfying (0.3.102) with Λ_k independent of τ'. In particular,

$$\|u\|_{\mathcal{P}^{2m+k,k}(\tau)} \leq \Lambda_k \,. \tag{0.3.103}$$

By (0.3.7), (0.3.103) also implies that

$$\|u\|_{C([0,\tau];H^{m+k})} \leq C\,\Lambda_k \,. \tag{0.3.104}$$

We wish to show the existence of Λ_{k+1} independent of τ', such that (0.3.102) holds with k replaced by $k + 1$; that is, explicitly,

$$\|u\|_{\mathcal{P}^{2m+k+1,k+1}(\tau')} \leq \Lambda_{k+1} \,. \tag{0.3.105}$$

To this end we multiply equation (0.1.10) in L^2 by $\Delta^{m+k+1}u + \Delta^{k+1}u_t$ to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|^2_{\overline{m+k+1}} + \|u\|^2_{\overline{2m+k+1}} + \|u_t\|^2_{\overline{k+1}} \\ &\quad = \langle \nabla^{k+1} N(f, u^{(m-1)}), \nabla^{k+1}(\Delta^m u + u_t)\rangle \\ &\quad \leq \|N(f, u^{(m-1)})\|_{\overline{k+1}} \left(\|u\|_{\overline{2m+k+1}} + \|u_t\|_{\overline{k+1}}\right) \\ &\quad \leq \|N(f, u^{(m-1)})\|^2_{\overline{k+1}} + \tfrac{1}{2}\left(\|u\|^2_{\overline{2m+k+1}} + \|u_t\|^2_{\overline{k+1}}\right) . \end{aligned} \tag{0.3.106}$$

To estimate $N(f, u^{(m-1)})$ we distinguish three cases: 1) $k = 0$; 2) $m > 2$ and $k \geq 1$, or $m = 2$ and $k \geq 2$; 3) $m = 2$, $k = 1$.

1) If $k = 0$, we improve on the estimates of Theorem 0.3.2 as follows. We use (0.1.26) of Lemma 0.1.2, which, together with (0.1.47) for $h = 2$ and $h = 1$, yields

$$\|\nabla N(f, u^{(m-1)})\|_0 \leq C\,\|f\|_{\overline{m+2}}\,\|u\|_{\overline{m+1}}^{m-1} + C\,\|f\|_{\overline{m+1}}\,\|u\|_{\overline{m+2}}\,\|u\|_{\overline{m+1}}^{m-2}$$
$$\leq C\,\|u\|_m^{m-2}\,\|u\|_{\overline{m+1}}^{m+1} + C\,\|u\|_m^{m-1}\,\|u\|_{\overline{m+1}}^{m-1}\,\|u\|_{\overline{m+2}}\,. \tag{0.3.107}$$

By interpolation, and (0.3.104) for $k = 0$,

$$\begin{aligned} \|u\|_m^{m-2}\,\|u\|_{\overline{m+1}}^{m+1} &= \|u\|_m^{m-2}\,\|u\|_{\overline{m+1}}^{m}\,\|u\|_{\overline{m+1}} \\ &\leq C\,\|u\|_m^{2m-3}\,\|u\|_{\overline{2m}}\,\|u\|_{\overline{m+1}} \\ &\leq C\,\Lambda_0^{2m-3}\,\|u\|_{\overline{2m}}\,\|u\|_{\overline{m+1}}\,; \end{aligned} \tag{0.3.108}$$

likewise,

$$\begin{aligned} \|u\|_m^{m-1}\,\|u\|_{\overline{m+1}}^{m-1}\,\|u\|_{\overline{m+2}} &= \|u\|_m^{m-1}\,\|u\|_{\overline{m+1}}^{m-2}\,\|u\|_{\overline{m+1}}\,\|u\|_{\overline{m+2}} \\ &\leq C\,\|u\|_m^{2m-3}\,\|u\|_{\overline{m+1}}\,\|u\|_{\overline{2m}} \\ &\leq C\,\Lambda_0^{2m-3}\,\|u\|_{\overline{2m}}\,\|u\|_{\overline{m+1}}\,. \end{aligned} \tag{0.3.109}$$

Inserting (0.3.108) and (0.3.109) into (0.3.106) for $k = 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_{\overline{m+1}}^2 + \tfrac{1}{2}\left(\|u\|_{\overline{2m+1}}^2 + \|\nabla u_t\|_0^2\right) \leq C\,\Lambda_0^{2(2m-3)}\,\|u\|_{\overline{2m}}^2\,\|u\|_{\overline{m+1}}^2\,. \tag{0.3.110}$$

From this, by Gronwall's inequality, and recalling (0.3.103) for $k = 0$, we obtain that for all $t \in [0, \tau']$,

$$\begin{aligned} &\|u(t)\|_{\overline{m+1}}^2 + \tfrac{1}{2}\int_0^t \left(\|u\|_{\overline{2m+1}}^2 + \|\nabla u_t\|_0^2\right)\,\mathrm{d}\theta \\ &\quad\leq \|u_0\|_{m+1}^2\,\exp\left(C\,\Lambda_0^{2(2m-3)}\int_0^\tau \|u\|_{2m}^2\,\mathrm{d}\theta\right) \\ &\quad\leq \|u_0\|_{m+1}^2\,\exp\left(C\,\Lambda_0^{4(m-1)}\right)\,. \end{aligned} \tag{0.3.111}$$

Since the right side of (0.3.111) is independent of τ', and we already know that $u \in \mathcal{P}^{2m,0}(\tau)$, we conclude that (0.3.105) does hold for $k = 0$.

2) If $m > 2$ and $k \geq 1$, or $m = 2$ and $k \geq 2$, we estimate $N(f, u^{(m-1)})$ by means of (0.1.29) of Lemma 0.1.3 with $h = k + 1$, obtaining

$$\|N(f, u^{(m-1)})\|_{\overline{k+1}} \leq C\,\max\left\{\|f\|_{\overline{m}}, \|f\|_{\overline{m+k+1}}\right\}\,\|u\|_{m+k+1}^{m-1}\,. \tag{0.3.112}$$

By (0.1.44) and (0.3.104),

$$\|f\|_{\overline{m}} \le C\,\|u\|_m^m \le C\,\Lambda_0^m\,; \tag{0.3.113}$$

likewise, by (0.1.49),

$$\|f\|_{\overline{m+k+1}} \le C\,\|u\|_m^{m-1}\,\|u\|_{m+k+1} \le C\,\Lambda_0^{m-1}\,\|u\|_{m+k+1}\,. \tag{0.3.114}$$

Inserting (0.3.113) and (0.3.114) into (0.3.112), by Hölder's inequality and interpolation we deduce that

$$\begin{aligned} \|N(f,u^{(m-1)})\|_{\overline{k+1}} &\le C\,\Lambda_0^m\,\|u\|_{m+k+1}^{m-1} + C\,\Lambda_0^{m-1}\,\|u\|_{m+k+1}^m \\ &\le C\,\Lambda_0^{2m-1} + 2\,C\,\Lambda_0^{m-1}\,\|u\|_{m+k+1}^m \\ &\le C\,\Lambda_0^{2m-1} + 2\,C\,\Lambda_0^{m-1}\,\|u\|_{m+k}^{m-1}\,\|u\|_{2m+k}\,. \end{aligned} \tag{0.3.115}$$

Replacing this into (0.3.106) and recalling (0.3.104) we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_{\overline{m+k+1}}^2 + \tfrac{1}{2}\,\Big(\|u\|_{\overline{2m+k+1}}^2 + \|u_t\|_{\overline{k+1}}^2\Big) \\ &\qquad \le\; C\,\Lambda_0^{2(2m-1)} + C\,\Lambda_0^{2(m-1)}\,\Lambda_k^{2(m-1)}\,\|u\|_{2m+k}^2\,, \end{aligned} \tag{0.3.116}$$

from which, by (0.3.103), we deduce that for all $t \in [0,\tau']$,

$$\begin{aligned} &\|u(t)\|_{\overline{m+k+1}}^2 + \tfrac{1}{2}\int_0^t \Big(\|u\|_{\overline{2m+k+1}}^2 + \|u_t\|_{\overline{k+1}}^2\Big)\,\mathrm{d}\theta \\ &\quad \le\; \|u_0\|_{m+k+1}^2 + C\,\Lambda_0^{2(2m-1)}\,T \\ &\qquad + C\,\Lambda_0^{2(m-1)}\,\Lambda_k^{2(m-1)}\int_0^\tau \|u\|_{2m+k}^2\,\mathrm{d}\theta \\ &\quad \le\; \|u_0\|_{m+k+1}^2 + C\,\Lambda_0^{2(2m-1)}\,T + C\,\Lambda_0^{2(m-1)}\,\Lambda_k^{2m-1}\,. \end{aligned} \tag{0.3.117}$$

Again, the right side of (0.3.117) is independent of τ', and we already know that $u \in \mathcal{P}^{2m,0}(\tau)$; thus, we conclude that (0.3.105) holds.

3) Finally, if $m = 2$ and $k = 1$, we use (0.1.30) of Lemma 0.1.3, to obtain

$$\|\Delta N(f,u^{(m-1)})\|_0 \le C\,\max\{\|f\|_{\overline{2}}, \|f\|_{\overline{5}}\}\,\max\{\|u\|_{\overline{2}}, \|u\|_{\overline{4}}\}\,. \tag{0.3.118}$$

Acting as before, by (0.1.47), (0.1.48) with $h = 3$, and (0.3.104),

$$\|f\|_{\overline{2}} \le C\,\|u\|_{\overline{2}}^2 \le C\,\Lambda_0^2\,, \tag{0.3.119}$$

$$\|f\|_{\overline{5}} \le C\,\max\{\|u\|_{\overline{4}}, \|u\|_{\overline{2}}\} \le C\,\max\{\|u\|_{\overline{4}}^2, \Lambda_0^2\}\,; \tag{0.3.120}$$

thus, from (0.3.118),

$$\|\Delta N(f,u^{(m-1)})\|_0 \le C\,\Lambda_0^3 + C\,\|u\|_{\overline{4}}^3\,. \tag{0.3.121}$$

Inserting this into (0.3.106) and using interpolation we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_4^2+\tfrac12\,\big(\|u\|_6^2+\|u_t\|_2^2\big)\\
&\quad\le\; C\,\Lambda_0^6+C\,\|u\|_4^4\,\|u\|_4^2 \qquad\qquad (0.3.122)\\
&\quad\le\; C\,\Lambda_0^6+C\,\|u\|_3^2\,\|u\|_5^2\,\|u\|_4^2\,.
\end{aligned}
$$

Integrating, by (0.3.104) with $k=1$ and Gronwall's inequality, we obtain that for all $t\in[0,\tau']$,

$$
\begin{aligned}
&\|u(t)\|_4^2+\tfrac12\int_0^t\big(\|u\|_6^2+\|u_t\|_2^2\big)\,\mathrm{d}\theta\\
&\quad\le\;\big(\|u_0\|_4^2+C\,\Lambda_0^6\,T\big)\,\exp\left(C\,\Lambda_1^2\int_0^\tau\|u\|_5^2\,\mathrm{d}\theta\right) \qquad\qquad (0.3.123)\\
&\quad\le\;\big(\|u_0\|_4^2+C\,\Lambda_0^6\,T\big)\,\mathrm{e}^{C\,\Lambda_1^4}\,,
\end{aligned}
$$

which allows us, as above, to deduce that (0.3.105) also holds for $m=2$ and $k=1$. This concludes the proof of Proposition 0.3.4. □

0.3.4. Almost Global Existence.

In this section we prove an almost global existence result for problem (P), in the same spirit of the one given in Theorem 0.2.3 for problem (H); the proof, however, is somewhat different.

Theorem 0.3.4. *In the same assumptions of Theorem* 0.3.1*, given arbitrary* $T>0$ *there is* $\delta>0$*, depending only on* $\|\varphi\|_{\mathcal{Y}^{2m,0}(T)}$*, such that if* $\|u_0\|_m\le\delta$*, problem* (P) *admits a unique solution* $u\in\mathcal{P}^{2m,0}(T)$*, with* $f(u)\in\mathcal{Y}^{2m,0}(T)$*.*

Proof. 1) We follow a standard ODE extension method, similar to the one we described in the first sections of Chapter 4, to which we refer for details. In brief, the argument runs as follows. The local existence Theorem 0.3.1 yields a solution $u\in\mathcal{P}^{2m,0}(\tau)$ of problem (P), for some $\tau\in\,]0,T]$. If $\tau<T$, by repeated applications of Theorem 0.3.1 we can extend (see section 4.2) u to a sequence, possibly finite, of expanding intervals $[0,\tau_n]\subseteq[0,T]$, $\tau_1:=\tau<\tau_2<\dots$, with $u\in\mathcal{P}^{2m,0}(\tau_n)$ (with slight abuse of notation, we keep denoting by u the successive extensions of the local solution; this is justified by the fact that strong solutions of problem (P) are unique wherever they are defined). If there is $n>1$ such that $\tau_n=T$, there is nothing more to prove, because this means that $u\in\mathcal{P}^{2m,0}(T)$, as desired. Otherwise, the essential point to note is that, ultimately, the dependence of each length $\tau_{n+1}-\tau_n$ (with $\tau_0=0$) on n depends only on $\|u(\tau_n)\|_m$. In particular, for $n=0$, the dependence of $\tau_1=\tau$ on $\|u_0\|_m$ is seen in (0.3.85), (0.3.86), and (0.3.87).

From this observation it follows that, in order to show that we can extend the local solution of problem (P) to a global one, it is sufficient to prove that we can bound the norm of u in $C([0,\tau];H^m)$, $\tau \in]0,T]$, independently of τ.

2) To this end, we set $C_\varphi := C\,\|\varphi\|^{m-1}_{C([0,T];H^m)}$, where C is the universal constant appearing in (0.3.130) below; we define

$$M_\varphi := \exp\left(\tfrac{1}{2}\,C_\varphi\left(T + C_\varphi\int_0^T \|\varphi\|^2_{2m}\,\mathrm{d}\theta\right)\right)\,, \tag{0.3.124}$$

and we claim that there is $\delta \in]0,1]$ such that for all $\tau \in]0,T]$ for which problem (P) has a solution $u \in \mathcal{P}^{2m,0}(\tau)$, with $\|u(0)\|_m \le \delta$, u satisfies the estimate

$$\max_{0\le t\le\tau} \|u(t)\|_m \le M_\varphi\,\delta\,. \tag{0.3.125}$$

Since the right side of (0.3.125) is independent of τ, this yields the desired time-independent estimate on $\|u(\,\cdot\,)\|_m$.

3) We prove our claim by contradiction. Thus, assume that for all $\delta \in]0,1]$ there is $\tau_\delta \in]0,T]$ such that problem (P) has a solution $u \in \mathcal{P}^{2m,0}(\tau_\delta)$, with $\|u(0)\|_m \le \delta$ but

$$M_\delta := \max_{0\le t\le\tau_\delta} \|u(t)\|_m > M_\varphi\,\delta\,. \tag{0.3.126}$$

If $M_\delta > 2\,M_\varphi\,\delta$, noting that $\|u(0)\|_m \le \delta < M_\varphi\,\delta$ we deduce by continuity that there is $\theta_\delta \in]0,\tau_\delta]$ such that for all $t \in [0,\theta_\delta]$,

$$\|u(t)\|_m \le 2\,M_\varphi\,\delta = \|u(\theta_\delta)\|_m\,. \tag{0.3.127}$$

If instead $M_\delta \le 2\,M_\varphi\,\delta$, we set $\theta_\delta := \tau_\delta$, so that the first inequality of (0.3.127) still holds. We now multiply equation (0.1.10) in L^2 by $2\,(\Delta^m u + u)$, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|^2_m + 2\,\|u\|^2_{\overline{2m}} + 2\,\|u\|^2_{\overline{m}} \\ &\qquad = \; 2\,\langle N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u), \Delta^m u + u\rangle\,, \end{aligned} \tag{0.3.128}$$

where, as usual, $f := f(u)$. By (0.1.23), (0.1.47) and interpolation,

$$\begin{aligned} 2\,|\langle N(f,u^{(m-1)}),\Delta^m u\rangle| &\le 2\,\|N(f,u^{(m-1)})\|_0\,\|u\|_{\overline{2m}} \\ &\le \; C\,\|f\|_{\overline{m+1}}\,\|u\|^{m-1}_{\overline{m+1}}\,\|u\|_{\overline{2m}} \le C\,\|u\|^{m-1}_m\,\|u\|^m_{\overline{m+1}}\,\|u\|_{\overline{2m}} \\ &\le \; C\,\|u\|^{2(m-1)}_m\,\|u\|^2_{\overline{2m}}\,. \end{aligned} \tag{0.3.129}$$

Likewise,

(0.3.130)

$$2\,|\langle N(\varphi^{(m-1)},u),\Delta^m u\rangle| \le 2\,\|N(\varphi^{(m-1)},u)\|_0\,\|u\|_{\overline{2m}}$$

$$\le\ C\,\|\varphi\|_{m+1}^{m-1}\,\|u\|_{\overline{m+1}}\,\|u\|_{\overline{2m}} \le C\,\|\varphi\|_m^{\frac{(m-1)^2}{m}}\ \|\varphi\|_{2m}^{\frac{m-1}{m}}\ \|u\|_m^{\frac{m-1}{m}}\ \|u\|_{\overline{2m}}^{\frac{m+1}{m}}$$

$$\le\ C_\varphi^2\,\|\varphi\|_{2m}^2\,\|u\|_m^2 + \|u\|_{\overline{2m}}^2\,.$$

Next, recalling (0.1.2), by (0.1.25),

$$\begin{aligned} &2\,|\langle N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u),u\rangle| \\ &\quad\le\ 2\,|I(f,u^{(m)})| + 2\,|I(\varphi^{(m-1)},u,u)| \\ &\quad\le\ C\,\|f\|_{\overline{m}}\,\|u\|_{\overline{m}}^m + C\,\|\varphi\|_m^{m-1}\,\|u\|_{\overline{m}}^2 \\ &\quad\le\ C\,\|u\|_{\overline{m}}^{2m} + C_\varphi\,\|u\|_m^2\,. \end{aligned} \tag{0.3.131}$$

Inserting (0.3.129), (0.3.130), and (0.3.131) into (0.3.128), and recalling the inequality of (0.3.127), we obtain that for $t\in[0,\theta_\delta]$,

(0.3.132)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_m^2 + \|u\|_{\overline{2m}}^2 + \|u\|_{\overline{m}}^2$$

$$\le\ C_*\,\|u\|_m^{2(m-1)}\left(\|u\|_{\overline{2m}}^2 + \|u\|_{\overline{m}}^2\right) + C_\varphi\left(1 + C_\varphi\,\|\varphi\|_{2m}^2\right)\,\|u\|_m^2$$

$$\le\ C_*\,(2M_\varphi\delta)^{2(m-1)}\left(\|u\|_{\overline{m}}^2 + \|u\|_{\overline{2m}}^2\right) + C_\varphi\left(1 + C_\varphi\,\|\varphi\|_{2m}^2\right)\,\|u\|_m^2\,.$$

Thus, if we choose $\delta\in\,]0,1]$ so small that

$$C_*\,(2M_\varphi\delta)^{2(m-1)} \le 1\,, \tag{0.3.133}$$

we deduce from (0.3.132) that for all $t\in[0,\theta_\delta]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_m^2 \le C_\varphi\left(1 + C_\varphi\,\|\varphi\|_{2m}^2\right)\,\|u\|_m^2\,, \tag{0.3.134}$$

and, by Gronwall's inequality,

$$\|u(t)\|_m^2 \le \|u_0\|_m^2\,\exp\left(C_\varphi\left(T + C_\varphi\int_0^T \|\varphi\|_{2m}^2\,\mathrm{d}\theta\right)\right) \le M_\varphi^2\,\delta^2\,. \tag{0.3.135}$$

If $\theta_\delta = \tau_\delta$, (0.3.135) contradicts (0.3.126), while if $\theta_\delta < \tau_\delta$, (0.3.135) for $t = \theta_\delta$ contradicts (0.3.127). Consequently, (0.3.125) holds, and we can conclude the proof of Theorem 0.3.4. □

Bibliography

[1] M. S. Berger, *On Von Karman's Equations and the Buckling of a Thin Elastic Plate, I.* Comm. Pure Appl. Math., 20 (1967), 687-719.

[2] P. Cherrier and A. Milani, *Equations of Von Karman Type on Compact Kähler Manifolds.* Bull. Sci. Math., 2e série, 116 (1992), 325-352.

[3] P. Cherrier and A. Milani, *Parabolic Equations of Von Karman Type on Compact Kähler Manifolds.* Bull. Sci. Math., 131 (2007), 375-396.

[4] P. Cherrier and A. Milani, *Parabolic Equations of Von Karman Type on Compact Kähler Manifolds, II.* Bull. Sci. Math., 133 (2009), 113-133.

[5] P. Cherrier and A. Milani, *Hyperbolic Equations of Von Karman Type on Compact Kähler Manifolds.* Bull. Sci. Math., 136 (2012), 19-36.

[6] I. Chuesov and I. Lasiecka, *Von Karman Evolution Equations: Well-Posedness and Long-Time Dynamics.* Springer-Verlag, New York, 2010.

[7] P. Ciarlet, *A Justification of the von Karman Equations.* Arch. Rat. Mech. Anal., 73 (1980), 349-389.

[8] P. Ciarlet, *Mathematical Elasticity, Theory of Shells.* North-Holland, Amsterdam, 2000.

[9] P. Ciarlet and P. Rabier, *Les Equations de von Karman.* Springer-Verlag, Berlin, 1980.

[10] A. Favini and M. A. Horn and I. Lasiecka and D. Tataru, *Global Existence of Solutions to a von Karman System with Nonlinear Boundary Dissipation"* J. Diff. Int. Eqs., 9/2 (1996), 267-294.

[11] A. Favini and M. A. Horn and I. Lasiecka and D. Tataru, *Addendum to the Paper "Global Existence of Solutions to a von Karman System with Nonlinear Boundary Dissipation"* J. Diff. Int. Eqs., 10/1 (1997), 197-200.

[12] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires.* Dunod-Gauthier-Villars, Paris, 1969.