Lecture Notes on Functional Analysis

Review of Notation and Solutions to Homework Problems

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Review of main notation

$\mathbb{R}$ the field of real numbers.

$\mathbb{C}$ the field of complex numbers.

$\mathbb{K}$ a field of numbers, either $\mathbb{R}$ or $\mathbb{C}$.

$Re\, z$, $Im\, z$ the real and imaginary part of a complex number z.

$\bar{z} = a - ib$ the complex conjugate of the number $z = a + ib \in \mathbb{C}$.

$[a, b]$ a closed interval, $]a, b[$ an open interval, $]a, b]$, $[a, b[$ half-open intervals.

$\mathbb{R}^n$ the n-dimensional Euclidean space.

$\langle \cdot, \cdot \rangle$ scalar product on the Euclidean space $\mathbb{R}^n$.

$|v| \doteq \sqrt{(v, v)}$ the Euclidean length of a vector $v \in \mathbb{R}^n$.

$A \setminus B \doteq \{x \in A\,,\ \ x \notin B\}$ a set-theoretic difference.

$\overline{A}$ the closure of a set A.

∂A the boundary of a set A.

$\chi_{_A}$ the indicator function of a set A. $\quad \chi_{_A}(x) = \begin{cases} 1 & \text{if } x \in A\,, \\ 0 & \text{if } x \notin A\,. \end{cases}$

$f : A \mapsto B$ a mapping from a set A into a set B,

$a \mapsto b = f(a)$ the function f maps the element $a \in A$ to the element $b \in B$.

$\doteq$ equal, by definition.

$\Longleftrightarrow$ if and only if

$\mathcal{C}(E) = \mathcal{C}(E, \mathbb{R})$ vector space of all continuous, real valued functions on the metric space E.

$\mathcal{C}(E, \mathbb{C})$ vector space of all continuous, complex valued functions on the metric space E.

$\mathcal{B}C(E)$ space of all bounded, continuous, real valued functions $f : E \mapsto \mathbb{R}$, with norm $\|f\| = \sup_{x\in E} |f(x)|$.

ℓ^1, ℓ^p, ℓ^∞ spaces of sequences of real (or complex) numbers.

$\mathbf{L}^1(\Omega)$, $\mathbf{L}^p(\Omega)$, $\mathbf{L}^\infty(\Omega)$ Lebesgue spaces.

$W^{k,p}(\Omega)$ Sobolev space of functions whose weak partial derivatives up to order k lie in $\mathbf{L}^p(\Omega)$, for some open set $\Omega \subseteq \mathbb{R}^n$.

$H^k(\Omega) = W^{k,2}(\Omega)$ Hilbert-Sobolev space.

$\mathcal{C}^{k,\gamma}(\Omega)$ Hölder space of functions $u : \Omega \mapsto \mathbb{R}$ whose derivatives up to order k are Hölder continuous with exponent $\gamma \in]0, 1]$.

$\|\cdot\| = \|\cdot\|_X$ norm on a vector space X.

$(\cdot, \cdot) = (\cdot, \cdot)_H$ inner product on a Hilbert space H.

X^* is the dual space of X, i.e. the space of all continuous linear functionals $x^* : X \mapsto \mathbb{K}$.

$\langle x^*, x\rangle = x^*(x)$ duality product of $x^* \in X^*$ and $x \in X$.

$x_n \to x$ strong convergence in norm; this means $\|x_n - x\| \to 0$.

$x_n \rightharpoonup x$ weak convergence.

$\varphi_n \overset{*}{\rightharpoonup} \varphi$ weak-star convergence.

$f * g$ convolution of two functions $f, g : \mathbb{R}^n \mapsto \mathbb{R}$.

$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ gradient of a function $u : \mathbb{R}^n \mapsto \mathbb{R}$.

$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$ a partial differential operator of order $|\alpha| \doteq \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

$meas(\Omega)$ the Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$.

$\fint_\Omega f\,dx \doteq \frac{1}{meas(\Omega)} \int_\Omega f\,dx$ the average value of f over the set Ω.

Solutions to Homework Problems

Chapter 2

1. (i) It is not a normed space. (N2) fails when $\lambda < 0$.

(ii) It is a normed space but not complete. The sequence $\mathbf{x}_n = \Big(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\Big)$ is Cauchy but does not have a limit in X.

(iii) It is a normed space but not complete. The sequence $p_n(x) \doteq \sum_{k=0}^{n} \frac{x^n}{n!}$ is Cauchy, but does not converge to any element of X. Indeed, its uniform limit on the interval $[0,1]$ is the function $f(x) = e^x$, which is not a polynomial.

(iv) It is a normed space of dimension 3. It is also a Banach space. Every finite dimensional normed space is complete.

(v) It is a normed space, but not complete. Indeed, consider the functions

$$f(x) \doteq \begin{cases} 0 & \text{if } x \in \left[0,\ \frac{1}{2}\right], \\ 1 & \text{if } x \in \left]\frac{1}{2},\ 1\right], \end{cases} \qquad f_n(x) \doteq \begin{cases} 0 & \text{if } x \in \left[0,\ \frac{1}{2}\right], \\ n\Big(x - \frac{1}{2}\Big) & \text{if } x \in \left[\frac{1}{2},\ \frac{1}{2} + \frac{1}{n}\right], \\ 1 & \text{if } x \in \left[\frac{1}{2} + \frac{1}{n},\ 1\right]. \end{cases}$$

Then the sequence of continuous functions $(f_n)_{n\geq 1}$ is a Cauchy sequence without any limit in the space X. Namely, $\|f - f_n\|_{\mathbf{L}^1([0,1])} = \frac{1}{2n} \to 0$ as $n \to \infty$, but $f \notin X$.

(vi) For $p < 1$ this is not a norm, because (N3) fails. For example, in $\mathbb{R}^2$ take $x = (1,0)$, $y = (0,1)$. Then $\|x\| = \|y\| = 1$, but $\|x + y\| = 2^{1/p} > 2$.

2. (N1) $\|(x,y)\| = \max\{\|x\|,\ \|y\|\} \geq 0$.

$\|(x,y)\| = 0$ if and only if $\max\{\|x\|,\ \|y\|\} = 0$ if and only if $\|x\| = \|y\| = 0$ if and only if $x = y = 0$.

(N2) $\|(\lambda x, \lambda y)\| = \max\{\|\lambda x\|,\ \|\lambda y\|\} = \max\{|\lambda|\,\|x\|,\ |\lambda|\,\|y\|\} = |\lambda| \max\{\|\lambda x\|,\ \|\lambda y\|\} = |\lambda|\,\|(x,y)\|$.

(N3) $\|(x+\tilde{x},\ y+\tilde{y})\| = \max\{\|x+\tilde{x}\|,\ \|y+\tilde{y}\|\} \leq \max\{\|x\|+\|\tilde{x}\|\,,\ \|y\|+\|\tilde{y}\|\} \leq \max\{\|x\|,\ \|y\|\} + \max\{\|\tilde{x}\|,\ \|\tilde{y}\|\} = \|(x,y)\| + \|(\tilde{x},\tilde{y})\|$.

To prove that $X \times Y$ is a Banach space, let $(x_n, y_n)_{n\geq 1}$ be a Cauchy sequence in $X \times Y$. Then $(x_n)_{n\geq 1}$ is a Cauchy sequence in X and $(y_n)_{n\geq 1}$ is a Cauchy sequence in Y. Since X, Y are complete, there exist the limits $x_n \to x$, $y_n \to Y$. This implies $(x_n, y_n) \to (x,y)$, showing that $X \times Y$ is complete as well.

3. (i) Let $(\bar{x}, \bar{y}) \in X \times X$ and $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/2$. If $\|(x,y) - (\bar{x}, \bar{y})\| \leq \delta$ then

$$\|(x+y) - (\bar{x} + \bar{y})\| \;\leq\; \|x - \bar{x}\| + \|y - \bar{y}\| \;\leq\; \delta + \delta \;=\; \varepsilon.$$

This proves that the map $(x,y) \mapsto x + y$ is continuous at the point $(\bar{x}, \bar{y})$.

(ii) Let $\alpha \in \mathbb{K}$, $x \in X$, and $\varepsilon > 0$ and be given. Choose

$$\delta \;\doteq\; \min\left\{1,\; \frac{\varepsilon/2}{1 + \|x\|},\; \frac{\varepsilon/2}{1 + |\alpha|}\right\}.$$

Assume $|\beta - \alpha| \leq \delta$, $\|y - x\| \leq \delta$. Then $\|y\| \leq 1 + \|x\|$ and

$$\|\beta y - \alpha x\| \;\leq\; \|\beta y - \alpha y\| + \|\alpha y - \alpha x\| \;\leq\; |\beta - \alpha|\,(1 + \|x\|) + |\alpha|\,\|y - x\| \;\leq\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon\,.$$

4. If V is a subspace of X it is clear that the properties (N1)–(N3) of the norm remain valid. Now assume that X is a Banach space and that V is a closed subset of X. If $(v_n)_{n\geq 1}$ is a Cauchy sequence in V, then it is also a Cauchy sequence in X. Since X is complete, one has the convergence $x_n \to x$ for some point $x \in X$. Since V is closed, $x \in V$, proving that V is complete as well.

5. Let X be a Banach space and assume that $\sum_{n\geq 1} \|x_n\| < \infty$. Consider the sequence of partial sums $y_n \doteq \sum_{k=1}^n x_k$. This is a Cauchy sequence. Indeed, for $m < n$ we have

$$\|y_m - y_n\| \;=\; \left\|\sum_{m<k\leq n} x_k\right\| \;\leq\; \sum_{m<k\leq n} \|x_k\| \;\to\; 0 \qquad\qquad \text{as} \;\; m, n \to \infty.$$

Since X complete, this sequence has a limit: $y_n \to y = \sum_{k=1}^{\infty} x_k$.

Viceversa, assume that every absolutely convergent series of elements of X has a sum. Let $(y_n)_{n\geq 1}$ be a Cauchy sequence of elements in X. For each integer $k \geq 1$, choose an index N_k such that

$$\|y_m - y_n\| \;\leq\; 2^{-k} \qquad\qquad \text{whenever } m, n \geq N_k\,.$$

It is not restrictive to assume $N_1 < N_2 < N_3 < \cdots$. Consider the series $\sum_k x_k$, where

$$x_1 = y_{N_1}, \quad x_2 = y_{N_2} - y_{N_1}, \qquad \cdots \qquad x_k = y_{N_k} - N_{k-1}, \quad \cdots$$

This is absolutely convergent, because $\|x_k\| \leq 2^{1-k}$ for all $k \geq 2$. By assumption, there exist the sum $y = \sum_k x_k$. We claim that the sequence $(y_n)_{n\geq 1}$ converges to y. Indeed, let $\varepsilon > 0$ be given and choose k large enough so that $2^{2-k} < \varepsilon$. If $n \geq N_k$ then

$$\|y_n - y\| \;\leq\; \|y - y_{N_k}\| + \|y_{N_k} - y\| \;\leq\; 2^{-k} + \sum_{\nu \geq N_k} \|x_\nu\| \;\leq\; 2^{-k} + 2^{1-k} \;<\; \varepsilon.$$

6. (i) Let $x', x'' \in coA$, and assume $\lambda \in [0,1]$. We need to prove that $x = \lambda x' + (1-\lambda)x'' \in A$. By assumption we have

$$x' = \sum_{k=1}^{N'} \theta'_k a'_k\,, \qquad\qquad x'' = \sum_{k=1}^{N''} \theta''_k a''_k.$$

Choosing

$$N = N' + N'', \qquad\qquad \theta_k = \begin{cases} \lambda\theta'_k & \text{if} \quad k = 1, \ldots, N', \\ (1-\lambda)\theta''_{k-N'} & \text{if} \quad k = N'+1, \ldots, N'+N'', \end{cases}$$

we have the representation $x = \sum_{k=1}^{N} \theta_k a_k \in coA$.

(ii) Let $x = \sum_{k=1}^{N} \theta_k a_k \in coA$ and let K be any convex set containing A. Then K contains all elements $a_1, \ldots, a_N$. Being convex, K must also contain all convex combinations of these elements. In particular, $x \in K$.

The converse inclusion is trivial: since coA is a convex set containing A, the intersection of all convex sets that contain A cannot be larger than coA.

7. (i) Let A be open and consider any point $x = \sum_{k=1}^{N} \theta_k a_k \in coA$. We can assume $\theta_1 > 0$. Since $a_1 \in A$ and A is open, there exists a radius $r > 0$ such that $B(a_1, r) \subset A$. Then

$$B(x,\ \theta_1 r) \;=\; \Big\{\theta_1 y + \sum_{k=2}^{N} \theta_k a_k\,; \quad y \in B(a_1, r)\Big\} \;\subset\; coA$$

This shows that x lies in the interior of coA, hence coA is open.

(ii) Let A be bounded, say $A \subseteq B(0, R)$ for some radius $R > 0$. Then $coA \subseteq B(0, R)$ because $B(0, R)$ is a bounded convex set containing A. Hence coA is bounded.

(iii) If $A \subseteq B(0, r_1)$ and $B \subseteq B(0, r_2)$, then $A + B \subseteq B(0,\ r_1 + r_2)$, showing that $A + B$ is bounded.

(iv) Let $(x_n)_{n\geq 1}$ be a sequence of points in $A+B$, with $x_n \to x$ as $n \to \infty$. We need to prove that $x \in A + B$. By assumption $x_n = a_n + b_n$, for some $a_n \in A$ and $b_n \in B$. Since B is compact, by possibly choosing a subsequence and relabeling we can assume $b_n \to b$ for some $b \in B$. We now have

$$\limsup_{m,n\to\infty} \|a_m - a_n\| \;=\; \limsup_{m,n\to\infty} \|x_m - b_m - (x_n - b_n)\| \;\leq\; \limsup_{m,n\to\infty} \Big(\|x_m - x_n\| + \|b_m - b_n\|\Big) \;=\; 0.$$

Indeed, the sequences $(x_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are both convergent, hence they are Cauchy sequences. This shows that the sequence $(a_n)_{n\geq 1}$ is Cauchy as well. Hence there exists the limit $a_n \to a$ for some $a \in A$, because X is a Banach space and $A \subseteq X$ is closed. We conclude that $x = a + b \in A + B$, proving that the sum $A + B$ is closed.

(v) In $\mathbb{R}^2$, consider the two closed subsets

$$A \;\doteq\; \{(x,y)\,; \quad x > 0,\ y > 0, \quad xy \geq 1\}, \qquad\qquad B \;\doteq\; \{(x,y)\,; \quad x < 0,\ y > 0, \quad xy \leq -1\}.$$

Then the set

$$A + B \;=\; \big\{(x,y)\,; \quad y > 0\big\}$$

is open, not closed.

(vi) The inclusion $A + A \subseteq 2A$ is always trivially true. On the other hand, if A is convex, then any element $a + a' \in A + A$ can be written as $\frac{a+a'}{2} + \frac{a+a'}{2} \in 2A$. Hence $A + A \subseteq 2A$.

(vii) Assume that A is closed and $A + A = 2A$. Let $a, b \in A$. By assumption there exists $c \in A$ such that $a + b = 2c$. This implies $\frac{a+b}{2} \in A$. Repeating the argument, we obtain that

$$\frac{1}{2} \cdot \frac{a+b}{2} + \frac{1}{2} \cdot b \;=\; \frac{1}{4}a + \frac{3}{4}b \;\in\; A, \qquad\qquad \frac{1}{2} \cdot a + \frac{1}{2} \cdot \frac{a+b}{2} \;=\; \frac{3}{4}a + \frac{1}{4}b \;\in\; A\,.$$

By induction, we obtain

$$\frac{k}{2^n}a + \left(1 - \frac{k}{2^n}\right) b \;\in\; A \qquad\qquad \text{for all} \quad k = 0, 1, \ldots 2^n. \tag{1}$$

Since the set of all convex combinations of the form (1) are dense in the set of all convex combinations, and we are assuming that A is closed, we conclude that A is convex.

Notice that the result would fail if A were not closed. For example, let A be the set of all rational numbers contained inside $[0, 1]$.

8. (i) Let S be convex. Let $x, y \in \overline{S}$ and let $\theta \in [0, 1]$. We need to show that $\theta x + (1-\theta)y \in \overline{S}$. By assumption, there exist sequences $x_n \to x$, $y_n \to y$, with $x_n, y_n \in S$ for every $n \geq 1$. By convexity, $\theta x_n + (1-\theta)y_n \in S$. We now have the convergence $\theta x_n + (1-\theta)y_n \to \theta x + (1-\theta)y$, showing that $\overline{S}$ is convex.

(ii) Let S be symmetric and assume $x \in \overline{S}$. Then there exists a sequence $x_n \to x$, with $x_n \in S$ for every $n \geq 1$. Therefore $-x_n \in S$ and $-x_n \to -x$. This shows that $-x \in \overline{S}$, hence $\overline{S}$ is symmetric.

9. (i) Let S be convex. Assume $y_1, y_2 \in \Lambda(S)$ and $\theta \in [0, 1]$. Then there exist $x_1, x_2 \in S$ such that $y_1 = \Lambda(x_1)$, $y_2 = \Lambda x_2$. Since Λ is linear, this implies

$$\theta y_1 + (1-\theta)y_2 \;=\; \theta\Lambda(x_1) + (1-\theta)\Lambda(x_2) \;=\; \Lambda(\theta x_1 + (1-\theta)x_2) \;\in\; \Lambda(S),$$

because S is convex.

(ii) Let S be symmetric. Assume $y \in \Lambda(S)$. Then $y = \Lambda(x)$ for some $x \in S$. This implies $-y = \Lambda(-x) \in \Lambda(S)$ because S is symmetric.

10. If Λ is not continuous, then it is not bounded. Hence one can find a sequence of points $x_n \in X$ with $\|y_n\| \leq 1$ but $\|\Lambda(y_n)\| \geq n$ for every $n \geq 1$. Defining the sequence $x_n \doteq n^{-1/2}y_n$ we have $x_n \to 0$ but $\|\Lambda(x_n)\| \geq n^{1/2} \to \infty$, against the assumption.

11. (i) If S is a finite set, then $span(S)$ is a finite dimensional space. By Theorem 2.20, it is complete. Being complete, it must be a closed subset of X.

(ii) Since X is a Banach space, the closure $Y \doteq \overline{span(S)}$ is complete, hence it is a Banach space as well. For each $n \geq 1$, consider the finite dimensional subspace $V_n = span\{v_1, \ldots, v_n\} \subset Y$. Being finite dimensional, this subspace is complete, hence it must be closed. It is easy to see that V_n has empty interior. Indeed, since by assumptions the vectors $\{v_1, \ldots, v_{N+1}\}$ are linearly independent, for every $y \in V_n$ the sequence $y + \frac{1}{m}v_{n+1}$ does not lie in V_n and converges to y as $m \to \infty$.

According to Baire's category theorem, the complement

$$Y \setminus \bigcup_{n\geq 1} V_n \;=\; \bigcap_{n\geq 1} (Y \setminus V_n)$$

is everywhere dense in Y. Observing that $span(S) = \cup_{n\geq 1} V_n$, this shows that $span(S) \neq Y = \overline{span(S)}$.

12. If $x, y \in X$ and $\alpha \in \mathbb{R}$, then

$$Re(\Phi(x+y)) \;=\; Re(\Phi(x)) + Re(\Phi(y))\,, \qquad\qquad Re(\Phi(\alpha x)) \;=\; \alpha Re(\lambda(x))\,.$$

Moreover, $Im(\Phi(x)) \;=\; Re(-i\Phi(x)) \;=\; -\,Re(\Phi(ix))$. Hence

$$\Phi(x) \;=\; Re(\Phi(x)) + Im(\Phi(x)) \;=\; \phi(x) - i\phi(ix)\,.$$

13. If X is finite dimensional, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis. Then every linear functional is continuous and is a linear combination of the n linear functionals $\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*$ where $\mathbf{e}_i^*(\mathbf{e}_j) = \delta_{ij}$.

On the other hand, if X is infinite dimensional, let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ be a countable family of linearly independent vectors. For each n, consider the subspace $V_n = span\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Let $\varphi_n : V_n \mapsto \mathbb{R}$ be the linear functional such that

$$\varphi_n(\mathbf{e}_n) = 1\,, \qquad\qquad \varphi_n(\mathbf{e}_k) \;=\; 0 \qquad \text{for all} \;\; k < n\,.$$

Since V_n is finite dimensional, φ_n is continuous. By Hahn-Banach, it can be extended to a continuous linear functional $\varphi_n : X \mapsto \mathbb{R}$.

It remains to check that the countable set of functionals $\{\varphi_n\,;\;\; n \geq 1\}$ is linearly independent. If

$$\sum_{k=1}^{N} a_k \varphi_k(x) \;=\; 0 \qquad\qquad \text{for all} \;\; x \in X\,,$$

let j be the smallest index such that $a_j \neq 0$. Then

$$0 \;=\; \sum_{k=1}^{N} a_k \varphi_k(\mathbf{e}_j) \;=\; a_j\,,$$

reaching a contradiction.

14. If $\mathbf{x} = (x_1, x_2, \ldots) \in \ell^p$, then

$$\|\mathbf{x}\|_{\ell^\infty} \;=\; \sup_n |x_n| \;=\; \Big(\sup_n |x_n|^p\Big)^{1/p} \;\leq\; \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p} \;=\; \|\mathbf{x}\|_{\ell^p}\,.$$

Hence $\mathbf{x} \in \ell^\infty$ and the embedding $\ell^p \subseteq \ell^\infty$ is bounded, with operator norm 1. Moreover, if $p < q < \infty$, then

$$\|\mathbf{x}\|_{\ell^q} \;=\; \left(\sum_{n=1}^{\infty} |x_n|^q\right)^{1/q} \;=\; \left(\sum_{n=1}^{\infty} |x_n|^p |x_n|^{q-p}\right)^{1/q} \;\leq\; \left(\sum_{n=1}^{\infty} |x_n|^p \cdot \|\mathbf{x}\|_{\ell^\infty}^{q-p}\right)^{1/q}$$

$$=\; \|\mathbf{x}\|_{\ell^p}^{p/q} \cdot \|\mathbf{x}\|_{\ell^\infty}^{(q-p)/q} \;\leq\; \|\mathbf{x}\|_{\ell^p}^{p/q} \cdot \|\mathbf{x}\|_{\ell^p}^{(q-p)/q} \;=\; \|\mathbf{x}\|_{\ell^p}\,.$$

15. Consider first the case $1 \le p < \infty$. Let $\mathbf{x} = (x_1, x_2, \ldots) \in \ell^p$. Define the sequence of linear combinations

$$\mathbf{y}_n \doteq (x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \sum_{k=1}^n x_k \mathbf{e}_k \in V \doteq span\{\mathbf{e}_k\,;\ \ k \ge 1\}. \tag{2}$$

Then

$$\|\mathbf{x} - \mathbf{y}_n\|_{\ell^p}^p = \sum_{k>n} |x_k|^p \to 0 \qquad \text{as} \quad n \to \infty.$$

Next, assume $\mathbf{x} = (x_1, x_2, \ldots) \in \ell^\infty$. Define the sequence of approximations $\mathbf{y}_n$ as in (2). If $\mathbf{x} \in c_0$, i.e. if $\lim_{k\to\infty} x_k = 0$, then

$$\|\mathbf{x} - \mathbf{y}_n\|_{\ell^\infty} = \sup_{k>n} |x_k| \to 0 \qquad \text{as} \quad n \to \infty.$$

Hence $V \doteq span\{\mathbf{e}_k\,;\ \ k \ge 1\}$ is dense on c_0. To prove that c_0 coincides with the closure of V, it remains to prove that c_0 is closed. Consider a sequence of elements $\mathbf{x}_m = (x_{m,1}, x_{m,2}, x_{m,3}, \ldots)$, with $\mathbf{x}_m \in c_0$ for every $m \ge$. Assume that $\lim_{m\to\infty} \|\mathbf{x}_m - \mathbf{x}\|_{\ell^\infty} = 0$. We claim that $\mathbf{x} \in c_0$ as well. Indeed, let $\varepsilon > 0$ be given. By assumption, there exists an integer $\overline{m}$ large enough so that $\|\mathbf{x}_{\overline{m}} - \mathbf{x}\|_{\ell^\infty} < \varepsilon/2$. Since $\mathbf{x}_{\overline{m}} \in c_0$, there exists N large enough so that $|x_{\overline{m},n}| < \varepsilon/2$ for all $n > N$. Hence

$$|x_n| \le |x_n - x_{\overline{m},n}| + |x_{\overline{m},n}| \le \|\mathbf{x} - \mathbf{x}_{\overline{m}}\|_{\ell^\infty} + |x_{\overline{m},n}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves $\lim_{k\to\infty} x_k = 0$.

16. To fix the ideas, we assume $1 \le p < \infty$. The case $p = \infty$ is similar.

(i) If $M \doteq \sup_k |\lambda_k|$, then

$$\|\Lambda \mathbf{x}\|_{\ell^p} = \left(\sum_k |\lambda_k x_k|^p\right)^{1/p} \le M \cdot \left(\sum_k |x_k|^p\right)^{1/p} = M\|\mathbf{x}\|_{\ell^p}.$$

This shows that $\|\Lambda\| \le M$. On the other hand, given any $\varepsilon > 0$, choose k such that $|\lambda_k| > M - \varepsilon$. Then

$$\|\Lambda \mathbf{e}_k\|_{\ell^p} = \|\lambda_k \mathbf{e}_k\|_{\ell^p} = |\lambda_k|\|\mathbf{e}_k\|_{\ell^p} \ge (M - \varepsilon)\|\mathbf{e}_k\|_{\ell^p}.$$

This shows that $\|\Lambda\| \ge M - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\|\Lambda\| = M$.

To prove (ii), assume that the sequence $(\lambda_k)_{k\ge1}$ is unbounded. Then

$$\|\Lambda\| \ge \sup_{k\ge1} \|\Lambda \mathbf{e}_k\|_{\ell^p} \ge \sup_{k\ge1} |\lambda_k| = \infty.$$

In this case, we can extract a subsequence with $|\lambda_{n_k}| \ge 3^k$. Consider the vector $\mathbf{v} = \sum_{k=1}^\infty 2^{-k} \mathbf{e}_{n_k}$. Observe that this sum is well defined, because the series is absolutely convergent:

$$\sum_{k\ge1} \|2^{-k} \mathbf{e}_{n_k}\|_{\ell^p} = \sum_{k\ge1} |2^{-k}| = 1.$$

We now have

$$\Lambda \mathbf{v} \;=\; \sum_k \lambda_{n_k} 2^{-k} \mathbf{e}_{n_k} \;\notin\; \ell^p$$

Indeed,

$$\lim_{m\to\infty} \sum_{k=1}^m |\lambda_{n_k} 2^{-k}| \;\geq\; \lim_{m\to\infty} \sum_{k=1}^m (3/2)^k \;=\; \infty\,.$$

17. To fix the ideas, assume $1 \leq p < \infty$.

(i) Let $g \in \mathbf{L}^\infty(\Omega)$, say with $\|g\|_{\mathbf{L}^\infty} = K$. If $f \in \mathbf{L}^p(\Omega)$, then

$$\|fg\|_{\mathbf{L}^p} \;=\; \left(\int_\Omega |fg|^p\,dx\right)^{1/p} \;\leq\; \left(\int_\Omega K^p |f|^p\,dx\right)^{1/p} \;\leq\; K\|f\|_{\mathbf{L}^p}\,.$$

Hence the norm of the multiplication operator satisfies $\|M_g\| \leq K$. On the other hand, for any $\varepsilon > 0$ there exists a bounded measurable subset $A \subseteq \Omega$ with strictly positive measure such that $|g(x)| \geq K - \varepsilon$ for all $x \in A$. Let $f = \chi_A$ be the characteristic function of the set A. Namely, $f(x) = 1$ if $x \in A$ and $f(x) = 0$ if $x \notin A$. Then

$$\|M_g f\|_{\mathbf{L}^p} \;=\; \|fg\|_{\mathbf{L}^p} \;=\; \left(\int_A |g|^p\,dx\right)^{1/p} \;\geq\; \left(\int_A (K-\varepsilon)^p\,dx\right)^{1/p}$$

$$=\; (K-\varepsilon)\cdot\Big(meas(A)\Big)^{1/p} \;=\; (K-\varepsilon)\cdot\|f\|_{\mathbf{L}^p}\,.$$

This proves that $\|M_g\| \geq K - \varepsilon$.

(ii) Next, assume $g \notin \mathbf{L}^\infty$. Then for every constant $K \geq 1$ we can find a bounded measurable set A with strictly positive measure such that $|g(x)| \geq K - \varepsilon$ for all $x \in A$. Arguing as before, taking $f = \chi_A$ we conclude that $\|M_g\| \geq K$. Since K is now arbitrary, we conclude that the operator M_g is unbounded.

To obtain a function $f \in \mathbf{L}^p$ such that $fg \notin \mathbf{L}^p$, we first construct a sequence of disjoint, bounded, measurable subsets $A_n \subset \Omega$ with the following properties.

1) Each A_n has strictly positive measure.

2) $|g(x)| \geq 3^n$ for every $x \in A_n$.

3) For every $n \geq 1$ one has $g \notin \mathbf{L}^\infty(\Omega \setminus \cup_{j=1}^n A_j)$.

This sequence of subsets can be constructed by induction on n. Taking

$$f(x) \;\doteq\; \sum_{n=1}^\infty 2^{-n} \frac{\chi_{A_n}}{\|\chi_{A_n}\|_{\mathbf{L}^p}}$$

we have that $f \in \mathbf{L}^p(\Omega)$ but $fg \notin \mathbf{L}^p(\Omega)$.

18. For any given $a < b$, the map

$$f \mapsto \varphi(f) \doteq \int_a^b f(x)\,dx$$

is a bounded linear functional on $\mathbf{L}^p(\mathbb{R})$. By definition, the weak convergence $f_n \rightharpoonup f$ implies $\varphi(f_n) \to \varphi(f)$.

19. (i) A straightforward computation yields

$$\|f_n\|_{\mathbf{L}^\infty} = \frac{1}{n}, \qquad \|f_n\|_{\mathbf{L}^p} = n^{\frac{1-p}{p}}.$$

As $n \to \infty$, for every $p > 1$ the $\mathbf{L}^p$ norm of f_n approaches zero.

(ii) On the other hand, $\|f_n\|_{\mathbf{L}^1} = 1$ for every n.

Consider any subsequence, say f_{n_k}, with $n_1 < n_2 < n_3 < \cdots$ By choosing a further subsequence and relabeling, it is not restrictive to assume $n_{k+1} > 3n_k$.

Define the function $g \in \mathbf{L}^\infty(\mathbb{R})$ by setting

$$\begin{aligned} g(x) &= 0 \qquad \text{if} \quad n_{2k-1} < x \le n_{2k} \\ g(x) &= 1 \qquad \text{if} \quad n_{2k} < x \le n_{2k+1}. \end{aligned}$$

Then the map $f \mapsto \varphi(f) \doteq \int fg\,dx$ is a bounded linear functional linear on $\mathbf{L}^1(\mathbb{R})$. The above definition implies

$$\varphi(f_{n_{2k}}) \ge \frac{2}{3}, \qquad \varphi(f_{n_{2k+1}}) \le \frac{1}{3}.$$

Therefore the subsequence $\varphi(f_{n_k})$ has no limit. As a consequence, the original subsequence does not converge weakly.

20. For every $y \in Y$, let $\Lambda y = (\Lambda_1 y, \Lambda_2 y, \ldots, \Lambda_n y) \in \mathbb{R}^n$. By the Hahn-Banach extension theorem, each linear functional $\Lambda_j : Y \mapsto \mathbb{R}$ can be extended to a bounded linear functional $\widetilde{\Lambda}_j : X \mapsto \mathbb{R}$ with norm $\|\widetilde{\Lambda}_j\| = \|\Lambda_j\| \le \|\Lambda\|$. For $x \in X$, define the extension $\widetilde{\Lambda}x = (\widetilde{\Lambda}_1 x, \widetilde{\Lambda}_2 x, \ldots, \widetilde{\Lambda}_n x) \in \mathbb{R}^n$. This is a bounded linear operator, with norm $\|\widetilde{\Lambda}\| \le n^{1/2}\|\Lambda\|$. Indeed, if $\|x\| \le 1$, then

$$|\widetilde{\Lambda}x| = \left(\sum_{j=1}^n |\widetilde{\Lambda}_j x|^2\right)^{1/2} \le \Big(n\|\Lambda\|^2\Big)^{1/2} = n^{1/2}\|\Lambda\|.$$

21. Let $\phi(x_1, x_2) = ax_1 + bx_2$. To prove (i) we write

$$\|\phi\|_\infty \doteq \sup_{|x_1|+|x_2|\le 1} |ax_1 + bx_2| \le \sup_{|x_1|+|x_2|\le 1} (|ax_1| + |bx_2|) \le \max\{|a|, |b|\}.$$

Choosing

$$(\bar{x}_1, \bar{x}_2) = \begin{cases} \left(\dfrac{a}{|a|}, 0\right) & \text{if } |a| \ge |b|, \\[2ex] \left(0, \dfrac{b}{|b|}\right) & \text{if } |a| \le |b|, \end{cases}$$

one obtains the converse inequality

$$\|\phi\|_\infty \;\geq\; |a\bar{x}_1 + b\bar{x}_2| \;=\; \max\{|a|\,,\; |b|\}\,.$$

To prove (ii) we write

$$\|\phi\|_\infty \;\doteq\; \sup_{|x_1|\leq 1,\,|x_2|\leq 1} |ax_1 + bx_2| \;\leq\; \sup_{|x_1|\leq 1,\,|x_2|\leq 1} (|ax_1| + |bx_2|) \;\leq\; |a| + |b|\,.$$

Choosing

$$(\bar{x}_1, \bar{x}_2) \;=\; \Big(\frac{a}{|a|}, \frac{b}{|b|}\Big),$$

one obtains the converse inequality

$$\|\phi\|_\infty \;\geq\; |a\bar{x}_1 + b\bar{x}_2| \;=\; |a| + |b|\,.$$

Using the discrete Hölder inequality, (iii) is proved by

$$\|\phi\|_p \;\doteq\; \sup_{(|x_1|^q+|x_2|^q)^{\frac{1}{q}}\leq 1} |ax_1 + bx_2| \;\leq\; \sup_{(|x_1|^q+|x_2|^q)^{\frac{1}{q}}\leq 1} (|ax_1| + |bx_2|) \;\leq\; (|a|^p + |b|^p)^{\frac{1}{p}}\,.$$

Choosing

$$(\bar{x}_1, \bar{x}_2) \;=\; \Big(|a|^q + |b|^q\Big)^{\frac{1-q}{q}} \cdot \Big(|a|^{q-1}\text{sign } a\,,\; |b|^{q-1}\text{sign } b\Big),$$

one obtains the converse inequality

$$\|\phi\|_p \;\geq\; |a\bar{x}_1 + b\bar{x}_2| \;=\; (|a|^p + |b|^p)^{\frac{1}{p}}\,.$$

22. By assumption, $B \subset X$ is a convex set with the following property. For every non-zero vector $x \in X$, there exists $\eta_x > 0$ such that

$$B \,\cap\, \{t\,x;\;\; t \in \mathbb{K}\} \;=\; \{t\,x;\;\; |t| \leq \eta_x\}\,. \tag{3}$$

This clearly implies $B = -B$ and $0 \in B$. Moreover, the functional

$$\|x\| \;\doteq\; \min\{r \geq 0;\quad x \in rB\} \;=\; \begin{cases} 0 & \text{if } \; x = 0\,, \\[2ex] \dfrac{1}{\eta_x} & \text{if } \; x \neq 0\,, \end{cases} \tag{4}$$

is well defined and satisfies

$$\|x\| \leq 1 \qquad \text{if and only if} \qquad x \in B\,.$$

It remains to prove the properties (N1)–(N3) of a norm. From (3)-(4) one immediately obtains

$$\|x\| = 0 \qquad \text{if and only if} \qquad x = 0\,,$$

$$\lambda\alpha x \;\in\; B \quad\iff\quad |\lambda|\,|\alpha| \;\leq\; \eta_x \quad\iff\quad |\alpha| \;\leq\; \theta_{\lambda x} \;=\; \theta_{|\lambda| x}\,.$$

Hence

$$\theta_{|\lambda| x} \;=\; \frac{\eta_x}{|\lambda|}\,, \qquad\qquad \|\lambda x\| \;=\; |\lambda|\,\|x\|\,.$$

To prove the sub-additivity of the norm, consider any $x, y \neq 0$. Observe that

$$x \;\in\; \|x\|\,B\,, \qquad y \;\in\; \|y\|\,B\,.$$

Since B is a convex set containing the origin, this implies

$$x+y \;\in\; (\|x\|+\|y\|)B\,,$$

$$\|x+y\| \;\doteq\; \min\{r\,;\;\; x+y\in rB\} \;\leq\; \|x\|+\|y\|\,.$$

23. (i) $\Longrightarrow$ (iii) Let $(f_j)_{j\geq 1}$ be a Cauchy sequence for the distance d. Let $K\subset\Omega$ be any compact subset and let $\varepsilon>0$ be given. Since the open sets A_j cover K, there exists N large enough such that $K\subset\bigcup_{k=1}^N A_k$. This implies

$$\limsup_{j,\ell\to\infty}\left(\sup_{x\in K}|f_j(x)-f_\ell(x)|\right) \;\leq\; 2^N \lim_{j,\ell\to\infty} d(f_j,f_\ell) \;=\; 0.$$

Therefore, the sequence $(f_j)_{j\geq 1}$ converges uniformly on the compact set K.
An identical argument shows that (ii) $\Longrightarrow$ (iii).

(iii)$\Longrightarrow$ (ii) Assume that the sequence $(f_j)_{j\geq 1}$ converges uniformly on compact subsets of Ω. Let $\varepsilon>0$ be given. Choose N so that $2^{-N}<\varepsilon$. Then the set $K\doteq\cup_{1\leq k\leq N}\overline{A_k}$ is a compact subset of Ω. We thus have the estimate

$$\limsup_{j,\ell\to\infty} d(f_j,f_\ell) \;=\; \limsup_{j,\ell\to\infty}\sum_{k=1}^N 2^{-k}\frac{p_k(f_j-f_\ell)}{1+p_k(f_j-f_\ell)} + \limsup_{j,\ell\to\infty}\sum_{k=N+1}^\infty 2^{-k}\frac{p_k(f_j-f_\ell)}{1+p_k(f_j-f_\ell)}$$

$$\leq\; 0+2^{-N} \;<\; \varepsilon\,.$$

Hence the sequence $(f_j)_{j\geq 1}$ is Cauchy w.r.t. the distance $d(\cdot,\cdot)$.
An identical argument shows that (iii) $\Longrightarrow$ (ii).

24. (i) There is no guarantee that the series $\sum_{k=1}^\infty 2^{-k}p_k(f)$ converges. For example, take $\Omega=]0,1[$ and define

$$p_k(f) \;\doteq\; \max\left\{|f(x)|\,;\quad x\in A_k=\left[\frac{1}{k}\,,\;\frac{k-1}{k}\right]\right\}.$$

If $f(x)=e^{1/x}$ then

$$\sum_{k=1}^\infty 2^{-k}p_k(f) \;=\; \sum_{k=1}^\infty 2^{-k}e^k \;=\; \infty\,.$$

(ii) Choose N large enough so that $2^{-N}<\varepsilon$. Then, if $p_1(f)=\cdots=p_N(f)=0$, then for any $\lambda>0$ we have

$$\sum_{n=1}^\infty 2^{-n}\frac{p_n(\lambda^{-1}f)}{1+p_n(\lambda^{-1}f)} \;\leq\; \sum_{n=N+1}^\infty 2^{-n}\frac{p_n(\lambda^{-1}f)}{1+p_n(\lambda^{-1}f)} \;<\; 2^{-N} \;<\; \varepsilon\,.$$

Hence $\lambda^{-1} f \in B_\varepsilon$ for every $\lambda > 0$. By definition, this implies $\|f\|_\varepsilon = 0$, showing that the property (N1) of the norm can fail.

25. Assume $x \in span(S)$, so that

$$x = \sum_{k=1}^{n} \alpha_k x_k , \qquad \alpha_k \in \mathbb{K}, \quad x_k \in S .$$

Choose an integer $m > n \sum_k \|\alpha x_k\|$. Then

$$x = \left(\frac{\alpha_1}{m} x_1 + \frac{\alpha_2}{m} x_2 + \cdots + \frac{\alpha_m}{m} x_n\right) + \cdots + \left(\frac{\alpha_1}{m} x_1 + \frac{\alpha_2}{m} x_2 + \cdots + \frac{\alpha_m}{m} x_n\right) ,$$

where the right hand side contains m equal groups of summands. By the choice of m, each partial sum has norm $\leq 2\|x\|$. We thus achieve the desired representation (2.42), with $N = mn$.

26. The implication (ii) $\Longrightarrow$ (i) is clear.

To prove the converse implication (i) $\Longrightarrow$ (ii) we proceed as follows. Assume $x \in \overline{span(S)}$, so that $x = \lim_{n\to\infty} y_n$ with $y_n \in span(S)$ for every n. It is not restrictive to assume $\|y_{n+1} - y_n\| < 2^{-n}$ for every $n \geq 1$. We can write x as an absolutely convergent sum:

$$x = y_1 + \sum_{n=1}^{\infty} (y_{n+1} - y_n) , \qquad y_{n+1} - y_n \in span(S).$$

Using problem **25**, we can write

$$y_{n+1} - y_n = \sum_{i=1}^{N(n)} c_{ni} x_{ni} , \qquad c_{ni} \in \mathbb{K} , \quad x_{ni} \in S ,$$

with

$$\left\| \sum_{i=1}^{k} c_{ni} x_{ni} \right\| \leq 2\|y_{n+1} - y_n\| \leq 2^{1-n} \qquad \text{for every } k \in \{1, \ldots, N(n)\}. \tag{5}$$

We now arrange all terms in the double series[1]

$$\sum_{n=1}^{\infty} \sum_{i=1}^{N(n)} c_{ni} x_{ni}$$

into a single sequence, namely

$$c_{11}x_{11} + \cdots + c_{1,N(1)} x_{1,N(1)} + c_{21}x_{21} + \cdots + c_{2,N(2)} x_{2,N(2)} + c_{31}x_{31} + \cdots$$

Thanks to (5) we obtain a series converging to x.

[1] Note that, in general, this double series may not be absolutely convergent. Hence different rearrangements of its terms may yield different limits. It is the property (5) that guarantees that the sequence converges to x.

27. (i) Let $V \subset \ell^\infty$ be the subspace of all sequences $x = (x_1, x_2, \ldots)$ such that $\lim_{n\to\infty} x_n$ exists. For $x \in V$ define the linear operator $F(x) \doteq \lim_{n\to\infty} x_n$. Observing that $|F(x)| = |\lim_{n\to\infty} x_n| \leq \sup_n |x_n| = \|x\|_\ell^\infty$. it is clear that $f : V \mapsto \mathbb{R}$ is a continuous linear functional with norm $\|F\| = 1$. Using the Hahn-Banach extension theorem, we can extend F to the entire space ℓ^∞, still with the same norm $\|F\| = 1$.

(ii) Assume $m = \liminf x_n < \limsup x_n = M$. If $F(x) = M + \delta$ for some $\delta > 0$, a contradiction is achieved as follows. Choose N large enough so that $m - \frac{\delta}{2} < x_n < M + \frac{\delta}{2}$ for all $n > N$. Consider the sequence $y = (x_1, x_2, \ldots, x_N, m, m, m, \ldots)$. Then

$$F(x) - F(y) \;=\; (M+\delta) - m \;>\; M + \frac{\delta}{2} - m \;\geq\; \|x - y\|_{\ell^\infty}\,.$$

This is a contradiction, because $\|F\| = 1$, hence we should have $F(x) - F(y)| \leq \|x - y\|_{\ell^\infty}$. The case $F(x) < m$ is handled in a similar way.

(iii) Using linearity and then (ii), we obtain

$$F(y) - F(x) \;=\; F(y - x) \;\geq\; \liminf_{n\to\infty} y_n - x_n \;\geq\; 0\,.$$

(iv) If $a = (a_1, a_2, \ldots) \in \ell^1$, then there exists N large enough so that $\sum_{n>N} |a_n| < 1$. Consider the sequence $x = (0, 0, \ldots, 0, 1, 1, 1, \ldots)$, where the first N entries are zero and all other entries are equal to 1. Then

$$\sum_{n=1}^{\infty} a_n x_n \;=\; \sum_{n>N} a_n \;<\; 1 \;=\; \lim_{n\to\infty} x_n \;=\; F(x).$$

(v) The functional $\widetilde{F}$ satisfies (2.43)-(2.44), but it is not linear. For example, take $x = (1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots)$, $y = (0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$. Then $x + y = (1, 0, 1, 1, 0, 1, 1, 0, 1, \ldots)$ and

$$F(x+y) \;=\; \frac{1}{2} \;\neq\; F(x) + F(y) \;=\; \frac{1}{2} + \frac{1}{2}\,.$$

28. The map $f \mapsto f(0)$ is a bounded linear operator on V, with norm

$$\sup_{f \text{ continuous},\ \|f\|_{\mathbf{L}^\infty} \leq 1} |f(0)| \;=\; 1\,.$$

By the Hahn-Banach theorem it can be extended to a linear functional Λ defined on the entire space $\mathbf{L}^\infty(\mathbb{R})$, with norm $\|\Lambda\| = 1$.

Next, assume that there exists $g \in \mathbf{L}^1(\mathbb{R})$ such that $\Lambda f = \int fg\,dx$ for all $f \in \mathbf{L}^\infty(\mathbb{R})$. The dominated convergence theorem yields

$$\lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} |g(x)|\,dx \;=\; 0.$$

Hence we can choose $\delta > 0$ so that $\int_{-\delta}^{\delta} |g(x)|\,dx \leq 1/2$. Let $\phi : \mathbb{R} \mapsto [0, 1]$ be a smooth function, with $\phi(0) = 1$ and with support contained in the interval $[-\delta, \delta]$. We now reach a contradiction by writing

$$1 \;=\; \phi(0) \;=\; \Lambda(\phi) = \int_{\mathbb{R}} g(x)\phi(x)\,dx \;=\; \int_{-\delta}^{\delta} g(x)\phi(x)\,dx \;\leq\; \max_x |\phi(x)| \cdot \int_{-\delta}^{\delta} |g(x)|\,dx \;\leq\; \frac{1}{2}\,.$$

29. (i) Assuming that $x \in \lambda_1\Omega$, $y \in \lambda_2\Omega$ for some $\lambda_1, \lambda_2 > 0$, we claim that $x+y \in (\lambda_1+\lambda_2)\Omega$. Indeed, by assumption there exist $\omega_1, \omega_2 \in \Omega$ such that $x = \lambda_1\omega_1$, $y = \lambda_2\omega_2$. Using the assumption that Ω we obtain

$$x+y \;=\; \lambda_1\omega_1 + \lambda_2\omega_2 \;=\; (\lambda_1+\lambda_2)\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\omega_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\omega_2\right) \;\in\; (\lambda_1+\lambda_2)\Omega\,.$$

The identity $p(tx) = tp(x)$ for $t > 0$ follows from the fact that $x \in \lambda\Omega$ if and only if $tx \in t\lambda\Omega$.

(ii) We have

$$p(x) \;=\; \inf\{\lambda > 0\,;\;\; x \in \lambda\Omega\} \;\leq\; \inf\{\lambda > 0\,;\;\; x \in \lambda B_r\} \;=\; \frac{\|x\|}{r}\,.$$

(iii) If $\Omega = B_1$, then

$$p(x) \;=\; \inf\{\lambda > 0\,;\;\; x \in \lambda B_1\} \;\leq\; \inf\{\lambda > 0\,;\;\; \|x\| < \lambda\} \;=\; \|x\|\,.$$

30. (i) If $(f_n)_{n\geq 1}$ is a sequence of continuous functions such that $f_n \to f$ uniformly on $[0,1]$ and $f_n(0) = 0$ for every $n \geq 1$, then $f(0) = 0$. Hence X is a closed subspace.

(ii) The map $f \mapsto \Lambda f \doteq \int_0^1 f(x)\,dx$ is a linear functional on X. for every $f \in X$ with $\|f\| = \max_{x\in[0,1]} |f(x)| \leq 1$ we have $|\Lambda f| = |\int_0^1 f(x)\,dx| < 1$, because $f(0) = 0$, hence $|f(x)| < 1/2$ for x in some interval $[0,\delta]$. On the other hand, choosing $f_n = x^{1/n}$ we have $f_n \in X$, $\|f_n\| = 1$ and $\Lambda f_n = \frac{n}{n+1} \to 1$ as $n \to \infty$. Hence

$$\|\Lambda\| \;\doteq\; \sup_{f\in X\,,\;\|f\|\leq 1} \left|\int_0^1 f(x)\,dx\right| \;=\; 1,$$

but the supremum is not attained.

31. Being the intersection of a family of closed, convex sets, the set Ω^{**} is also closed and convex. It remains to show that Ω^{**} is contained in the closure of Ω. Assume, on the contrary, that there exists $x_0 \in \Omega^{**}$ and a radius $r > 0$ such that the open ball $B(x_0, r)$ does not intersect Ω. By the separation theorem, there exists a bounded linear functional $F : X \mapsto \mathbb{R}$ and $c > 0$ such that

$$F(x) \;\leq\; c \quad \text{for all} \;\; x \in \Omega, \qquad\qquad F(x) \;>\; c \quad \text{for all} \;\; x \in B(x_0, r)\,.$$

Defining $\phi(x) = \frac{1}{c}F(x)$ we obtain

$$\phi(x) \;\leq\; 1 \quad \text{for all} \;\; x \in \Omega\,.$$

This yields a contradiction, because $\phi \in \Omega^*$ but $\phi(x_0) > 1$.

32. On the one dimensional space $U = \{t\xi\,;\;\; t \in \mathbb{R}\}$ define the functional $F(t\xi) = t$. This is a bounded linear functional with norm $\|F\| = \|\xi\|$. Using the Hahn-Banach Theorem, we extend F to a bounded linear functional $F : X \mapsto \mathbb{R}$ with the same norm. Let $V = ker(F)$. We claim that all conclusions are satisfied.

Since F is continuous linear functional, V is a closed subspace. Given $x \in X$, consider the projections

$$\pi_U x \doteq F(x)\,\xi\,, \qquad \pi_V x \doteq x - F(x)\xi\,.$$

These are bounded linear operators, and satisfy

$$\pi_U x \in U\,, \qquad \pi_V x \in V\,, \qquad \pi_U x + \pi_V x = x\,.$$

33. Consider the subspace $Y = \{ax+b\,;\ \ a,b \in \mathbb{R}\}$ of all polynomials of degree 1. Restricted to Y, the linear functional φ is continuous and has norm

$$\|\varphi\| = \sup_{f\in Y,\, f\neq 0} \frac{|\varphi(f)|}{\|f\|_{\mathbf{L}^1}} = \sup_{(a,b)\neq(0,0)} \frac{a}{\displaystyle\int_{-1}^{1} |ax+b|\,dx} = 1$$

Indeed, for a given a, the minimum of the denominator is achieved when $b = 0$.

Using the Hahn-Banach theorem, the functional φ can then be extended to the entire space $\mathbf{L}^1([-1,1])$, with the same norm.

An explicit form of this functional is

$$\varphi(f) = \int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx\,.$$

34. It is clear that each $p_k(\cdot)$ is a seminorm. If $f \in \mathcal{C}^m(\Omega)$, $f \neq 0$, then $f(x) \neq 0$ for some point $x \in \Omega$. If $x \in A_k$, then $p_k(f) > 0$.

To show that the space $\mathcal{C}^m(\Omega)$ is complete for the corresponding distance

$$d(f,g) \doteq \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(f-g)}{1+p_k(f-g)}\,,$$

let $(f_j)_{j\geq 1}$ be a Cauchy sequence. Then for every multi-index α with $|\alpha| \leq m$ the sequence of continuous functions $(D^\alpha f_j)_{j\geq 1}$ is uniformly convergent on each open set A_k. Therefore, it has a continuous limit, say

$$f(x) \to g(x)\,, \qquad D^\alpha f_j(x) \to g_\alpha(x) \qquad \text{for all } \ x \in \Omega,\ |\alpha| \leq m\,,$$

for some continuous functions g, g_α. Since the limit is uniform on compact sets, it follows $g_\alpha = D^\alpha g$ for every $|\alpha| \leq m$. Hence $d(f_j, g) \to 0$, proving that this Cauchy sequence has a limit.

35. Consider the subspace $\widetilde{Y} \doteq \{\tilde{y} = \ y + tx_0\,;\ \ y \in Y, t \in \mathbb{R}\}$ be the space spanned by Y together with $x+0$. Let $\alpha \doteq d(x_0, Y) > 0$. Define the functional $\varphi : \widetilde{Y} \mapsto \mathbb{R}$ by setting

$$\varphi(y + tx_0) \doteq \alpha t\,.$$

This clearly implies $\varphi(x_0) = \alpha = d(x_0, Y)$ and $\varphi(y) = 0$ for $y \in Y$.

We claim that φ is a bounded operator with norm $\|\varphi\| = 1$. Indeed, assume that $\|y_0 + tx_0\| < \alpha t$ for some choice of $y_0 \in Y$ and $t \neq 0$. This leads to a contradiction, because

$$\| - y_0 - tx_0\| \;=\; \|y_0 + tx_0\| \;<\; \alpha t\,,$$

$$\left\| - \frac{y_0}{t} - x_0\right\| \;<\; \alpha \;=\; \inf_{y\in Y} \|y - x_0\|\,.$$

Using the Hahn-Banch theorem, the functional φ can then be extended to the entire space X.

36. (i) If $1 \le p < \infty$, every continuous linear functional on ℓ^p has the form $x \mapsto \varphi(x) = \sum_n a_n x_n$, for some sequence $a = (a_n)_{n\ge 1} \in \ell^q$. In this case,

$$\lim_{k\to\infty} \varphi(\mathbf{e}_k) \;=\; \lim_{k\to\infty} a_k \;=\; 0\,.$$

proving the weak convergence $\mathbf{e}_k \rightharpoonup 0$ in the space ℓ^p.

(ii) Now consider the case $p = \infty$. Take any subsequence $(\mathbf{e}_{n_k})_{k\ge 1}$. Define the element $a = (a_1, a_2, \ldots) \in \ell^\infty$ as follows.

$$a_j \;=\; \begin{cases} 1 & \text{if } j = n_k \text{ for some odd integer } k,\\ 0 & \text{otherwise.}\end{cases}$$

Consider the bounded linear functional on ℓ^1 defined by $\varphi(x) \doteq \sum_j a_j x_j$. Then

$$\liminf_{k\to\infty} \varphi(\mathbf{e}_{n_k}) \;=\; 0 \;<\; 1 \;= \limsup_{k\to\infty} \varphi(\mathbf{e}_{n_k})\,,$$

showing that the subsequence $(\mathbf{e}_{n_k})_{k\ge 1}$ is not weakly convergent.

37. In the case $1 \le p < \infty$, assume $\phi'(x) \ge c > 0$ for all $x \in \mathbb{R}$. For every $f \in \mathbf{L}^p(\mathbb{R})$, setting $y = \phi(x)$ one obtains

$$\left(\int |f(\phi(x))|^p\,dx\right)^{1/p} \;=\; \left(\int |f(y)|^p\,\frac{dy}{\phi'(x)}\right)^{1/p} \;\le\; \frac{1}{c^{1/p}}\cdot \|f\|_{\mathbf{L}^p}\,.$$

Hence, in this case, Λ is a bounded linear operator on $\mathbf{L}^p$ with norm $\|\Lambda\| \le c^{-1/p}$.

In the case $p = \infty$, it suffices to assume that $\phi'(x) > 0$. Then Λ is a bounded operator with norm $\|\Lambda\| = 1$. Indeed, if $f \in \mathbf{L}^\infty$, let $M \doteq \|f\|_{\mathbf{L}^\infty}$. Consider the set $A \;\doteq\; \Big\{x\,;\;\; f(\phi(x)) > M\Big\}$. If $meas(A) > 0$, we obtain a contradiction by writing

$$0 \;=\; meas\Big(\phi(A)\Big) \;=\; \int_A \phi'(x)\,dx \;>\; 0.$$

Hence $\|\Lambda f\|_{\mathbf{L}^\infty} \le M$. This proves that $\|\Lambda\| \le 1$. On the other hand, taking the function $g(x) \equiv 1$, one has $\|\Lambda g\|_{\mathbf{L}^\infty} = \|g\|_{\mathbf{L}^\infty} = 1$.

38. If the Banach space X is infinite dimensional, the closed unit ball is not precompact. In particular, we can find a sequence of points $x_n \in X$ such that $\|x_n\| \le 1$, $\|x_m - x_n\| \ge 1/2$ whenever $m \neq n$.

Choose $\bar{x} \in \Omega_0$ and $r > 0$ such that the ball $B(\bar{x}, r)$ is contained in Ω_0. By assumption, this implies $0 < \mu(B(\bar{x}, r)) \leq \mu(\Omega_0) < \infty$.

Consider the countably many open balls $B_n \doteq B\left(\frac{r}{2}x_n\,,\ \frac{r}{8}\right)$. These are mutually disjoint, and all contained in the open ball $B(0, r)$. Hence, by countable additivity we have

$$\mu(B(0,r)) \ \geq\ \mu\left(\bigcup_{n\geq 1} B_n\right) \ =\ \sum_{n\geq 1}\mu(B_n) \ =\ \infty$$

because all balls B_n have the same radius and the same strictly positive measure. This yields a contradiction, because by translation invariance $\mu(B(0,r)) = \mu(B(\bar{x},r)) < \infty$.

39. (i) This is an immediate consequence of Theorem 2.33 (ii), taking $A = \{y\}$ and $B = S$.

(ii) Assume $y \notin S$. Then there exists a bounded linear functional $\varphi \in X^*$ such that $\varphi(y) < \inf_{x\in S}\varphi(x)$.

The weak convergence $x_n \rightharpoonup y$ implies $\varphi(x_n) \to \varphi(y)$. But this yields a contradiction because

$$\lim_{n\to\infty}\varphi(x_n) \ \geq\ \inf_{x\in S}\varphi(x) \ >\ \varphi(y).$$

40. (i) Let $V \subset \mathbf{L}^\infty(\mathbb{R})$ be the subspace of all functions f such that $\text{ess-lim}_{x\to 0} f(x)$ exists. We claim that the functional $\Phi(f) \ \doteq\ \text{ess-lim}_{x\to 0} f(x)$ is a bounded linear functional on V. The linearity is obvious. If $\Phi(f) = \lambda$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - \lambda| < \varepsilon$ for a.e. $x \in [-\delta, \delta]$. In particular, this implies $\|f\|_{\mathbf{L}^\infty} \geq |\lambda| - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude

$$|\Phi(f)| \ \leq\ \|f\|_{\mathbf{L}^\infty} \tag{6}$$

for all $f \in V$. By the Hahn-Banach theorem, we can extend the linear functional Φ to the entire space $\mathbf{L}^\infty(\mathbb{R})$, still satisfying the bound (6) for all $f \in \mathbf{L}^\infty(\mathbb{R})$.

(ii) In the space $\mathbf{L}^1(\mathbb{R})$ the conclusion fails because the functional $\Phi : V \mapsto \mathbb{R}$ is unbounded. Indeed, consider the sequence of continuous functions:

$$f_n(x) \ \doteq\ \begin{cases} n(1-nx) & \text{if } |x| \leq 1/n\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \in V$ and $\|f_n\|_{\mathbf{L}^1} = 1$ for every $n \geq 1$, but $\Phi(f_n) = f_n(0) = n$.

Chapter 3

1. Repeat steps **5 - 6** in the proof of the Stone-Weiertrass theorem.

2. (i) True. For every $n \geq 1$, let p_n be a polymomial such that $\max_{x\in[-n,n]}|f(x) - p_n(x)| < 2^{-n}$. Such a polynomial exists by the Stone-Weierstrass theorem. The sequence of polynomials $(p_n)_{n\geq 1}$ converges to f uniformly on bounded sets.

(ii) False. If p is a polynomial such that $\sup_{x\in\mathbb{R}} |p(x) - e^{-x^2}| < \infty$, then p must be bounded, hence constant. But there is no way to approximate the function $f(x) = e^{-x^2}$ with constant functions, uniformly on $\mathbb{R}$.

3. (i) Extend g to an even function defined on $[-\pi, \pi]$, and then by periodicity to an even function defined on the whole real line, periodic of period 2π. By Corollary 3.9 there exists a trigonometric polynomial of the form

$$p(x) \;=\; \sum_{k=1}^{N} a_k \sin kx + \sum_{k=0}^{N} b_k \cos kx$$

such that $|p(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Since g is even, this implies

$$\varepsilon \;>\; \left|\frac{p(x)+p(-x)}{2} - \frac{g(x)+g(-x)}{2}\right| \;=\; \left|\sum_{k=0}^{N} b_k \cos kx - g(x)\right| . \qquad \text{for all } \; x \in \mathbb{R}\,.$$

(ii) Extend g to an odd function defined on $[-\pi, \pi]$, and then by periodicity to an odd function defined on the whole real line, periodic of period 2π. By Corollary 3.9 there exists a trigonometric polynomial of the form

$$p(x) \;=\; \sum_{k=1}^{N} a_k \sin kx + \sum_{k=1}^{N} b_k \cos kx$$

such that $|p(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Since g is odd, this implies

$$\varepsilon \;>\; \left|\frac{p(x)-p(-x)}{2} - \frac{g(x)-g(-x)}{2}\right| \;=\; \left|\sum_{k=1}^{N} a_k \sin kx - g(x)\right| . \qquad \text{for all } \; x \in \mathbb{R}\,.$$

4. By the Stone-Weierstrass theorem, for any $n \geq 1$ there exists a polynomial q_n such that $|q_n(x) - f'(x)| \leq 2^{-n}$ for every $x \in [a, b]$. Define the polynomial

$$p_n(x) \;\doteq\; f(a) + \int_a^x q_n(y)\, dy.$$

This construction yields

$$|p_n(x) - f(x)| \;\leq\; 2^{-n}(x-a) \;\leq\; 2^{-n}(b-a)\,, \qquad\qquad |p_n'(x) - f'(x)| \;\leq\; 2^{-n}$$

for every $n \geq 1$ and $x \in [a, b]$.

5. The properties (i) and (ii) are easily checked. However, (iii) fails. For example, the function $f(\theta) = e^{-i\theta}$ cannot be uniformly approximated with functions in the algebra $\mathcal{A}$. Indeed, assume that

$$\left|f(\theta) - \sum_{n=0}^{N} c_n e^{in\theta}\right| \;\leq\; \frac{1}{2} \qquad\qquad \text{for all } \; \theta \in [0, 2\pi].$$

This leads to a contradiction because

$$\left| \int_0^{2\pi} \left(f(\theta) - \sum_{n=0}^N c_n e^{in\theta} \right) e^{i\theta}\, d\theta \right| \;\leq\; \int_0^{2\pi} \frac{1}{2} |e^{i\theta}|\, d\theta \;\leq\; \pi\,.$$

On the other hand, since $f(\theta) = e^{-i\theta}$, an explicit computation yields

$$\int_0^{2\pi} \left(e^{-i\theta} - \sum_{n=0}^N c_n e^{in\theta} \right) e^{i\theta}\, d\theta \;=\; \int_0^{2\pi} 1\, d\theta - \sum_{n=0}^N \int_0^{2\pi} c_n e^{i(n+1)\theta}\, d\theta \;=\; 2\pi\,.$$

6. Given $f \in \mathbf{L}^p(\Omega)$, extend f to the entire space $\mathbb{R}^n$ by setting $f(x) = 0$ if $x \notin \Omega$.

For any $\varepsilon > 0$, using a mollification we obtain a smooth function $\tilde{f} = f * J_\delta$ such that $\|\tilde{f} - f\|_{\mathbf{L}^p(\mathbb{R}^n)} < \varepsilon$.

Since Ω is bounded, its closure $\overline{\Omega}$ is compact. By Corollary 3.6, the continuous function $\tilde{f}$ can be uniformly approximated by polynomials on the compact set $\overline{\Omega}$. In particular, there exists a polynomial $p(x) = p(x_1, \ldots, x_n)$ such that

$$|p(x) - \tilde{f}(x)| \;\leq\; \varepsilon \qquad\qquad \text{for all} \;\; x \in \overline{\Omega}\,.$$

The previous inequalities imply

$$\|f - p\|_{\mathbf{L}^p(\Omega)} \;\leq\; \|f - \tilde{f}\|_{\mathbf{L}^p(\Omega)} + \|\tilde{f} - p\|_{\mathbf{L}^p(\Omega)} \;\leq\; \varepsilon + \left(\int_\Omega \varepsilon^p\, dx \right)^{1/p} \;=\; \varepsilon + \varepsilon (meas(\Omega))^{1/p}.$$

Since $\varepsilon > 0$ was arbitrary, the result is proved.

7. (i) Choose $\delta > 0$ such that $|f(x,y) - f(x',y)| \leq \varepsilon$ whenever $|x - x'| \leq \delta$. Construct a continuous partition of unity $\{g_1, g_2, \ldots, g_N\}$ on the interval $[0,a]$ such that each g_i is supported on some interval $I_i \subseteq [a,b]$ of length $\leq \delta$. That means

$$\sum_{i=1}^N g_i(x) \;=\; 1 \qquad \text{for each } x \in [0,a], \qquad\qquad g_i(x) = 0 \quad \text{for } x \notin I_i\,.$$

For each $i \in \{1, \ldots, N\}$, choose a point $x_i \in I_i$. For every $(x,y) \in Q = [0,a] \times [0,b]$ we now have

$$\left| f(x,y) - \sum_{i=1}^N g_i(x)\, f(x_i,y) \right| \;\leq\; \sum_{i=1}^N g_i(x) \Big| f(x,y) - f(x_i,y) \Big| \;\leq\; \varepsilon. \tag{7}$$

Indeed, the only nonzero terms in the second summation are those for which $x \in I_i$, hence $|x - x_i| \leq \delta$. By construction, this implies $|f(x,y) - f(x_i,y)| \leq \varepsilon$.
Defining $h_i(y) \doteq f(x_i, y)$, all requirements are satisfied.

(ii) Let $0 < \varepsilon \leq 1$ be given. If f vanishes on the boundary of Q, then in the above construction we can choose $x_i = 0$ if $0 \in I_i$ and $x_i = a$ if $a \in I_i$. In both cases, $f(x_i, y) = 0$. Therefore, in the sum (7) such terms $g_i(x) f(x_i, y)$ can be discarded, because these vanish identically. As a result, it is not restrictive to assume that in (7) every g_i is a continuous function with support contained strictly inside the open interval $]0,a[$.

Since all functions g_i vanish at $x = 0$ and at $x = a$, and all functions $f(x_i, y)$ vanish at $y = 0$ and at $y = b$, using problem **3.**(ii), we can now find coefficients A_{im}, B_{im} and an integer M such that

$$\left| g_i(x) - \sum_{m=1}^{M} A_{im} \sin \frac{\pi m x}{a} \right| \leq \varepsilon \qquad \text{for all } \; x \in [0, a]\,,$$

$$\left| f(x_i, y) - \sum_{n=1}^{M} B_{in} \sin \frac{\pi n y}{b} \right| \leq \varepsilon \qquad \text{for all } \; y \in [0, b]\,.$$

This implies

$$\left| g_i(x) f(x_i, y) - \sum_{m,n=1}^{M} A_{im} B_{in} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b} \right|$$

$$\leq \left| g_i(x) - \sum_{m=1}^{M} A_{im} \sin \frac{\pi m x}{a} \right| |f(x_i, y)| + \left| \sum_{m=1}^{M} A_{im} \sin \frac{\pi m x}{a} \right| \cdot \left| f(x_i, y) - \sum_{n=1}^{M} B_{in} \sin \frac{\pi n y}{b} \right|$$

$$\leq \varepsilon \cdot \|f\|_{\mathcal{C}^0} + (1 + \|g_i\|_{\mathcal{C}^0}) \cdot \varepsilon \leq (2 + \|f\|_{\mathcal{C}^0})\, \varepsilon\,.$$

Therefore, setting $c_{mn} \doteq \sum_{i=1}^{N} A_{im} B_{in}$, we obtain

$$\left| \sum_{i=1}^{N} g_i(x) f(x_i, y) - \sum_{m,n=1}^{M} c_{mn} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b} \right| \leq N\, (2 + \|f\|_{\mathcal{C}^0})\, \varepsilon\,. \tag{8}$$

Since $\varepsilon > 0$ was arbitrary, by (7) and (8) the result is proved.

8. Let $(f_n)_{n \geq 1}$ be a sequence of functions in the Hölder space $\mathcal{C}^{0,\gamma}(\Omega)$, with $\|f_n\|_{\mathcal{C}^{0,\gamma}} \leq C$ for some constant C and every $n \geq 1$.

Being Hölder continuous, each function f_n is uniformly continuous and hence can be uniquely extended by continuity to the closure $\overline{\Omega}$, which is a compact set.

Since the functions f_n are uniformly bounded and equicontinuous on the compact set $\overline{\Omega}$, by Ascoli's theorem we can extract a subsequence converging to some continuous function f, uniformly for $x \in \overline{\Omega}$.

9. (i) $f_1 \in C^{0,1/3}$. For any $0 < a < b$ we have

$$f_1(b) - f_1(a) = \int_a^b \frac{1}{3} x^{-2/3}\, dx \leq \int_a^b \frac{1}{3} (x-a)^{-2/3}\, dx = (b-a)^{1/3}.$$

Hence

$$\sup_{0<a<b<1} \frac{|f(b) - f(a)|}{|b-a|^{1/3}} \leq 1\,.$$

On the other hand, choosing $a_n = n^{-2}$, $b_n = n^{-1}$, for every $\gamma > 1/3$ we have

$$\lim_{n\to\infty} \frac{b_n^{1/3} - a_n^{1/3}}{|b_n - a_n|^{1/3} \cdot |b_n - a_n|^{\gamma - 1/3}} = \lim_{n\to\infty} \frac{1}{|b_n - a_n|^{\gamma - 1/3}} = \infty.$$

Therefore $f_1 \notin C^{0,\gamma}$ for any $\gamma > 1/3$.

(ii) $f_2 \notin C^{0,\gamma}$ for any $\gamma > 1$. Indeed, choose $a_n = \frac{1}{(n+1)\pi}$ $b_n = \frac{1}{n\pi}$. observing that $\sin\frac{1}{a_n} = -\sin\frac{1}{b_n} = \pm 1$, we compute

$$\frac{|f_2(b_n) - f_2(a_n)|}{|b_n - a_n|^{1/4}} \;=\; \frac{\sqrt{1/(n+1)} + \sqrt{1/n}}{(n+n^2)^{-1/4}} \;\geq\; \frac{1}{2}\,.$$

This already shows that $f_2 \notin C^{0,\gamma}$ for any $\gamma > 1/4$. The fact that $f_2 \in C^{0,1/4}$ is proved by estimating the ratio

$$\frac{|f_2(b) - f_2(a)|}{|b-a|^{1/4}}$$

separately in two cases:

CASE 1: $b \in \left[\frac{1}{n\pi}\,,\ \frac{1}{(n-1)\pi}\right], \quad a \in \left[\frac{1}{(n+1)\pi}\,,\ b\right]$.

CASE 2: $b \in \left[\frac{1}{n\pi}\,,\ \frac{1}{(n-1)\pi}\right], \quad a < \frac{1}{(n+1)\pi}$.

The first case is handled simply by integrating $|f_2'|$ on the interval $[a,b]$, observing that $|f_2'(x)| \leq x^{-1/2} + x^{-3/2}$. Case 2 follows from Case 1, using the inequality

$$|f_2(b) - f_2(a)| \;\leq\; \sup\left\{|f_2(b) - f_2(x)|\,;\quad x \in \left[\frac{1}{(n+1)\pi}\,,\ b\right]\right\}.$$

(iii) $f_3 \in \mathcal{C}^{0,\gamma}$ for every $\gamma \in]0,1[$. Indeed, for any $0 < a < b < 1$ one has

$$f_3(b) - f_3(a) \;=\; \int_a^b (|\ln x| - 1)\,dx \;\leq\; \int_a^b C_\gamma x^{\gamma-1}\,dx \;\leq\; \int_a^b C_\gamma (x-a)^{\gamma-1}\,dx \;\leq\; \frac{C_\gamma}{\gamma}(b-a)^\gamma,$$

for some constant C_γ.

10. Ascoli's theorem guarantees the existence of a subsequence $(f_{n_k})_{k\geq 1}$ which converges to some limit function $f \in \mathcal{C}^{0,\gamma}([0,1])$, uniformly on $[0,1]$. This means $\|f_{n_k} - f\|_{\mathcal{C}^0} \to 0$. However, in general it is not true that $\|f_{n_k} - f\|_{\mathcal{C}^{0,\gamma}} \to 0$. In other words, the convergence takes place only in the norm of $\mathcal{C}^0$, not in the stronger norm of $\mathcal{C}^{0,\gamma}$.

11. To show that $\|\cdot\|_\varphi$ is a norm, we need to check that it satisfies the conditions (N1)–(N3) in the definition. The first two conditions are clear. To prove (N3) we observe that, for any $x, y \in \Omega$, $x \neq y$, one has

$$|(f+g)(x)| \;\leq\; |f(x)| + |g(x)|\,,$$

$$\frac{|f(x) + g(x) - f(y) - g(y)|}{\varphi(|x-y|)} \;\leq\; \frac{|f(x) - f(y)|}{\varphi(|x-y|)} + \frac{|g(x) - g(y)|}{\varphi(|x-y|)}\,.$$

Taking the supremum over all x, y we obtain $\|f+g\|_\varphi \;\leq\; \|f\|_\varphi + \|g\|_\varphi$.

It remains to show that the space $\mathcal{C}^\varphi$ is complete. For this purpose, let $(f_n)_{n\geq 1}$ be a Cauchy sequence w.r.t. the norm $\|\cdot\|_\varphi$. Then this sequence is Cauchy w.r.t. the $\mathcal{C}^0$ norm, hence it converges uniformly to a continuous function $f : \Omega \mapsto \mathbb{R}$. To show that $\|f_n - f\|_\varphi \to 0$, observe that, for any $\varepsilon > 0$ there exists N large enough so that

$$\sup_{x \neq y} \frac{\Big| f_n(x) - f_m(x) - f_n(y) + f_m(y) \Big|}{|x-y|} \; < \; \varepsilon \qquad \text{for all} \;\; m, n \geq N\,.$$

Keeping n fixed and letting $m \to \infty$, this implies

$$\sup_{x \neq y} \frac{\Big| f_n(x) - f(x) - f_n(y) + f(y) \Big|}{|x-y|} \; < \; \varepsilon \qquad \text{for all} \;\; n \geq N\,.$$

Hence $\|f_n - f\|_\varphi \to 0$.

12. One simply needs to retrace all steps in the proof of Ascoli's theorem. Since E is compact, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x,y) \; \leq \; \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| \; \leq \; \varepsilon \quad \text{for all } x, y \in E \text{ and } f \in \mathcal{F}.$$

Since K is compact, it is precompact. Hence we can choose points $\alpha_1, \ldots, \alpha_m \in K$ such that $K \subseteq \bigcup_{j=1}^m B(\alpha_j, \varepsilon)$. The remaining steps of the proof can be repeated without change.

Chapter 4

1. (1) The operator Λ is linear and bounded, with $\|\Lambda\| = 1$, but not compact.

(2) This operator is not linear: $(\Lambda(2f))(x) = \sin(2f(x)) \neq 2\sin(f(x)) = 2(\Lambda f)(x)$.

(3) Λ is linear and bounded, with $\|\Lambda\| = 1$, but not compact.

(4) Λ is linear, bounded, and compact. Indeed, Λf is always a polynomial of degree ≤ 1. Hence the range of Λ is two-dimensional.

(5) This operator Λ is linear, bounded, and compact. Indeed,

$$(\Lambda f)(x) \;=\; \int_0^x e^{y-x} f(y)\, dy\,,$$

Hence, for any $x \in [0,1]$,

$$|(\Lambda f)(x)| \;\leq\; \int_0^x |f(y)|\, dy \;\leq\; \|f|_{\mathcal{C}}\,.$$

To see that Λ is compact, let $(f_n)_{n \geq 1}$ be a sequence of continuous functions with $\|f_n\|_{\mathcal{C}} \leq 1$ for every n. Then all functions $\Lambda(f_n)$ are Lipschitz continuous with constant 2. Indeed, from the differential equation it follows

$$|y'(x)| \;\leq\; |f(x)| + |y(x)| \;\leq\; 1 + 1\,.$$

By Ascoli's theorem, the sequence (f_n) admits a uniformly convergent subsequence.

2. To show that E_n is closed, assume $\|f_k - f\|_{\mathbf{L}^1} \to 0$ and $\|f_k\|^2_{\mathbf{L}^2} \le n$ for every k. By possibly taking a subsequence we can assume $f_k(x) \to f(x)$ for a.e. $x \in [0,1]$. Fatou's lemma now yields

$$\int_0^1 |f(x)|^2 dx \;=\; \int_0^1 \liminf_{n\to\infty} |f_k(x)|^2 dx \;\le\; \liminf_{n\to\infty} \int_0^1 |f_k(x)|^2 dx \;\le\; n\,.$$

Hence $f \in E_n$ as well.

To prove that E_n has empty interior in $\mathbf{L}^1([0,1])$, consider any $f \in E_n$. For any $\varepsilon > 0$, define

$$f_\varepsilon(x) \;\doteq\; \begin{cases} f(x) + \varepsilon x^{-1/2} & \text{if } f(x) \ge 0\,, \\ f(x) - \varepsilon x^{-1/2} & \text{if } f(x) < 0\,. \end{cases}$$

Then $\|f_\varepsilon - f\|_{\mathbf{L}^1} = 2\varepsilon$ but $|f_\varepsilon(x)| \ge \varepsilon x^{-1/2}$ for every $x \in]0,1]$, and hence $f_\varepsilon \notin \mathbf{L}^2([0,1])$.

3. Let $\pi_n : \ell^\infty \mapsto \mathbb{R}$ be the projection operator, so that $\pi_n(x_1, x_2, \ldots) = x_n$. If $\Lambda : X \mapsto \ell^\infty$ is bounded, then the composition $\Lambda_n = \pi_n \circ \Lambda$ is bounded as well.

Viceversa, assume that each linear functional Λ_n is bounded. By assumption, for every $x \in X$ one has

$$\sup_{n\ge 1} |\Lambda_n(x)| \;<\; \infty\,.$$

The uniform boundedness principle thus implies $\|\Lambda\| = \sup_{n\ge1} \|\Lambda_n\| < \infty$.

4. By the closed graph theorem, it suffices to prove that Λ has closed graph. Toward this goal, consider a sequence of functions $f_n \in \mathbf{L}^p([0,1])$ such that

$$\|f_n - f\|_{\mathbf{L}^p} \;\to\; 0\,, \qquad\qquad \|\Lambda(f_n) - g\|_{\mathbf{L}^p} \;\to\; 0\,,$$

for some functions $f, g \in \mathbf{L}^p([0,1])$. We need to show that $\Lambda f = g$. By choosing a subsequence and relabeling, we can assume the pointwise convergence

$$f_n(x) \to f(x)\,, \qquad\qquad (\Lambda f_n)(x) \to g(x) \qquad\qquad \text{for a.e. } x \in [0,1]\,.$$

By assumption, this implies $(\Lambda f_n)(x) \to (\Lambda f)(x)$ for a.e. x. Hence $(\Lambda f)(x) = g(x)$ for a.e. $x \in [0,1]$.

5. Assume that the range $Y = K(X)$ is a closed subspace of X. Then Y itself is complete. The linear operator K is a continuous bijection between X and the Banach space Y. Hence it has a continuous inverse K^{-1}.

Being the composition of a compact operator and a continuous one, the identity map $\iota(x) = x = K^{-1}K(x)$ is a compact operator from X into itself. But this is impossible, because X is infinite dimensional.

6. Let $(x_n)_{n\ge1}$ be a bounded sequence of points in X.

CASE 1: Λ_1 compact. Then the sequence $\Lambda_1(x_n)$ admits a convergent subsequence, say $\Lambda_1(x_{n_k}) \to y$. Since Λ_2 is continuous, this implies the convergence $\Lambda_2 \circ \Lambda_1(x_{n_k}) \to \Lambda_2(y)$.

CASE 2: Λ_2 is compact. Since Λ_1 is continuous, the sequence $(\Lambda_1(x_n))_{n\geq 1}$ is bounded in Y. By the compactness of Λ_2, the sequence $(\Lambda_2\circ\Lambda_1(x_n))_{n\geq 1}$ admits a convergent subsequence.

7. If $Y \doteq Range(\Lambda)$ is an infinite dimensional normed space, then the unit ball in Y is not precompact. As shown in the proof of Theorem 2.22, there exist a sequence of points $y_n \in Y$ such that $\|y_n\| \leq 1$, $\|y_n - y_m\| \geq 1/2$ for all $m, n \geq 1$ $m \neq n$. Hence from this sequence $(y_n)_{n\geq 1}$ one cannot extract any convergent subsequence.

This leads a contradiction, because the assumptions imply that Λ is compact and $\Lambda(y) = y$ for every $y \in Y$.

8. First of all, observe that the assumptions imply $0 \in U$. Call $V \doteq -U \cap U$. Since $V \subseteq U$, it suffices to prove that V contains a neighborhood of the origin. Observe that

(i) V is closed, convex.

(ii) $V = -V$, $0 \in V$.

(iii) $\bigcup_{n\geq 1} nV = X$.

Indeed, (i)-(ii) are clear. To prove (iii), consider any $x \in X$. Then there exist integers n_1, n_2 large enough so that $x \in n_1 U$, $-x \in n_2 U$. Calling $n \doteq \max\{n_1, n_2\}$ we have $\frac{x}{n} \in U$, $-\frac{x}{n} \in U$. By convexity, this implies

$$A \doteq co\left\{\frac{x}{n}, -\frac{x}{n}\right\} = \left\{\theta x\,;\quad |\theta| \leq \frac{1}{n}\right\} \subseteq U.$$

Since $A = -A$, we conclude that $A \subseteq V$. Hence $x \in nA \subseteq nV$.

Since X is a Banach space, $X = \bigcup_{n\geq 1} nV$, and each set nV is closed, by Baire's category theorem at least one of the sets nV must have non-empty interior. By homogeneity, V itself has non-empty interior. If $B(x, r) \subseteq V$, then (ii) implies $B(-x, r) \subseteq V$. By convexity,

$$B(0, r) = \frac{1}{2}B(x, r) + \frac{1}{2}B(-x, r) \subseteq V.$$

9. Assume that K is surjective. By the open mapping theorem, K is an open map. In particular, the image of the unit ball $B_1 \doteq \{x \in X\,;\ \|x\| < 1\}$ contains a neighborhood of the origin. But this is impossible, because the closure $\overline{K}(B_1)$ is compact, and cannot contain any open set in the infinite dimensional space Y.

In the particular example, the point $\mathbf{y} = \left(1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\right)$ lies in ℓ^1 but not in the image $K(\ell^1)$.

10. Fix any $x \in \mathbb{R}$ and consider a sequence $x_n \to x$. Since f is bounded, there exists the upper and lower limits

$$m \doteq \liminf_{y\to x} f(y) \leq \limsup_{y\to x} f(y) \doteq M.$$

We claim that $m = M = f(x)$. Indeed, assume $m \neq f(x)$. Then we can find a sequence $x_n \to x$ such that $f(x_n) \to m$. Hence the point (x, m) lies in the closure of the graph of f. If

$f(x) \neq m$, this graph is not closed and we reach a contradiction. The proof that $f(x) = M$ is entirely similar.

To show that the result fails without the boundedness assumption, consider the function $f(x) = x^{-1}$ if $x \neq 0$, $f(0) = 0$. Then f has closed graph but is not continuous at $x = 0$.

11. Let $f : X \mapsto Y$ be continuous. Assume that $x_n \to x$ and $f(x_n) \to y$. By continuity, $f(x) = \lim_{n\to\infty} f(x_n) = y$, hence the graph of f is closed.

To construct the example, choose any function $0 \neq f \in \mathbf{L}^1(\mathbb{R})$. Define the map $g : t \mapsto g^t(\cdot)$ from $\mathbb{R}$ into $\mathbf{L}^1(\mathbb{R})$ by setting

$$g^t(x) \doteq \begin{cases} 0 & \text{if} \quad t \leq 0, \quad x \in \mathbb{R}\,, \\ f\Big(x - \dfrac{1}{t}\Big) & \text{if} \quad t > 0, \quad x \in \mathbb{R}\,. \end{cases}$$

Then g is not continuous at $t = 0$ because, for $t > 0$, $\|g^t - g^0\|_{\mathbf{L}^1} = \|f\|_{\mathbf{L}^1}$ and does not approach zero as $t \to 0+$. On the other hand, the map g has closed graph. To see this, simply observe that, for any sequence $t_k \to 0+$, the sequence f^{t_k} has no limit.

12. Assume that $\lim_{k\to\infty} \lambda_k \to 0$. For each $n \geq 1$ consider the truncated operator

$$\Lambda_n(x_1, x_2, \ldots) = (\lambda_1 x_1,\ \lambda_2 x_2, \ldots, \lambda_n x_n, 0, 0, \ldots). \tag{9}$$

Each Λ_n is continuous and has finite dimensional range, hence is compact. Observing that $\|\Lambda - \Lambda_n\| = \sup_{j>n} |\lambda_j| \to 0$ as $n \to \infty$, by Theorem 4.10 we conclude that Λ is compact.

On the other hand, assume that $\limsup_{n\to\infty} |\lambda_n| > 0$. Then there exists $\varepsilon > 0$ and a subsequence $(\lambda_{n_k})_{k\geq 1}$ such that $|\lambda_{n_k}| \geq \varepsilon$ for every $k \geq 1$. Consider the unit vectors

$$\mathbf{e}_1 = (1, 0, 0, 0, \ldots), \qquad \mathbf{e}_2 = (0, 1, 0, 0, \ldots), \qquad \mathbf{e}_3 = (0, 0, 1, 0, \ldots), \qquad \ldots$$

Then the sequence $(\mathbf{e}_{n_k})_{k\geq 1}$ is bounded, but the sequence $\Lambda(\mathbf{e}_{n_k}) = \lambda_{n_k}\mathbf{e}_{n_k}$ does not admit any convergent subsequence. Indeed,

$$\|\Lambda \mathbf{e}_{n_j} - \Lambda \mathbf{e}_{nk}\| = \|\lambda_{n_j}\mathbf{e}_{n_j} - \lambda_{n_k}\mathbf{e}_{nk}\| \geq \|\lambda_{n_k}\mathbf{e}_{nk}\| \geq \varepsilon\,.$$

Hence Λ is not compact.

13. Let B be continuous at the origin. Then there exists $\delta > 0$ such that $|B(x, y)| \leq 1$ whenever $\|x\| \leq \delta$ and $\|y\| \leq \delta$.

For any $(x, y) \in X \times Y$ we now have

$$|B(x, y)| = \frac{1}{\delta^2} \cdot \|x\|\,\|y\| \cdot B\left(\frac{\delta x}{\|x\|}\,,\ \frac{\delta y}{\|y\|}\right) \leq C \cdot \|x\|\,\|y\|\,,$$

with $C = \delta^{-2}$.

14. Given any polynomial p, we have

$$\left|\int_0^1 p(t)\,q(t)\,dt\right| \leq \max_{x\in[0,1]} |p(x)| \cdot \int_0^1 |q(t)|\,dt.$$

Hence the map $\Lambda_p : q \mapsto B(p,q)$ is a bounded linear operator from X into itself, with norm $\|\Lambda_p\| = \|p\|_{\mathcal{C}([0,1])}$.

To prove that the bilinear map B is not continuous on the product space $X \times X$, we proceed as follows. Choose a sequence of numbers $x_n > 0$ such that $|\ln x_n| > n^2$. Define the continuous functions

$$f_n(x) \doteq \begin{cases} \dfrac{1}{\sqrt{nx}} & \text{if } x \in [x_n, 1]\,, \\[2ex] \dfrac{1}{\sqrt{nx_n}} & \text{if } x \in [0, x_n]\,. \end{cases}$$

Observe that

$$\int_0^1 f_n(x)\,dx \;<\; \frac{1}{\sqrt{n}} \int_0^1 \sqrt{x}\,dx \;=\; \frac{2}{\sqrt{n}}\,.$$

$$\int_0^1 f_n^2(x)\,dx \;>\; \int_{x_n}^1 \frac{1}{nx}\,dx \;>\; \frac{1}{n}\,|\ln x_n| \;\geq\; n\,.$$

By the Stone-Weierstrass theorem, each function f_n can be uniformly approximated by a polynomial. We can thus construct a sequence of polynomials $(p_n)_{n\geq 1}$ such that

$$\int_0^1 p_n(x)\,dx \;<\; \frac{2}{\sqrt{n}}\,, \qquad\qquad \int_0^1 p_n^2(x)\,dx \;>\; n$$

for every $n \geq 1$. We thus have $(p_n, p_n) \to 0$ in $X \times X$, but $B(p_n, p_n) \to \infty$.

15. Since S is bounded, there exists M such that $\|x\| \leq M$ for all $x \in S$. Let $\varepsilon \in]0,1]$ be given. Let Y_ε be a finite dimensional space such that $d(x, Y_\varepsilon) \leq \varepsilon$ for all $x \in S$.

In the finite dimensional space Y_ε, the closed ball $B \doteq \{y \in Y_\varepsilon\,;\;\; \|y\| \leq 1 + M\}$ centered at the origin with radius $1 + M$ is compact. Hence it can be covered with finitely many balls of radius ε, say $B(y_n, \varepsilon)$, $n = 1, \ldots, N$. We claim that

$$S \;\subseteq\; \bigcup_{n=1}^N B(y_n, 2\varepsilon)\,.$$

Indeed, for every $x \in S$ there exists $y_x \in Y_\varepsilon$ such that $\|x - y_x\| \leq \varepsilon$. Clearly, we must have $\|y_x\| \leq M + \varepsilon$. Hence $y_x \in B(y_n, \varepsilon)$ for some n. This implies $x \in B(y_n, 2\varepsilon)$.

The previous analysis shows that S is precompact. Namely, for every $\varepsilon > 0$ is can be covered by finitely many balls of radius ε. By assumption, S is a closed subset of a Banach space, hence it is complete. We thus conclude that S is compact.

16. If $A \neq \lambda I$, consider two cases.

CASE 1: There exists a vector $f \in X$ such that $g = Af$ is not a scalar multiple of f.

Clearly, the two vectors f, g must be linearly independent.

Consider the two-dimensional subspace $V = span\{f, g\}$. Let $\varphi : V \mapsto \mathbb{R}$ be the linear functional such that $\varphi(f) = 0$, $\varphi(g) = 1$. Otherwise stated,

$$\varphi(\alpha f + \beta g) \;\doteq\; \beta \qquad\qquad \text{for all } \;\alpha, \beta \in \mathbb{K}\,.$$

Using the Hahn-Banach extension theorem, this functional can be extended to a bounded linear functional, still called φ, defined on the entire space X.

We then define the operator $K : X \mapsto X$ by setting $K(x) \doteq \varphi(x)\, g$. Notice that this implies

$$K(f) = \varphi(f)\, g = 0, \qquad\qquad K(g) = \varphi(g)\, g = g\,.$$

Moreover, K has one-dimensional range, hence it is compact.

We now check that $A(Kf) = 0$ while $K(Af) = Kg = g$. Hence $AK \neq KA$.

CASE 2: There exists two nonzero vectors $f, g \in X$ such that $Af = \lambda f$, $Ag = \lambda' g$ with $\lambda \neq \lambda'$. Then $A(f+g) = \lambda f + \lambda' g$ is not a scalar multiple of $f+g$, and Case 1 again applies.

17. Since V is an infinite dimensional normed space, we can find a sequence of points $(x_n)_{n\geq 1}$ in V such that $\|x_n\| \leq 1$, $\|x_m - x_n\| \geq 1/2$ for all $m \neq n$. Then the sequence $(\Lambda x_n)_{n\geq 1}$ does not admit any convergent subsequence. Indeed, by assumption

$$\|\Lambda x_m - \Lambda x_n\| \;\geq\; \varepsilon \|x_m - x_n\| \;\geq\; \frac{\varepsilon}{2} \qquad\qquad \text{for all } \; m \neq n.$$

Therefore, the operator Λ cannot be compact.

18. Consider the family $\{\varphi_n\,;\;\; n \geq 1\}$ of bounded linear functionals on X^*, defined by

$$\varphi_n(x^*) \;=\; \sum_{k=1}^{n} \langle x^*,\, x_k\rangle.$$

By assumption,

$$\lim_{n\to\infty} \varphi_n(x^*) \;=\; \varphi(x^*)$$

exists for every $x^* \in X^*$. Hence $\sup_{n\geq 1} |\varphi_n(x^*)| \;<\; \infty$ for every $x^* \in X^*$.

By the uniform boundedness principle, $M \doteq \sup_{n\geq 1} \|\varphi_n\| < \infty$. Therefore

$$\|\varphi\| \;\doteq\; \sup_{\|x^*\|\leq 1} |\varphi(x^*)| \;\leq\; \sup_{\|x^*\|\leq 1}\, \sup_{n\geq 1} |\varphi_n(x^*)| \;\leq\; \sup_{n\geq 1} \|\varphi_n\| \;=\; M,$$

showing that φ is bounded.

19. For every continuous function $f : [0,1] \mapsto \mathbb{R}$, the mean value theorem implies

$$\lim_{t\to 0} \frac{1}{t}\int_0^t f(s)\, ds \;=\; f(0)\,.$$

Hence the function Λf is continuous as well.

(i) It is clear that the operator Λ is linear. Observing that the average value of f over the interval $[0,t]$ satisfies

$$\left|\frac{1}{t}\int_0^t f(s)\, ds\right| \;\leq\; \max_{s\in[0,t]} |f(s)| \;\leq\; \|f\|_{\mathcal{C}([0,1]}\,,$$

we conclude that Λ is a bounded operator, with norm $\|\Lambda\| \leq 1$.

(ii) If $(\Lambda f)(x) = 0$ for every $x \in [0,1]$, then $\int_0^t f(s)\,ds = 0$ for every $t > 0$ and hence $f(s) = 0$ for every $x \in [0,1]$. This proves that Λ is one-to-one.

If $g = \Lambda f$ for some continuous function f, taking derivatives we obtain

$$g'(t) \;=\; \frac{f(t) - g(t)}{t} \qquad \text{for all} \;\; t \in \,]0,1[\,.$$

Hence g is continuously differentiable. Any continuous function $g : [0,1] \mapsto \mathbb{R}$ which is not continuously differentiable cannot be in the range of Λ.

(iii) To prove that Λ is not compact, consider the sequence of functions

$$f_n(x) \;\doteq\; \begin{cases} \sin 2^n x & \text{if} \quad x \in [0,\, 2^{1-n}\pi]\,, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\Lambda f_n(x) = 0$ for $x \geq 2^{1-n}\pi$. Hence, for $m < n$ a direct computation yields

$$\|\Lambda f_m - \Lambda f_n\|_{\mathcal{C}^0} \;\geq\; \Big|\Lambda f_m(2^{-m}\pi) - \Lambda f_n(2^{-m}\pi)\Big| \;=\; \Big|\frac{2}{\pi} - 0\Big| \;=\; \frac{2}{\pi}\,.$$

As a consequence, the sequence $(\Lambda f_n)_{n\geq 1}$ does not admit any uniformly convergent subsequence.

20. (i) The multiplication operator M_f is one-to-one if and only if $g(x) \neq 0$ for a.e. $x \in \Omega$.

(ii) In general, the range of M_g is not closed. For example, let $\Omega = \,]0,1[$ and $g(x) = x$. Then the function $h(x) \equiv 1$ is not in the range of M_g, because the function $f(x) = h(x)/g(x) = 1/x$ does not lie in $\mathbf{L}^p(\Omega)$. However, if $1 \leq p < \infty$, then the characteristic functions $f_n \doteq \chi_{[1/n\,,\,1]}$ lie in the range of M_g. Moreover, $\|f_n - f\|_{\mathbf{L}^p} \to 0$ whenever $1 \leq p < \infty$.

To see a positive result, assume that there exists $\varepsilon > 0$ such that the domain Ω can be decomposed as $\Omega = \Omega_0 \cup \Omega_\varepsilon$, with Ω_0, Ω_ε disjoint measurable sets, and

$$g(x) \;=\; 0 \quad \text{for a.e. } x \in \Omega_0\,, \qquad\qquad |g(x)| \;\geq\; \varepsilon \quad \text{for a.e. } x \in \Omega_\varepsilon\,.$$

Then Λ has closed range. Indeed,

$$\mathit{Range}(\Lambda) \;=\; \Big\{ f \in \mathbf{L}^p(\Omega)\,; \;\; f(x) = 0 \quad \text{for a.e.} \quad x \in \Omega_0 \Big\}.$$

(iii) If g does not coincide a.e. with the zero function, then the operator M_g is not compact.

To see this, assume that, for some $\varepsilon > 0$, the set $A_\varepsilon \doteq \{x \in \Omega\,; \;\; |g(x)| \geq \varepsilon\}$ has strictly positive measure. Then we can find countably many disjoint measurable sets A_n such that

$$A_n \;\subset\; A_\varepsilon\,, \qquad\qquad 0 \;<\; \mathit{meas}(A_n) \;<\; 1 \qquad \text{for all} \;\; n \geq 1\,.$$

Define the sequence of functions

$$f_n(x) \;\doteq\; \begin{cases} c_n & \text{if} \;\; x \in A_n\,, \\ 0 & \text{otherwise,} \end{cases}$$

choosing the constants c_n so that $\|f_n\|_{\mathbf{L}^p} = 1$. By construction, the sequence $(f_n)_{n\geq 1}$ is bounded. However, the sequence $(M_g f_n)_{n\geq 1}$ does not admit any convergent subsequence. Indeed, for any $m \neq n$, since f_n and f_m have disjoint support, one has

$$\|f_n g - f_m g\|_{\mathbf{L}^p} \;\geq\; \|f_n g\|_{\mathbf{L}^p} \;\geq\; \varepsilon\, \|f_n\|_{\mathbf{L}^p} \;=\; \varepsilon.$$

21. In the space ℓ^1, consider the compact operator

$$\Lambda(x_1, x_2, x_3, \ldots) \;=\; \Big(\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\Big).$$

Then Λ is one-to-one and its range is dense in ℓ^1. However, given any $\varepsilon > 0$, choose $n > \varepsilon^{-1}$ and consider the truncated operator

$$\Lambda_n(x_1, x_2, x_3, \ldots) \;=\; \Big(\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, 0, 0, \ldots\Big).$$

Then $\|\Lambda_n - \Lambda\| < \varepsilon$ but Λ_n is not one-to-one and its range is not dense in ℓ^1.

22. (i) By the open mapping theorem, the functional Λ has a continuous inverse. Hence there exists a constant $C > 0$ such that $\|\Lambda^{-1} y\| \leq C\,\|y\|$ for every $y \in Y$. Taking $\beta = C^{-1}$, this implies $\beta\|x\| \leq \|\Lambda x\|$ for all $x \in X$.

(ii) We check that, if $\gamma \doteq \|\Psi\| < 1/\beta$, then the map $u \mapsto \Lambda^{-1}(f - \Psi u)$ is a strict contraction. Indeed, for any $u, v \in X$ one has

$$\|\Lambda^{-1}(f-\Psi u) - \Lambda^{-1}(f-\Psi v)\| \;\leq\; \|\Lambda^{-1}(\Psi v) - \Lambda^{-1}(\Psi u)\| \;\leq\; \|\Lambda^{-1}\|\cdot\|\Psi\|\,\|u-v\| \;\leq\; \beta\cdot\gamma\|u-v\|\,.$$

Since $\beta \cdot \gamma < 1$, this is a strict contraction and therefore has a unique fixed point.

(iii) Let $\Lambda : X \mapsto Y$ be a continuous bijection, and let $\beta > 0$ be such that $\|\Lambda x\| \geq \beta\|x\|$ for all $x \in X$. We claim that, if $\Psi : X \mapsto Y$ is any bounded linear operator with $\|\Psi\| < \beta^{-1}$, then the sum $\Lambda + \Psi$ is still a bijection. Indeed, by (ii) for any $f \in Y$ there exists a unique element $u \in X$ such that $u = \Lambda^{-1}(f - \Psi u)$. This implies $\Lambda u = f - \Psi u$, hence $(\Lambda + \Psi)u = f$. showing that the operator Λ_Ψ is a bijection.

23. (i) From the definition it follows

$$|y_n| \;=\; \left|\frac{1}{n}\sum_{i=1}^n x_i\right| \;\leq\; \max_{1\leq i\leq n} |x_i| \;\leq\; \|x\|_{\ell^\infty}\,.$$

Hence this averaging operator has norm $\|\Lambda\| = 1$.

To prove that Λ is not compact, consider the sequences $\mathbf{x}_k = (x_{k1}, x_{k2}, x_{k3}, \ldots) \in \ell^\infty$ defined by

$$x_{ki} \;\doteq\; \begin{cases} 1 & \text{if } 2^{k-1} < i \leq 2^k\,, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\|\mathbf{x}_k\|_{\ell^\infty} = 1$ for every k. However, this bounded sequence does not admit any subsequence $(\mathbf{x}_{k_\ell})_{\ell\geq 1}$ such that $\Lambda\mathbf{x}_{k_\ell}$ converges. Indeed, if $j < k$, then

$$\|\Lambda\mathbf{x}_j - \Lambda\mathbf{x}_k\|_{\ell^\infty} \;\geq\; |x_{j,2^j} - x_{k,2^j}| \;=\; \left|\frac{1}{2} - 0\right| \;=\; \frac{1}{2}\,.$$

(ii) Λ is not a bounded operator from ℓ^1 into ℓ^1. For example, if $x = (1, 0, 0, \ldots) \in \ell^1$, then $\Lambda x = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \notin \ell^1$.

24. (i) If $Y = X$, let $\iota : Y \mapsto X$ be the identity map. This is a bijective, continuous, linear map from Y onto X. By Corollary 4.5, the inverse map $\iota^{-1} : X \mapsto Y$ is continuous as well. Hence there exists a constant C' such that $\|x\|_Y \leq C' \|x\|_X$ for all $x \in X$.

(ii) Assume $Y \neq X$. Consider the sets $S_n \doteq \{y \in Y\,;\;\; \|y\|_Y \leq n\}$, and let $\overline{S}_n$ be the closure of S_n in the space X. We claim that

(C) Each $\overline{S}_n$ has empty interior.

If (C) holds, then

$$Y \;=\; \bigcup_{n\geq 1} S_n \;\subseteq\; \bigcup_{n\geq 1} \overline{S}_n\,.$$

Hence Y is of first category, being contained in the union of countably many closed sets with empty interior.

To prove (C), we argue by contradiction. If some $\overline{S}_n$ has nonempty interior, by positive homogeneity the set $\overline{S}_1$ has nonempty interior as well. To fix the ideas, assume that the open ball $B(x, r) \subset \overline{S}_1$ for some $x \in \overline{S}_1$ and $r > 0$. Since $\overline{S}_1 = -\overline{S}_1$, we also have $B(-x, r) \subset \overline{S}_1$. By convexity, this implies

$$B(0, r) \;=\; \frac{1}{2} B(x, r) + \frac{1}{2} B(-x, r) \;\subset\; \overline{S}_1\,.$$

In turn, by positive homogeneity this yields

$$B(0, 2^{-n} r) \;\subset\; \overline{S}_{2^{-n}} \;\doteq\; \overline{\{y \in Y\,;\;\; \|y\|_Y \leq 2^{-n}\}}\,.$$

Repeating the argument in step **3.** of the proof of Theorem 4.4 (the Open Mapping Theorem), we conclude that $B(0, r/2) \subset S_1$.

For any $x \in X$ we thus have

$$x \;=\; \frac{3\,\|x\|_X}{r} \cdot \frac{rx}{3\|x\|_X} \;\in\; \frac{3\,\|x\|_X}{r} \cdot B(0, r/2) \;\subset\; \frac{3\,\|x\|_X}{r} \cdot S_1 \;\subset\; Y.$$

This shows that $X = Y$, against the assumption.

Chapter 5

1. Using the properties of inner products one obtains

$$(x,\, y + z) \;=\; \overline{(y + z, x)} \;=\; \overline{(y, x)} + \overline{(z, x)} \;=\; (x, y) + (x, z).$$

$$(x, \alpha y) \;=\; \overline{(\alpha y, x)} \;=\; \overline{\alpha (y,x)} \;=\; \bar{\alpha}(x,y)\,.$$

Saying that the vectors x, y are mutually orthogonal means that $(x,y) = 0$. This implies

$$\|x+y\|^2 \;=\; (x+y,\, \overline{x+y}) \;=\; (x,\bar{x}) + (x,\bar{y}) + (y,\bar{x}) + (y,\bar{y}) \;=\; \|x\|^2 + 0 + 0 + \|y\|^2.$$

2. Let $x, y \in H$ and $\varepsilon > 0$ be given. Let $M \doteq \max\{\|x\|,\, \|y\|\}$. Choose $0 < \delta < 1$, such that $\delta = \varepsilon/(1+2M)$. If $\|x'-x\| \leq \delta$ and $\|y'-y\| \leq \delta$, then

$$\begin{aligned} |(x',y') - (x,y)| \;&=\; \Big|(x'-x,\, y'-y) + (x'-x,\, y) + (x,\, y'-y)\Big| \\ &\leq\; \|x'-x\|\,\|y'-y\| + \|x'-x\|\,\|y\| + \|x\|\,\|y'-y\| \;\leq\; \delta^2 + 2M\delta \;\leq\; \varepsilon\,. \end{aligned}$$

3. (i) The parallelogram identity is proved by writing

$$\|x+y\|^2 + \|x-y\|^2 \;=\; (x+y,\, x+y) + (x-y,\, x-y) \;=\; 2(x,x) + 2(y,y) \;=\; 2\|x\|^2 + 2\|y\|^2\,. \tag{10}$$

(ii) Assume that the norm $\|\cdot\|$ satisfies the parallelogram identity. We claim that the assignment

$$(x,y) \;\doteq\; \frac{1}{2}\Big(\|x+y\|^2 - \|x\|^2 - \|y\|^2\Big) \;=\; \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) \tag{11}$$

satisfies all properties of an inner product. Note that the equality between the last two terms in (11) is an easy consequence of (10).

By (11), the identity $(x,y) \;=\; (y,x)$ is obvious.

To check that $(x+x', y) = (x,y) + (x',y)$, using the parallelogram identity we obtain

$$\begin{aligned} \|x+x'+y\|^2 \;&=\; 2\|x+y\|^2 + 2\|x'\|^2 - \|x+y-x'\|^2 \\ &=\; 2\|x'+y\|^2 + 2\|x\|^2 - \|x'+y-x\|^2 \\ &=\; \|x\|^2 + \|x'\|^2 + \|x+y\|^2 + \|x'+y\|^2 - \frac{1}{2}\|x+y-x'\|^2 - \frac{1}{2}\|x'+y-x\|^2\,. \end{aligned}$$

This implies

$$\begin{aligned} (x+x',y) \;&=\; \frac{1}{4}\Big(\|x+x'+y\|^2 - \|x+x'-y\|^2\Big) \\ &=\; \frac{1}{4}\left(\|x\|^2 + \|x'\|^2 + \|x+y\|^2 + \|x'+y\|^2 - \frac{1}{2}\|x+y-x'\|^2 - \frac{1}{2}\|x'+y-x\|^2\right) \\ &\qquad -\frac{1}{4}\left(\|x\|^2 + \|x'\|^2 + \|x-y\|^2 + \|x'-y\|^2 - \frac{1}{2}\|x-y-x'\|^2 - \frac{1}{2}\|x'-y-x\|^2\right) \\ &=\; \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) + \frac{1}{4}\Big(\|x'+y\|^2 - \|x'-y\|^2\Big) \;=\; (x,y) + (x',y)\,. \end{aligned}$$

In particular, from the above identity by induction it follows

$$(nx, y) \;=\; (x + x + \cdots + x, y) \;=\; n(x, y)\,.$$

In turn, this implies $\left(\frac{1}{n}x, y\right) \;=\; \frac{1}{n}(x, y)$, and hence

$$(\lambda x, y) \;=\; \lambda(x, y) \tag{12}$$

for every rational number $\lambda = \frac{m}{n}$. By continuity, we conclude that (12) remains valid for every $\lambda \in \mathbb{R}$.

Finally, we check that if $x \neq 0$, then

$$(x, x) \;=\; \frac{1}{4}\left(\|x + x\|^2 - \|x - x\|^2\right) \;=\; \|x\|^2 \;>\; 0.$$

Hence the inner product is positive definite, and $\sqrt{(x, x)} \;=\; \|x\|$.

4. (i) Choose $\mathbf{x} = (1, 0)$, $\mathbf{y} = (0, 1)$. Then $\|\mathbf{x}\|_p = \|\mathbf{y}\|_p = 1$, $\|\mathbf{x} + \mathbf{y}\|_p = \|\mathbf{x} - \mathbf{y}\|_p = 2^{1/p}$. If $p \neq 2$, then

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 \;=\; 2^{2/p} + 2^{2/p} \;\neq\; 2 + 2 \;=\; 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

(ii) Choose $f(x) = \sin \pi x$, $g(x) = 1 - \sin \pi x$. Then $\|f\| = \|g\| = \|f + g\| = \|f - g\| = 1$ and the parallelogram identity fails.

5. Since the monomials $1, x$ are mutually orthogonal in $\mathbf{L}^1([-1, 1])$, the first two polynomials in an orthonormal basis are

$$p_0(x) \;=\; \frac{1}{\|1\|_{\mathbf{L}^2}} \;=\; \frac{1}{\sqrt{2}}\,, \qquad\qquad p_1(x) \;=\; \frac{x}{\|x\|_{\mathbf{L}^2}} \;=\; \sqrt{\frac{3}{2}}\,x\,.$$

To find p_2, we compute the inner product

$$(p_0, x^2)_{\mathbf{L}^2} \;=\; \int_{-1}^{1} p_0(x)\, x^2\, dx \;=\; \int_{-1}^{1} \frac{x^2}{\sqrt{2}}\, dx \;=\; \frac{\sqrt{2}}{3}\,.$$

Hence $q(x) = x^2 - (p_0, x^2)_{\mathbf{L}^2} \cdot p_0(x) \;=\; x^2 - \frac{1}{3}$ is perpendicular to p_0, and to p_1 as well. We compute

$$\|q\|^2_{\mathbf{L}^2} \;=\; \int_{-1}^{1} \Big(x^2 - \frac{1}{3}\Big)^2 dx \;=\; 2\int_0^1 \Big(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\Big)\, dx \;=\; 2\Big(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\Big) \;=\; \frac{8}{45}\,.$$

Therefore

$$p_2(x) \;=\; \frac{q(x)}{\|q\|_{\mathbf{L}^2}} \;=\; \sqrt{\frac{45}{8}}\,\Big(x^2 - \frac{1}{3}\Big).$$

6. For every $x \in H$, Bessel's inequality (5.10) implies $\sum_{n=1}^{\infty} |(x, e_n)|^2 \;\leq\; \|x\|^2$. Hence $\lim_{n\to\infty} |(x, e_n)| = 0$. By definition, this means that $e_n \rightharpoonup 0$.

7. Since H is infinite dimensional, for a given vector $x \in H$ we can find an orthonormal sequence $(e_k)_{k\geq 1}$ such that $(x, e_k) = 0$ for every k. Choose $\lambda \doteq \sqrt{1 - \|x\|^2}$. Then the sequence $x_n \doteq x + \lambda e_n$ satisfies all requirements. Indeed, by Pythagora's theorem $\|x_n\|^2 = \|x\|^2 + \lambda^2 = 1$. Moreover, for any $y \in H$ by the previous problem **6.** we have

$$\lim_{n\to\infty} (y, x_n) \;=\; (y, x) + \lim_{n\to\infty} \lambda(y, e_n) \;=\; (y, x)\,.$$

8. (i) By assumption, $\sum_k \|\alpha_k v_k\| < \infty$, hence the series is absolutely convergent. Since H is complete, the series converges.

(ii) Let $(v_k)_{k\geq 1}$ be an orthonormal sequence. Then (assuming $m \leq n$)

$$\limsup_{m,n\to\infty} \left| \sum_{m<k\leq n} \alpha_k v_k \right| \;=\; \limsup_{m,n\to\infty} \sum_{m<k\leq n} |\alpha_k|^2\,.$$

This shows that the sequence of partial sums is Cauchy if and only if $\sum_k |\alpha_k|^2 < \infty$.

9. For any $f \in \mathbf{L}^2(\mathbb{R}^n)$, performing the change of variable $y = \phi(x)$ and using the fact that $det D\phi \equiv 1$, we compute

$$\int_{\mathbb{R}^n} |(\Lambda f)(x)|^2\,dx \;=\; \int_{\mathbb{R}^n} |f(\phi(x))|^2\,dx \;=\; \int_{\mathbb{R}^n} |f(y)|^2\,dy.$$

Hence $\|\Lambda f\|_{\mathbf{L}^2} = \|f\|_{\mathbf{L}^2}$ for every $f \in \mathbf{L}^2(\mathbb{R}^n)$, proving that $\|\Lambda\| = 1$.

The adjoint operator Λ^* is defined by the identity

$$\int_{\mathbb{R}^n} f(y)\cdot(\Lambda^* g)(y)\,dy \;=\; \int_{\mathbb{R}^n} (\Lambda f)(x)\cdot g(x)\,dx \;=\; \int_{\mathbb{R}^n} f(\phi(x))\cdot g(x)\,dx \;=\; \int_{\mathbb{R}^n} f(y)\cdot g(\phi^{-1}(y))\,dy\,.$$

Hence $(\Lambda^* g)(y) \;=\; g(\phi^{-1}(y)$, showing that $\Lambda^* = \Lambda^{-1}$.

10. The adjoint operator is defined by the identity

$$(\Lambda x, y) \;=\; \sum_{k=1}^{\infty} x_{i+1} y_i \;=\; \sum_{j=2}^{\infty} x_j y_{j-1} \;=\; (x, \Lambda^* y).$$

Hence $\Lambda^*(y_1, y_2, y_3, \ldots) = (0, y_1, y_2, \ldots)$. Clearly, Λ is onto but not one-to-one, while Λ^* is one-to-one but not onto.

11. (i) Assume $x = \theta x_1 + (1-\theta)x_2$, for some $x_1 \neq x_2$, with $\|x_1\| \leq 1$, $\|x_2\| \leq 1$ and $0 < \theta < 1$.

By the parallelogram' identity

$$\|x_1 + x_2\|^2 \;=\; \|x_1\|^2 + \|x_2\|^2 + 2Re(x_1, x_2) \;=\; 2\|x_1\|^2 + 2\|x_2\|^2 - \|x_1 - x_2\|^2.$$

If $x_1 \neq x_2$, then we have the strict inequality

$$2Re(x_1, x_2) \;<\; \|x_1\|^2 + \|x_2\|^2.$$

Therefore

$$\begin{aligned}
\|x\|^2 &= \Big(\theta x_1 + (1-\theta)x_2\,,\ \theta x_1 + (1-\theta)x_2\Big)\\
&= \theta^2\|x_1\|^2 + (1-\theta)^2\|x_2\|^2 + 2\theta(1-\theta)Re(x_1,x_2)\\
&< \theta^2\|x_1\|^2 + (1-\theta)^2\|x_2\|^2 + \theta(1-\theta)(\|x_1\|^2 + \|x_2\|^2)\\
&\leq \theta^2 + (1-\theta)^2 + 2\theta(1-\theta) \ = \ 1.
\end{aligned}$$

This proves that every point x in the closed unit ball of a Hilbert space which is not extremal must have norm $\|x\| < 1$. Hence all points with $\|x\| = 1$ are extremal.

(ii) Next, assume $f \in \mathbf{L}^1([0,1])$ with $\|f\|_{\mathbf{L}^1} = 1$. Then we can find a set $A \subset [0,1]$ with positive measure such that $|f(x)| \geq 1/2$ for every $x \in A$. To fix the ideas, assume $f(x) \geq 1/2$ for all $x \in A$, with $meas(A) > 0$. Choose two disjoint measurable subsets $A_1, A_2 \subset [0,1]$ with $A_1 \cup A_2 = A$, $meas(A_1) = meas(A_2)$. Consider the function

$$g(x) \ = \ \begin{cases} \ \ 1/2 & \text{if } \ x \in A_1\,,\\ -1/2 & \text{if } \ x \in A_2\,,\\ 0 \quad \text{if } \ x \notin A_1 \cup A_2\,. \end{cases}$$

Then

$$\|f+g\|_{\mathbf{L}}^1 = \|f-g\|_{\mathbf{L}^1} \ = \ 1\,, \qquad\qquad f \ = \ \frac{f+g}{2} + \frac{f-g}{2}\,.$$

Hence f is not an extreme point of the unit ball in $\mathbf{L}^1$, being a convex combination of the two functions $f+g$ and $f-g$.

12. By taking a subsequence and relabeling, one can assume $\|x_n\| \to C$ as $n \to \infty$. Since this sequence is bounded, by Theorem 5.14 it admits a weakly convergent sequence, say $x_{n_j} \rightharpoonup x$ for some $x \in H$. We now compute

$$\|x\|^2 = (x,x) \ = \ \lim_{j\to\infty}(x, x_{n_j}) \ \leq \ \limsup_{j\to\infty}\|x\|\,\|x_{n_j}\| \ \leq \ C\|x\|\,.$$

Dividing by $\|x\|$ we conclude $\|x\| \leq C$.

13. By assumption, H admits a countable, everywhere dense set $\{x_1, x_2, x_3, \ldots\}$. Using the Gram-Schmidt procedure, from this set we construct a countable orthonormal basis $\{e_1, e_2, e_3, \ldots\}$.

Defining

$$\Lambda x \ \doteq \ \mathbf{a} = (a_1, a_2, a_3, \ldots) \qquad \text{with} \ \ a_k \doteq (x, e_k),$$

we obtain the desired bijection. Indeed,

$$\|\Lambda x\|_{\ell^2} \ = \ \sum_{k=1}^{\infty}|a_k|^2 \ = \ \sum_{k=1}^{\infty}|(x,e_k)|^2 \ = \ \|x\|\,.$$

14. (i) Assume $x \in V \doteq span\{v_1, \ldots, v_n\}$, so that

$$x \ = \ \sum_{k=1}^{n}\theta_k v_k$$

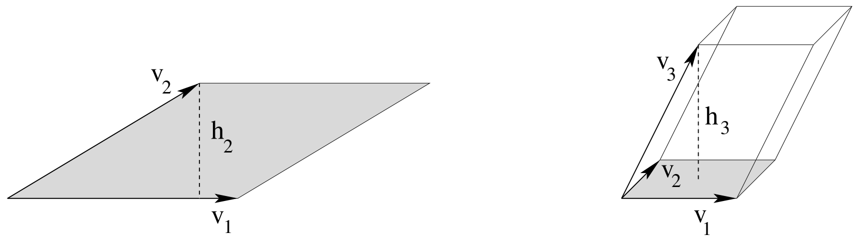

Figure 1: Computing the area of a parallelogram and the volume of a parallelepiped. Here $h_2 = d(v_2\,,\ span\{v_1\})$ while $h_3 = d(v_3\,,\ span\{v_1, v_2\})$.

for some coefficients θ_k. To actually compute these coefficients, we observe that, for every $j = 1, \ldots, n$, one must have

$$(x, v_j) \;=\; \sum_{k=1}^{n} \theta_k (v_k, v_j)\,.$$

Therefore the numbers $\theta_1, \ldots, \theta_n$ are obtained by solving the system of n linear equations

$$\begin{pmatrix} (v_1, v_1) & \cdots & (v_n, v_1) \\ \vdots & \ddots & \vdots \\ (v_1, v_n) & \cdots & (v_n, v_n) \end{pmatrix} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \;=\; \begin{pmatrix} (x, v_1) \\ \vdots \\ (x, v_n) \end{pmatrix}. \tag{13}$$

Take x to be the zero vector in (13). Then that the vectors $v_1, \ldots, v_n$ are linearly independent if and only if $\theta_1 = \theta_2 = \cdots = \theta_n = 0$ is the only solution of (13) when $x = 0$. This is the case if and only if the determinant $G((v_1, \ldots, v_n)$ of the $n \times n$ symmetric matrix in (13) is not zero.

(ii) If $x \in H$ and $y \in V$ then $G(x,\, v_1, v_2, \ldots, v_n) = G(x + y,\, v_1, v_2, \ldots, v_n)$. In particular, taking $y = P_V(x)$ and $z \;=\; x - y \;=\; P_{V^\perp}(x)$, we obtain

$$d(x, V) = \|z\|\,, \qquad G(x, v_1, v_2, \ldots, v_n) \;=\; G(z, v_1, v_2, \ldots, v_n) \;=\; (z, z)\, G(v_1, v_2, \ldots, v_n).$$

Therefore

$$d(x, V) \;=\; \|z\| \;=\; (z, z)^{1/2} \;=\; \sqrt{\frac{G(x, v_1, v_2, \ldots, v_n)}{G(v_1, v_2, \ldots, v_n)}}\,.$$

(iii) The volume of the parallelepiped with edges $v_1, \ldots, v_n$ can be computed as a product:

$$\|v_n\| \cdot d\Big(v_{n-1}\,;\ span\{v_n\}\Big) \cdot d\Big(v_{n-2}\,;\ span\{v_{n-1}, v_n\}\Big)\ \cdots\ d\Big(v_1\,;\ span\{v_2, \ldots, v_n\}\Big)$$

$$\sqrt{G(v_n)} \cdot \sqrt{\frac{G(v_{n-1}, v_n)}{G(v_n)}}\ \cdots\ \sqrt{\frac{G(v_1, v_2, \ldots, v_n)}{G(v_2, \ldots, v_n)}} \;=\; \sqrt{G(v_1, v_2, \ldots, v_n)},$$

where each factor was computed using (ii).

15. Let $U \subset \mathbf{L}^2(\mathbb{R})$ be the subspace of all even functions. This is a closed subspace, hence for every $f \in \mathbf{L}^2$ the perpendicular projection $g_0 = \pi_U f$ is well defined. We claim that

$$g_0(x) \;=\; \frac{f(x) + f(-x)}{2}\,.$$

Indeed, $g_0(x) = g_0(-x)$ for every $x \in \mathbb{R}$, hence $g_0 \in U$. It remains to check that $f - g_0$ is perpendicular to every function $g \in U$. This is indeed the case because, if g is even,

$$\begin{aligned}\int g(x)\left(f(x) - \frac{f(x)+f(-x)}{2}\right)dx &= \int g(x)\frac{f(x)}{2}\,dx - \int g(x)\frac{f(-x)}{2}\,dx \\ &= \int g(x)\frac{f(x)}{2}\,dx - \int g(-x)\frac{f(x)}{2}\,dx \;=\; 0\,.\end{aligned}$$

16. To prove that (i)$\Longrightarrow$(ii), assume that K is compact. Let $B_1 = \{x \in X\,;\ \ \|x\| \leq 1\}$ be the closed unit ball in X. Given any $\varepsilon > 0$, by assumption the image $K(B_1)$ is precompact, hence it can be covered with finitely many balls of radius ε. Say, $K(B_1) \subseteq \bigcup_{i=1}^n B(y_i, \varepsilon)$.

Call $Y \doteq span\{y_1, \ldots, y_n\}$ and set

$$K_\varepsilon(x) \;\doteq\; \pi_Y \circ K(x)\,.$$

Clearly, K_ε has finite dimensional range. Moreover,

$$\|K - K_\varepsilon\| \;=\; \sup_{\|x\|\leq 1} \|K(x) - K_\varepsilon(x)\| \;=\; \sup_{\|x\|\leq 1} \|K(x) - \pi_Y K(x)\| \;\leq\; \sup_{\|x\|\leq 1} d(K(x), Y) \;\leq\; \varepsilon\,.$$

Indeed,

$$d(K(x), Y) \;\leq\; \min_{1\leq i\leq n} \|K(x) - y_i\| \;\leq\; \varepsilon\,.$$

The converse implication (ii)$\Longrightarrow$ (i) is an immediate consequence of Theorem 4.10.

17. Assume that $\sum_n \|u_n - v_n\| < 1$. Let $(u_n)_{n\geq 1}$ be complete. If $(v_n)_{n\geq 1}$ is not complete, then there exists a unit vector $x \in H$ which is perpendicular to every v_n. A contradiction is obtained by writing

$$1 \;=\; \|x\|^2 \;=\; \sum_{n=1}^{\infty} (x,\, u_n)^2 \;=\; \sum_{n=1}^{\infty} (x,\, u_n - v_n)^2 \;\leq\; \sum_{n=1}^{\infty} \|x\|^2\, \|u_n - v_n\|^2 \;\leq\; \sum_{n=1}^{\infty} \|u_n - v_n\|\,.$$

18. (i) Let $(u_n)_{n\geq 1}$ be mutually orthogonal. Assume $\theta_0 y + \theta_1 u_1 + \cdots + \theta_N u_N \;=\; 0$, for some coefficients θ_j not all zero. Since the vectors $u_1, \ldots, u_N, u_{N+1}$ are linearly independent, we must have $\theta_0 \neq 0$ and $u_j \neq 0$ for every j. This yields a contradiction because

$$0 \;=\; (0,\, u_{N+1}) \;=\; (\theta_0 y + \theta_1 u_1 + \cdots + \theta_N u_N\;,\; u_{N+1}) \;=\; \theta_0 (y, u_{N+1}) = \theta_0 2^{-N-1} \|u_{N+1}\|^2\,.$$

(ii) Let $(u_n)_{n\geq 2}$ be an orthonormal set, and define $v \doteq \sum_{n=2}^{\infty} 2^{-n} u_n$, $\;u_1 \doteq v/\|v\|$. Observe that in this case the two vectors y, u_1 are parallel.

19. Assume that the strong convergence $\|x_n - x\| \to 0$ holds. Then for every $y \in H$ one has

$$|(y, x_n) - (y, x)| \;\leq\; \|y\| \cdot \|x_n - x\| \;\to\; 0 \qquad\qquad \text{as } n \to \infty.$$

Hence one has the weak convergence $x_n \rightharpoonup x$ as well. Moreover, if $\|x_n\| \geq \|x\|$, then

$$\Big| \|x_n\| - \|x\| \Big| \;\leq\; \|x_n - x\| \;\to\; 0 \qquad\qquad \text{as } n \to \infty.$$

Viceversa, assume $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then

$$\limsup_{n\to\infty} \|x_n - x\|^2 \;=\; \limsup_{n\to\infty} (x_n - x\,,\; x_n - x)$$

$$=\; \lim_{n\to\infty} \Big((x_n, x_n) - (x_n, x) - (x, x_n) + (x, x) \Big) \;=\; \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 \;=\; 0.$$

20. The orthogonal subspace is $U^\perp \;=\; W \;=\; \left\{ w \in \mathbf{L}^2(Q)\,; \quad \int_0^1 w(x,y)\,dx = 0 \quad \text{for a.e. } \; y \in [0,1] \right\}.$

Given $f \in \mathbf{L}^2(Q)$, we have the perpendicular decomposition $f = u + w$ with $u \in U$, $w \in W = U^\perp$. Here

$$u(x,y) \;=\; \varphi(y) \;=\; \int_0^1 f(s,y)\,ds\,, \qquad\qquad w(x,y) \;=\; f(x,y) - \int_0^1 f(s,y)\,ds\,.$$

Observe that u and w are perpendicular because

$$\int_Q u(x,y)w(x,y)\,dxdy \;=\; \int_0^1 \varphi(y) \left(\int_0^1 w(x,y)\,dx \right) dy \;=\; \int_0^1 \varphi(y) \cdot 0\,dy \;=\; 0\,.$$

In this case, the function $g \in U$ which has minimum distance from f is the perpendicular projection: $g = \pi_U(f) \;=\; u$.

21. Repeat the arguments in the proof of Theorem 4.2.

22. (i) The closure and convexity of $\Omega \subset \mathbf{L}^2(\mathbb{R})$ are straightforward. If $\|f_n - f\|_{\mathbf{L}^2} \to 0$, we can find a subsequence such that $f_n(x) \to f(x)$ for a.e. $x \in \mathbb{R}$. If $f_n(x) \leq e^x$ for every $n \geq 1$ and a.e. $x \in \mathbb{R}$, then also $f(x) \leq e^x$ for a.e. x. Hence Ω is a closed subset of $\mathbf{L}^2(\mathbb{R})$.

Moreover, if $f(x) \leq e^x$ and $g(x) \leq e^x$ for a.e. $x \in \mathbb{R}$, the same is true for any convex combination: $\theta f(x) + (1-\theta)g(x) \leq e^x$ for every $\theta \in [0,1]$ and a.e. x.

(ii) Clearly the function $g(x) = \min\{f(x), e^x\}$ lies in the convex set Ω. By the result proved in problem **21**, it suffices to check that

$$\int (\omega(x) - g(x)) \cdot (g(x) - f(x))\,dx \;\geq\; 0 \qquad\qquad \text{for all } \; \omega \in \Omega\,. \tag{14}$$

By assumption, $\omega(x) \leq e^x$ for a.e. x. For each $x \in \mathbb{R}$ we consider two cases.

Case 1: $f(x) > e^x$. Then $g(x) = e^x$ and therefore

$$\omega(x) - g(x) \;\leq\; 0\,, \qquad\qquad g(x) - f(x) \;<\; 0\,.$$

Case 2: $f(x) \leq e^x$. Then $g(x) = f(x)$ and therefore $g(x) - f(x) = 0$.

In both cases, $(\omega(x) - g(x)) \cdot (g(x) - f(x)) \geq 0$. Hence (14) holds.

23. (i) The operator Λ has norm $\|\Lambda\| = \sqrt{2}$. Indeed,

$$\begin{aligned} \|\Lambda f\|_{\mathbf{L}^2} &= \left(\int_{-\infty}^0 |f(-x)|^2\,dx + \int_0^\infty |f(x)|^2 dx\right)^{1/2} = \left(2\int_0^\infty |f(x)|^2 dx\right)^{1/2} \\ &\leq \left(2\|f\|^2_{\mathbf{L}^2}\right)^{1/2} = \sqrt{2}\cdot\|f\|_{\mathbf{L}^2}\,, \end{aligned}$$

with equality holding whenever f is supported on the positive half axis.

(ii) $Ker(\Lambda) = \{f \in \mathbf{L}^2\,;\ \ f(x) = 0 \ \text{ for a.e. } x > 0\}$,
$Range(\Lambda) = \{f \in \mathbf{L}^2\,;\ \ f(x) = f(-x) \ \text{ for a.e. } x \in \mathbb{R}\}$.

(iii) The adjoint operator Λ^* is defined by the identity

$$\begin{aligned} \int f(x)(\Lambda^* g)(x)\,dx &= \int (\Lambda f)(x) g(x) = \int_{-\infty}^0 f(-x)g(x)\,dx + \int_0^\infty f(x)g(x)dx \\ &= \int_0^\infty f(x)\Big(g(x) + g(-x)\Big)\,dx\,. \end{aligned}$$

This implies

$$(\Lambda^* g)(x) = \begin{cases} 0 & \text{if } \ x < 0\,, \\ g(-x) + g(x) & \text{if } \ x > 0\,. \end{cases}$$

24. (i) For any $f \in \mathbf{L}^2([0,\infty[)$, the substitution $y = e^x$ yields

$$\int_0^\infty |f(e^x)|^2\,dx = \int_1^\infty \frac{|f(y)|^2}{y} dy \leq \int_0^\infty |f(y)|^2 dy\,.$$

Hence $\|\Lambda f\|_{\mathbf{L}^2} \leq \|f\|_{\mathbf{L}^2}$ and $\|\Lambda\| \leq 1$. To prove the converse inequality consider the sequence of functions $f_n = \sqrt{n}\cdot\chi_{[1,1+1/n]}$. Then

$$\|f_n\|_{\mathbf{L}^2} = 1\,, \qquad \|\Lambda f_n\|^2 = \int_1^{1+1/n} \frac{n}{y}\,dy = n\,\ln\left(1 + \frac{1}{n}\right) \to 1 \qquad \text{as } \ n \to \infty.$$

Therefore $\|\Lambda\| = 1$.

(ii) $\quad Ker(\Lambda) = \Big\{f \in \mathbf{L}^2([0,\infty[)\,;\ \ f(x) = 0 \ \text{ for a.e. } x \in [0,1]\Big\}$.
$Range(\Lambda) = \left\{g \in \mathbf{L}^2([0,\infty[)\,;\ \ \int_1^\infty \frac{|g(\ln y)|^2}{y}\,dy < \infty\right\}$.

(iii) The adjoint operator Λ^* is characterized by the identity

$$\int_0^\infty f(x)(\Lambda^* g)(x)\,dx = \int_0^\infty (\Lambda f)(x) g(x)\,dx = \int_0^\infty f(e^x) g(x)\,dx = \int_1^\infty f(y)\,\frac{g(\ln y)}{y}\,dy\,.$$

Therefore

$$(\Lambda^* g)(y) = \begin{cases} 0 & \text{if } \ y \in [0,1]\,, \\ \dfrac{g(\ln y)}{y} & \text{if } \ y > 1\,. \end{cases}$$

25. Assume $f_n \rightharpoonup f$. Then for any $b \in [0,T]$, taking $g = \chi_{[0,b]} \in \mathbf{L}^2([0,T])$ we have

$$\lim_{n\to\infty} \int_0^b f_n(x)\,dx \;=\; \lim_{n\to\infty} \int_0^T f_n(x) g(x)\,dx \;=\; \int_0^T f(x) g(x)\,dx \;=\; \int_0^b f(x)\,dx\,.$$

Viceversa, assume that $\|f\|_{\mathbf{L}^2} \leq M$, $\|f_n\|_{\mathbf{L}^2} \leq M$ for all $n \geq 1$, and that

$$\lim_{n\to\infty} \int_0^b f_n(x)\,dx \;=\; \int_0^b f(x)\,dx \qquad\qquad \text{for every } b \in [0,T]\,.$$

If φ is a piecewise constant function of the form

$$\varphi \;=\; \sum_{k=1}^N c_k \cdot \chi_{[0,b_k]} \tag{15}$$

for some constants c_k, b_k, then by linearity we still have

$$\lim_{n\to\infty} \int_0^T f_n(x)\varphi(x)\,dx \;=\; \sum_{k=1}^N \lim_{n\to\infty} \int_0^{b_k} c_k f_n(x)\,dx \;=\; \sum_{k=1}^N \int_0^{b_k} c_k f(x)\,dx \;=\; \int_0^T f(x)\varphi(x)\,dx\,.$$

Now consider an arbitrary function $g \in \mathbf{L}^2([0,T])$. Given any $\varepsilon > 0$, we can find a piecewise constant function φ of the form (15) such that $\|g - \varphi\|_{\mathbf{L}^2} < \varepsilon$. This yields the estimate

$$\limsup_{n\to\infty} \left| \int_0^T (f_n - f) g\,dx \right| \;\leq\; \limsup_{n\to\infty} \left| \int_0^T (f_n - f)\varphi\,dx \right| + \sup_{n\geq 1} \left| \int_0^T (f_n - f)(\varphi - g)\,dx \right|$$

$$\leq\; 0 + \|f_n - f\|_{\mathbf{L}^2} \|\varphi - g\|_{\mathbf{L}^2} \;\leq\; 2M\,\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, this implies $\int_0^T f_n g\,dx \to \int_0^T fg\,dx$, for every $g \in \mathbf{L}^2([0,T]$, proving the weak convergence $f_n \rightharpoonup f$.

26. Consider first the case $f_n(x) \;=\; \sqrt{n} \cdot \cos nx$. Then for any $b \in [0,1]$ we have

$$\int_0^b f_n(x)\,dx \;=\; \frac{1}{\sqrt{n}} \sin nb \;\to\; 0 \qquad\qquad \text{as } \; n \to \infty\,.$$

By taking linear combinations, it is clear that

$$\lim_{n\to\infty} \int_0^1 f_n(x) g(x)\,dx \;=\; 0 \tag{16}$$

for any piecewise constant function g. In spite of the fact that piecewise constant functions are dense in $\mathbf{L}^2([0,1])$, the weak convergence $f_n \rightharpoonup 0$ FAILS, because the sequence $(f_n)_{n\geq 1}$ is not bounded. Indeed,

$$\int_0^1 |f_n(x)|^2\,dx \;=\; n \int_0^1 \cos^2 nx\,dx \;\to\; \infty \qquad\qquad \text{as } \; n \to \infty\,.$$

Next, consider the case

$$f_n(x) \;=\; \begin{cases} n^{2/3} & \text{if } \; x \in [0,\, n^{-1}]\,, \\ 0 & \text{if } \; x > n^{-1}\,. \end{cases}$$

For each fixed $0 < b \leq 1$, and all $n > b^{-1}$ we have

$$\int_0^b f_n(x)\,dx \;=\; \int_0^{1/n} n^{2/3}\,dx \;=\; n^{-1/3} \;\to\; 0 \qquad \text{as} \quad n \to \infty\,.$$

Hence, also in this case (16) holds for every piecewise constant function g. However,

$$\int_0^1 |f_n(x)|^2\,dx \;=\; \int_0^{1/n} |n^{2/3}|^2\,dx \;=\; n^{1/3} \;\to\; \infty \qquad \text{as} \quad n \to \infty\,.$$

Also in this case, the sequence $(f_n)_{n\geq 1}$ is not bounded in $\mathbf{L}^2([0,1])$, hence it cannot converge weakly. Notice that in this case the result proved in problem **25** cannot be used.

For example, take $g(x) = x^{-3/7}$. Then $g \in \mathbf{L}^2([0,1])$ but

$$\int_0^1 f_n(x)g(x)\,dx \;=\; \int_0^{1/n} n^{2/3}x^{-3/7}\,dx \;=\; n^{2/3}\cdot\frac{7}{4}n^{-4/7} \;\to\; \infty \qquad \text{as} \quad n \to \infty\,.$$

27. We only need to prove the implication (ii) $\Longrightarrow$ (i). By linearity, it is clear that the convergence $(x_n, y) \to (x, y)$ must hold for every $y \in span(S)$. By assumption, $\overline{span(S)}$ is a subspace whose closure has non-empty interior. Hence $\overline{span(S)} = H$.

Let $y \in H$ be given. For any $\varepsilon > 0$ we can find a point $\tilde{y} \in span(S)$ with $\|y - \tilde{y}\| < \varepsilon$. By assumption, there exists a constant M such that $\|x\| \leq M$ and $\|x_n\| \leq M$ for all $n \geq 1$. This yields the estimate

$$\limsup_{n\to\infty}\Big|(x_n - x,\, y)\Big| \;\leq\; \limsup_{n\to\infty}\Big|(x_n - x,\, y - \tilde{y})\Big| + \limsup_{n\to\infty}\Big|(x_n - x,\, \tilde{y})\Big|$$

$$\leq\; \sup_{n\geq 1}\|x_n - x\|\cdot\|y - \tilde{y}\| + 0 \;\leq\; 2M\cdot\varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary, this proves the weak convergence $x_n \rightharpoonup x$.

28. Use the result in problem **27**, choosing S as the set of all characteristic functions χ_Q, where $Q = [a_1, b_1] \times \cdots \times [a_N, b_N]$. Observe that the subspace spanned by all these functions is dense on $\mathbf{L}^2(\Omega)$.

29. If $y \notin S = \overline{co\{x_n\,;\;\; n \geq 1\}}$, then by Theorem 2.33 there exists a linear functional $\phi : H \mapsto \mathbb{R}$ that strictly separates the compact, closed convex set $\{y\}$ from the closed convex set S. By the Riesz' representation theorem, there exists an element $z \in H$ and constants $c_1 < c_2$ such that

$$(y, z) \;=\; \phi(y) \;\leq\; c_1 \;<\; c_2 \;\leq\; \phi(x_n) \;=\; (x_n, z) \qquad \text{for all} \quad n \geq 1\,.$$

This leads to a contradiction, because the weak convergence $x_n \rightharpoonup y$ implies $(x_n, z) \to (y, z)$.

Chapter 6

1. (i) Define

$$(\Lambda_1 f)(x) \doteq f(x+1), \qquad\qquad (\Lambda_2 f)(x) = \begin{cases} f(x-1) & \text{if } x > 1, \\ 0 & \text{if } x \in [0,1]. \end{cases}$$

Then

$$(\Lambda_1 \circ \Lambda_2 f)(x) = f(x), \qquad\qquad (\Lambda_2 \circ \Lambda_1 f)(x) \begin{cases} f(x) & \text{if } x > 1, \\ 0 & \text{if } x \in [0,1]. \end{cases}$$

(ii) If $\Lambda(I-K) = I$, then $Ker(I-K) = \{0\}$ hence the Fredholm operator $I-K$ is one-to-one. By Theorem 6.1, $I-K$ is onto. We conclude that $I-K$ is a bijection, and hence $\Lambda = (I-K)^{-1}$ commutes with $I-K$.

On the other hand, if $(I-K)\Lambda = I$, then $I-K$ must be onto. Hence by Theorem 6.1 the Fredholm operator $I-K$ is one-to-one. We conclude that $I-K$ is a bijection, and hence $\Lambda = (I-K)^{-1}$ commutes with $I-K$.

2. (i) Λ is a bounded operator, with norm $\|\Lambda\| = 2$. Indeed,

$$\|\Lambda f\|^2_{\mathbf{L}^2([0,\infty[)} = \int_0^\infty |(\Lambda f)(x)|^2\,dx = \int_0^\infty |2f(x+1)|^2\,dx = 4\int_1^\infty |f(y)|^2\,dy \leq 4\|f\|_{\mathbf{L}^2([0,\infty[)},$$

with equality holding if $f(x) = 0$ for a.e. $x \in [0,1]$.

Λ is not compact. Indeed, consider the sequence of characteristic functions $f_n = \chi_{[n,\,n+1]}$. Then

$$\|f_n\|_{\mathbf{L}^2} = 1, \qquad\qquad \|\Lambda f_n - \Lambda f_m\|_{\mathbf{L}^2} = 2\sqrt{2} \qquad \text{for all} \quad m \neq n.$$

Therefore, from the sequence $(\Lambda f_n)_{n\geq 1}$ one cannot extract any convergent subsequence.

(ii) To compute the adjoint operator we write

$$\int_0^\infty f(y)(\Lambda^* g)(y)\,dy = \int_0^\infty (\Lambda f)(x) g(x)\,dx = \int_0^\infty 2f(x+1)g(x)\,dx = \int_1^\infty f(y)\,2g(y-1)\,dy.$$

Therefore

$$(\Lambda^* g)(y) = \begin{cases} 2g(y-1) & \text{if } y > 1, \\ 0 & \text{if } y \in [0,1]. \end{cases}$$

(iii) One has $Ker\Lambda = \Big\{ f \in \mathbf{L}^2([0,\infty[;\ \ f(x) = 0 \text{ for a.e. } x \in [0,1] \Big\}$, while $Ker\Lambda^* = \{0\}$.

Of course, this implies that Λ is not a Fredholm operator. In particular, it cannot be written as $\lambda I + K$, with $\lambda \in \mathbb{R}$ and K compact.

3. Assume $\eta > M$. Then the operator $\eta I - \Lambda$ is strictly positive definite. Indeed,

$$(\eta x - \Lambda x\,,\ x) \geq \eta\|x\|^2 - \|\Lambda x\|\|x\| \geq (\eta - \|\Lambda\|)\,\|x\|^2.$$

An application of Theorem 5.12, with $\beta = \eta - \|\Lambda\|$, yields the existence of a continuous inverse $(\eta I - \Lambda)^{-1}$, having norm $\|(\eta I - \Lambda)^{-1}\| \leq 1/\beta$. Therefore, $\eta \in \sigma(\Lambda)$.

The case $\eta < -M$ is entirely similar.

4. (i) Let $M \doteq \max_{(x,y)\in[0,1]\times[0,1]} |K(x,y)|$. Then, by Hölder's inequality,

$$|\Lambda f(x)| \;=\; \left|\int_0^1 K(x,y)f(y)\,dy\right| \;\leq\; M\int_0^1 |f(y)|\,dy \;\leq\; M\,\|f\|_{\mathbf{L}^2([0,1])}\,.$$

This proves that Λ is bounded, with $\|\Lambda\| \leq 1$.

Since $K(x,y) = K(y,x)$, the fact that Λ is self-adjoint follows from

$$\begin{aligned}(\Lambda f,\, g)_{\mathbf{L}^2} \;&=\; \int_0^1\left(\int_0^1 K(x,y)f(y)\,dy\right)g(x)\,dx \;=\; \int_0^1\int_0^1 K(x,y)f(y)g(x)\,dxdy\\ &=\; \int_0^1\left(\int_0^1 K(y,x)g(x)dx\right)f(y)dy \;=\; (f,\,\Lambda g)_{\mathbf{L}^2}\,.\end{aligned}$$

To show that Λ is compact, follow the proof of Theorem 4.12. Observe that, if a sequence of continuous functions $g_n = \Lambda f_n$ converges uniformly on $[0,1]$, then it also converges in $\mathbf{L}^2([0,1])$.

(ii) Consider the special case where

$$u(x) \;=\; (\Lambda f)(x) \;=\; \int_0^x (1-x)y\,f(y)\,dy + \int_x^1 (1-y)xf(y)\,dy\,.$$

Differentiating twice w.r.t. x one obtains

$$\begin{aligned}u'(x) \;&=\; \int_0^x (-y)f(y)\,dy + \int_x^1 (1-y)f(y)\,dy\,,\\ u''(x) \;&=\; -\,f(x)\,.\end{aligned}$$

5. (i) The fact that Λ is self-adjoint is checked by writing

$$\int_0^\pi f(x)\,(\Lambda^* g)(x)\,dx \;=\; \int_0^\pi (\Lambda f)(x)\,g(x)\,dx \;=\; \int_0^\pi f(x)\,\sin x\;g(x)\,dx\,.$$

Hence $(\Lambda^* g)(x) = \sin x\,g(x)$.

(ii) For every $f \in \mathbf{L}^2([0,\pi])$ one has

$$\|\Lambda f\|^2_{\mathbf{L}^2} \;=\; \int_0^\pi \left|\sin x\;f(x)\right|^2 dx \;\leq\; \int_0^\pi \left|f(x)\right|^2 dx\,.$$

Hence $\|\Lambda\| \leq 1$. To prove the converse inequality, consider the set $A_n = \Big\{x \in [0,\pi]\,;\quad \sin x > 1-\frac{1}{n}\Big\}$ and define $f_n = \chi_{A_n}$. Then $\|\Lambda f_n\|_{\mathbf{L}^2} \geq \Big(1-\frac{1}{n}\Big)\|f_n\|_{\mathbf{L}^2}$. We thus conclude that $\|\Lambda\| = 1$. However, there is no function $f \in \mathbf{L}^2$ such that $\Lambda f = f$. Hence $1 \notin \sigma_p(\Lambda)$.

(iii) According to problem 20 in Chapter 4, the multiplication operator is not compact.

6. (i) Assume that $u \in \mathbf{L}^2([0,1])$, say with $\|u\|_{\mathbf{L}^2} \;=\; M$. Then, for any $0 \leq t_1 < t_2 \leq 1$, using Hölder's inequality we obtain

$$\Big|\Lambda u(t_2) - \Lambda u(t_1)\Big| \;\leq\; \int_{t_1}^{t_2} 1\cdot|u(s)|\,ds \;\leq\; \|1\|_{\mathbf{L}^2([t_1,t_2])}\cdot\|u\|_{\mathbf{L}^2} \;=\; \sqrt{t_2-t_1}\cdot\|u\|_{\mathbf{L}^2}\,.$$

Hence $\Lambda u \in \mathcal{C}^{0,1/2}$.

(ii) Consider any sequence $(f_n)_{n\geq 1}$ with $\|f_n\|_{\mathbf{L}^2} \leq 1$ for every n. Then the corresponding functions Λf_n are uniformly bounded and Hölder continuous, hence equicontinuous. By Ascoli's theorem, one can extract a subsequence $(f_{n_k})_{k\geq 1}$ that converges uniformly on $[0,1]$ to some continuous function f. In turn, this implies $\|f_{n_k} - f\|_{\mathbf{L}^2} \to 0$, proving that the operator Λ is compact.

(iii) Writing

$$\int_0^1 u(x)\,(\Lambda^* v)(x)\,dx \;=\; \int_0^1 (\Lambda u)(x)\,v(x)\,dx \;=\; \int_0^1 \left(\int_0^x u(s)\,ds\right) v(x)\,dx$$

$$=\; \left(\int_0^1 u(s)\,ds\right)\left(\int_0^1 v(s)\,ds\right) - \int_0^1 u(x)\left(\int_0^x v(s)\,ds\right)\,dx\,,$$

we conclude

$$(\Lambda^* v)(x) \;=\; \int_x^1 v(s)\,ds\,.$$

(iv) The operator $(I-K)$ is a Fredholm operator on $\mathbf{L}^2([0,1])$. If $u = Ku$, then $u(x) \;=\; \int_0^x u(s)\,ds\,.$ hence u is an absolutely continuous solution of the Cauchy problem

$$\frac{d}{dx}u(x) \;=\; u(x)\,, \qquad\qquad u(0) \;=\; 0\,.$$

By Gronwall's inequality, the only solution is $u(x) = 0$ for every x. We conclude that the operator $I - K$ is one-to-one. By Fredholm's theorem, $I - K$ is onto. Hence for every $g \in \mathbf{L}^2([0,1])$ there exists one and only one function u such that $u - Ku = g$. The inverse map $g \mapsto u = (I-K)^{-1}$ is also a bounded linear operator.

If g is continuously differentiable, then u is an absolutely continuous function that satisfies the Cauchy problem

$$u'(x) \;=\; g'(x) + u(x)\,, \qquad\qquad u(0) \;=\; g(0)\,.$$

To compute the solution $u(\cdot)$, we set $v = u - g$. The previous ODE yields

$$v'(x) \;=\; v + g\,, \qquad\qquad v(0) = 0\,,$$

$$v(x) \;=\; \int_0^x e^{x-y} g(y)\,dy\,.$$

This provides an explicit formula for the inverse operator:

$$u(x) \;=\; \Big((I-K)^{-1}g\Big)(x) \;=\; g(x) + \int_0^x e^{x-y} g(y)\,dy\,.$$

7. To fix the ideas, let $\mathbf{w}_1, \ldots, \mathbf{w}_N$ be the eigenvectors with corresponding eigenvalues $\lambda_1 = \cdots = \lambda_N = 1$, while $\lambda_k \neq 1$ for $k > N$. Then the equation $u - Ku = f$ admits a solution if and only if

$$(f, \mathbf{w}_k) \;=\; 0 \qquad\qquad \text{for every} \;\; k \in \{1, \ldots, N\}. \tag{17}$$

If (17) holds, then u is a solution if and only if

$$u = \sum_{k=1}^{N} c_k \mathbf{w}_k + \sum_{k>N} \frac{(f, \mathbf{w}_k)}{1-\lambda} \mathbf{w}_k ,$$

for arbitrary constants $c_1, \ldots, c_N$.

8. Assume $A = A^*$, $B = B^*$. Then for every $x, y \in H$ we have

$$(ABx,\, y) = (Bx,\, Ay) = (x,\, BAy).$$

Hence AB is self adjoint if and only $AB = BA$.

9. In the case where H is finite dimensional, the result is trivial: every operator is compact and (ii) is satisfied simply because the set of orthonormal sequences is empty. In the following we thus assume that H is infinite dimensional.

(i) $\Longrightarrow$ (ii). Assume that $\Lambda : H \mapsto H$ is compact, and let $\{v_1, v_2, \ldots\}$ be an orthonormal sequence of vectors in H.

If the sequence Λv_n does not converge to zero, by taking a subsequence we can assume $\|\Lambda v_n\| \geq \varepsilon > 0$ for all $n \geq 1$. A contradiction is obtained as follows.

By compactness, there exists a subsequence such that $\Lambda v_{n_k} \to w$ for some vector w. Our previous assumption implies $\|w\| \geq \varepsilon$.

By choosing a further subsequence, we can assume

$$\|\Lambda v_{n_k} - w\| < \frac{\|w\|}{2} \qquad \text{for all } \; k \geq 1\,. \tag{18}$$

Consider the vector $v = \sum_{k=1}^{\infty} v_{n_k}/k$. Since the unit vectors v_{n_k} are mutually orthogonal, the series is convergent and $\|v\|^2 = \sum_{k=1}^{\infty} k^{-2} < \infty$. Since Λ is a bounded operator, we should have

$$\sum_{k=1}^{N} \frac{\Lambda v_{n_k}}{k} = \Lambda\left(\sum_{k=1}^{N} \frac{v_{n_k}}{k}\right) \to \Lambda(v) \qquad \text{as } \; N \to \infty.$$

But this is impossible, because by (18)

$$\left\|\sum_{k=1}^{N} \frac{\Lambda v_{n_k}}{k}\right\| \geq \left\|\sum_{k=1}^{N} \frac{w}{k}\right\| - \sum_{k=1}^{N} \left\|\frac{w - \Lambda v_{n_k}}{k}\right\| \geq \sum_{k=1}^{N} \frac{1}{k} \left(\|w\| - \frac{\|w\|}{2}\right)$$

and the right hand side approaches infinity as $n \to \infty$.

(ii) $\Longrightarrow$ (i). Assume that Λ is not compact. Then the set $S \doteq \overline{\Lambda(B_1)}$ defined as the closure of the image of the unit ball, is not compact. By the result proved in problem **15**, Chapter 4, there exists $\varepsilon > 0$ such that

($\mathbf{P}_\varepsilon$) $\quad S$ is not contained in the ε-neighborhood of any finite-dimensional subspace of H.

We now inductively construct an orthonormal sequence $\{v_1, v_2, \ldots\}$, as follows.

- By $(\mathbf{P}_\varepsilon)$, there exists a vector v_1 such that $\|\Lambda v_1\| \geq \varepsilon$.
- Assume that the orthonormal set $\{v_1, \ldots, v_n\}$ has been constructed. Again by $(\mathbf{P}_\varepsilon)$, the set S is not contained in the ε-neighborhood of the finite dimensional space $V_n \doteq span\{\Lambda v_1, \ldots, \Lambda v_n\}$. Hence there exists a vector w such that $\|w\| \leq 1$ and $d(\Lambda w, V_n) \geq \varepsilon$. Define the vectors

$$w_{n+1} \doteq w - \sum_{k=1}^{n} (w, v_n) v_n , \qquad v_{n+1} \doteq \frac{w_{n+1}}{\|w_{n+1}\|} .$$

Observe that w_{n+1} is the perpendicular projection of w on $span\{v_1, \ldots, v_n\}^\perp$. Hence $0 < \|w_{n+1}\| \leq \|w\| \leq 1$. Moreover, observing that $\Lambda w_{n+1} - \Lambda w \in V_n$, we obtain

$$\|\Lambda v_{n+1}\| \geq \|\Lambda w_{n+1}\| \geq d(\Lambda w_{n+1}, V_n) = d(\Lambda w, V_n) \geq \varepsilon .$$

By induction, we thus obtain an orthonormal sequence $\{v_n ; \ n \geq 1\}$ such that $\|\Lambda v_n\| \geq \varepsilon > 0$ for every n. Hence (ii) fails.

To prove the last statement, consider the orthonormal sequence

$$w_n = 2^{-n/2} \sum_{2^n < k \leq 2^{n+1}} \mathbf{e}_k .$$

Then define the operator Λ by setting

$$\Lambda v \doteq \sum_{n=1}^{\infty} (w_n, v) \mathbf{e}_n .$$

This is a bounded linear operator with norm $\|\Lambda\| = 1$. It is not compact, because the image of the unit ball contains the entire orthonormal sequence $\{\mathbf{e}_n ; \ n \geq 1\}$. However, $\lim_{k\to\infty} \Lambda \mathbf{e}_k = 0$. Indeed, for $2^n < k \leq 2^{n+1}$ we have

$$\|\Lambda \mathbf{e}_k\| = |(w_n, \mathbf{e}_k)| = 2^{-n/2}.$$

10. Let $V \subseteq H$ be a closed subspace. Consider the orthogonal decomposition $V = V_1 + V_2$, where

$$V_1 \doteq V \cap Ker(I-K) , \qquad V_2 \doteq V \cap V_1^\perp .$$

Then $(I-K)(V) = (I-K)(V_2)$. By possibly replacing V with the closed subspace V_2, we can thus assume that $V \cap Ker(I-K) = \{0\}$, i.e. $I - K$ is one-to-one restricted to V.

We claim that there exists a constant $\beta > 0$ such that

$$\|u - Ku\| \geq \beta \|u\| \qquad \text{for all } u \in V. \tag{19}$$

Otherwise, we could find a sequence $(u_n)_{n\geq 1}$ such that

$$\|u_n\| = 1 , \qquad y_n \doteq u_n - Ku_n \to 0 .$$

Since K is compact, there exists a subsequence such that $Ku_{n_j} \to z$ for some $z \in H$. This yields

$$\lim_{n\to\infty} u_n \;=\; \lim_{n\to\infty} (y_n + Ku_n) \;=\; z \;\in\; V\,.$$

Hence

$$(I-K)z \;=\; \lim_{n\to\infty} (I-K)u_n \;=\; 0\,,$$

in contradiction with the assumption that $(I-K)$ is one-to-one restricted to V.

Now assume $v_n \in V$ for every $n \geq 1$, and

$$y_n \;=\; (I-K)v_n \;\to\; y.$$

We need to prove that $y \in (I-K)(V)$.

Using (19) we obtain

$$0 \;=\; \limsup_{m,n\to\infty} \|y_m - y_n\| \;\geq\; \beta \limsup_{m,n\to\infty} \|v_m - v_n\|\,.$$

Therefore the sequence $(v_n)_{n\geq 1}$ is Cauchy, and converges to some limit $v \in V$. By continuity, $y = (I-K)v$.

11. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ be an orthonormal basis of a real Hilbert space H, consisting of eigenvectors of a linear, compact, self-adjoint operator K. Let $\lambda_1, \lambda_2, \ldots$ be the corresponding eigenvalues.

Assume that $t \mapsto u(t)$ provides a solution to the Cauchy problem

$$\frac{d}{dt}u(t) \;=\; Ku(t)\,, \qquad\qquad u(0) \;=\; f\,, \tag{20}$$

for some $f \in H$. Taking the inner product of $u(t)$ with $\mathbf{w}_k$ we obtain

$$\frac{d}{dt}(u(t),\, \mathbf{w}_k) \;=\; (Ku(t),\, \mathbf{w}_k) \;=\; (u(t),\, K\mathbf{w}_k) \;=\; \lambda_k (u(t),\, \mathbf{w}_k)\,, \qquad\qquad (u(0),\, \mathbf{w}_k) \;=\; (f,\, \mathbf{w}_k)\,. \tag{21}$$

Therefore, the solution can be written as

$$u(t) \;=\; \sum_{k=1}^{\infty} c_k(t)\mathbf{w}_k\,, \tag{22}$$

where each coefficient $c_k(\cdot)$ is obtained by solving the linear scalar Cauchy problem

$$c_k'(t) \;=\; \lambda_k c_k(t)\,, \qquad\qquad c_k(0) \;=\; (f, \mathbf{w}_k)\,.$$

This yields $c_k(t) = e^{\lambda_k t}(f, \mathbf{w}_k)$, and hence

$$u(t) \;=\; \sum_{k\geq 1} e^{\lambda_k t}(f, \mathbf{w}_k)\, \mathbf{w}_k\,.$$

12. In the more general case of a non-homogenous equation, the coefficients $c_k(\cdot)$ satisfy the equations

$$c_k'(t) \;=\; \lambda_k c_k(t) + (g(t),\, \mathbf{w}_k)\,, \qquad\qquad c_k(0) \;=\; (f, \mathbf{w}_k)\,.$$

Hence the formula (22) remains valid, with

$$c_k(t) \;=\; e^{\lambda_k t}(f, \mathbf{w}_k) + \int_0^t e^{\lambda_k(t-s)}(g(t), \mathbf{w}_k)\, ds\,.$$

13. Taking the inner product of the solution with $\mathbf{w}_k$, in this case we find that the function $c_k(t) \doteq (u(t),\, \mathbf{w}_k)$ satisfies the second order initial value problem

$$c_k''(t) \;=\; \lambda_k c_k(t)\,, \qquad\qquad c_k(0) \;=\; (f, \mathbf{w}_k)\,, \quad c_k'(0) \;=\; (g, \mathbf{w}_k)\,.$$

14. Assume that $\eta I - \Lambda$ is a continuous bijection. Then there exists $\beta > 0$ such that, for every bounded linear operator Ψ with norm $\|\Psi\| < \beta$, the operator $\eta I - \Lambda + \Psi$ is a continuous bijection. Hence every $\tilde\eta \in \mathbb{R}$ with $|\tilde\eta - \eta| < \beta$ lies in the resolvent set $\rho(\Lambda)$.

Chapter 7

1. (i) The norm of the diagonal operator AS_t is computed by

$$\|AS_t\| \;=\; \sup_k \left|\lambda_k e^{\lambda_k t}\right|\,.$$

Using the assumption

$$\lambda_k \;=\; \alpha_k + i\beta_k \;=\; \omega - r_k(\cos\theta_k + i\sin\theta_k)$$

for some $r_k \geq 0$ and $|\theta_k| \leq \bar\theta < \pi/2$, we obtain

$$|\lambda_k e^{\lambda_k t}| \;\leq\; (|\omega| + r_k)e^{(\omega - r_k\cos\theta_k)t} \leq\; (|\omega| + r_k)e^{(\omega - \eta r_k)t} \qquad\qquad \eta \doteq \cos\bar\theta > 0\,.$$

Hence, for a fixed $t > 0$,

$$\|AS_t\| \;\leq\; \sup_{r\geq 0}\,(|\omega| + r)e^{(\omega - \eta r)t} \;<\; \infty\,.$$

(ii) Similarly,

$$\|A^n S_t\| \;=\; \sup_k \left|\lambda_k^n e^{\lambda_k t}\right| \;\leq\; \sup_{r\geq 0}\,(|\omega| + r)^n e^{(\omega - \eta r)t} \;<\; \infty\,.$$

2. This is an immediate consequence of the formula (7.31). Namely, if S is a semigroup of type ω, then

$$\|e^{tA_\lambda}\| \;\leq\; e^{2\omega t}$$

for all $t \geq 0$ and $\lambda \geq 2\omega$. If S is contractive semigroup, then S is of type $\omega = 0$.

3. The computation of the various integrals is an exercise in basic Calculus.

From the identities

$$\int_0^\infty W_n(t)\,dt \;=\; 1\,, \qquad \int_0^\infty tW_n(t) \;=\; T\,, \qquad \int_0^\infty (t-T)^2 W_n(t)\,dt \;=\; \frac{T^2}{n}$$

we deduce

$$\int_{|t-T|>\varepsilon} W_n(t)\,dt \;\leq\; \sqrt{\frac{T^2}{n}} \;\to\; 0 \qquad\qquad \text{as} \;\; n\to\infty.$$

This proves (7.39).

4. The formula

$$(I-hA)^{-n}u \;=\; \int_0^\infty w_n(t)S_t u\,dt \tag{23}$$

is proved by induction on n. The case $n=1$ is already known. Now assume (23) is valid. Using the variable $\tau = t+s$ we obtain

$$(I-hA)^{-(n+1)}u \;=\; (I-hA)^{-1}(I-hA)^{-n}u \;=\; \int_0^\infty w(t)S_t\Big((I-hA)^{-n}u\Big)\,dt$$

$$=\; \int_0^\infty w(t)S_t\left(\int_0^\infty w_n(s)S_s u\,ds\right)\,dt \;=\; \int_0^\infty (w*w_n)(\tau)S_r u\,d\tau\,.$$

Hence the same formula (23) is valid with n replaced by $n+1$.

The convergence

$$\left(I-\frac{T}{n}A\right)^{-n} u \;=\; \int_0^\infty W_n(t)S_t u\,dt \;\to\; S_T u \qquad \text{as} \;\; n\to\infty. \tag{24}$$

is proved by using (7.39). Indeed, let $\varepsilon>0$ By the continuity of the map $t\mapsto S_t u$ there exists $\delta>0$ such that

$$\|S_t u - S_T u\| \;\leq\; \varepsilon \qquad\qquad \text{for all} \;\; t\in[T-\delta,\,T+\delta]\,.$$

By (7.39), for all n large enough we have

$$\int_{|t-T|>\delta} W_n(t)\,dt \;<\; \varepsilon\,.$$

Recalling that $\|S_t u\|\leq\|u\|$ for every $t\geq 0$, we thus obtain

$$\left\|S_T u - \left(I-\frac{T}{n}A\right)^{-n} u\right\| \;\leq\; \left\|S_T u - \int_0^\infty W_n(t)S_t u\,dt\right\|$$

$$\leq\; \left\|S_T u - \int_0^\infty W_n(t)S_T u\,dt\right\| + \int_{|t-T|\leq\delta} W_n(t)\|S_t u - S_T u\|\,dt + \int_{|t-T|>\delta} W_n(t)\|S_t u - S_T u\|\,dt$$

$$\leq\; 0+\varepsilon+\varepsilon\,2\|u\|\,.$$

Since $\varepsilon>0$ was arbitrary, this proves (24).

5. (i) $\Longrightarrow$ (ii). Assume that $S_t u \in \Omega$ for every $u \in \Omega$ and $t \geq 0$. Then

$$(I - hA)^{-1}u \;=\; \int_0^\infty \frac{e^{-t/h}}{h} S_t u\, dt\,.$$

This shows that $(I-hA)^{-1}u$ is an integral average of points $S_t u \in \Omega$, with weight $w(t) \;=\; \frac{e^{-t/h}}{h}$ satisfying $\int_0^\infty w(t)\, dt \;=\; 1$. Approximating the integral with a finite sum, we see that $(I - hA)^{-1}u$ can be approximated by convex combinations of elements $S_{t_i} u \in \Omega$. Since Ω is closed and convex, we conclude $(I - hA)^{-1}u \in \Omega$.

(ii) $\Longrightarrow$ (i). Using the result of the previous problem **4**,

$$\left(I - \frac{T}{n}A\right)^{-n} \;=\; \int_0^\infty W_n(t) S_t u\, dt\,, \tag{25}$$

where W_n is a smooth averaging kernel, with $\int_0^\infty W_n(t)\, dt = 1$. Since Ω is closed and convex, the right hand side of (25) lies in Ω. Letting $n \to \infty$, by the previous problem **4** the left hand side of (25) converges to $S_T u$. Since Ω is closed, we conclude $S_T u \in \Omega$.

6. Let $\{S_t\,;\;\; t \geq 0\}$ be a semigroup of type ω, with generator A. Then

$$\|e^{\gamma t} S_t u\| \;=\; e^{\gamma t}\|S_t u\| \;\leq\; e^{\gamma t} e^{\omega t}\|u\|\,.$$

Hence $\{e^{\gamma t} S_t\,;\;\; t \geq 0\}$ is a semigroup of type $\gamma + \omega$. Moreover, if $u \in Dom(A)$, then

$$\lim_{h\to 0+} \frac{e^{\gamma h} S_h u - u}{h} \;=\; \lim_{h\to 0+} \frac{e^{\gamma h} S_h u - S_h u}{h} + \lim_{h\to 0+} \frac{S_h u - u}{h} \;=\; \gamma u + Au\,.$$

Calling A_γ the generator of the semigroup $\{e^{\gamma t} S_t\,;\;\; t \geq 0\}$, this proves that $Dom(A_\gamma) \supseteq Dom(A)$ and $A_\gamma(u) = Au + \gamma u$.

Inverting the roles of A, A_γ, we see that $Dom(A) \supseteq Dom(A_\gamma)$ and $Au = A_\gamma u - \gamma u$. This concludes the proof.

7. (i) The semigroup properties are easily checked:

$$(S_0 f)(x) \;=\; e^{-2\cdot 0} f(x+0) \;=\; f(x)\,,$$

$$(S_{t+s} f)(x) \;=\; e^{-2(t+s)} f(x+t+s) \;=\; e^{-2s}\Big(e^{-2t} f((x+t)+s)\Big) \;=\; (S_s(S_t f))(x)\,.$$

In the following, we first assume $1 \leq p < \infty$. By definition, the function $f \in \mathbf{L}^p(\mathbb{R})$ lies in the domain of the generator A if and only if the following limit exists in $\mathbf{L}^p(\mathbb{R})$:

$$\begin{aligned} Au \;&\doteq\; \lim_{h\to 0+} \frac{e^{-2h} f(x+h) - f(x)}{h} \;=\; \lim_{h\to 0+} \frac{e^{-2h} f(x+h) - f(x+h)}{h} + \lim_{h\to 0+} \frac{f(x+h) - f(x)}{h} \\ &=\; -2f(x) + \lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}\,. \end{aligned} \tag{26}$$

Hence

$$Dom(A) \;=\; \Big\{u\in \mathbf{L}^p(\mathbb{R})\,;\quad u \text{ is absolutely continuous and} \quad u_x\in \mathbf{L}^p(\mathbb{R})\Big\}$$

Moreover, $Au = -2u + u_x$ for every $u\in Dom(A)$.

(ii) On the space $\mathbf{L}^\infty(\mathbb{R})$ the above semigroup is not strongly continuous. For example, consider the function

$$u(x) \;=\; \begin{cases} 1 & \text{if } x\in [0,1],\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then

$$S_t u(x) \;=\; \begin{cases} e^{-2t} & \text{if } x\in [-t,\,1-t]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

As $t\to 0+$ we then have the convergence $\|S_t u - u\|_{\mathbf{L}^p(\mathbb{R})} \;\to\; 0$ for every $1\leq p<\infty$. However

$$\lim_{t\to 0+} \|S_t u - u\|_{\mathbf{L}^\infty(\mathbb{R})} \;=\; 1\,.$$

Hence the map $t\mapsto S_t u$ is not continuous from $[0,\infty[$ into the space $\mathbf{L}^\infty(\mathbb{R})$.

8. The generator A of the semigroup must be a linear operator on $\mathbb{R}^n$ with dense domain. The only dense subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. Hence $Dom(A)=\mathbb{R}^n$, and A must be described by an $n\times n$ matrix. In particular, the linear operator A is continuous, hence for every $u\in\mathbb{R}^n$ and $t\geq 0$ we must have

$$\frac{d}{dt} S_t u \;=\; AS_t u, \qquad\qquad S_0 u = 0\,.$$

Since $t\mapsto S_t u$ is the solution to the above Cauchy problem, we conclude that $S_t u = e^{tA}u$.

9. (i) On the space of all bounded continuous functions $w:[0,T]\mapsto X$, consider the Picard operator defined as

$$\Phi(w)(t) \;\doteq\; S_t\bar u + \int_0^t S_{t-s} f(s, w(s))\,ds\,.$$

Following the proof of Theorem 7.1, one checks that Φ is a strict contraction w.r.t. the equivalent norm

$$\|w\|_\dagger \;\doteq\; \max_{t\in[0,T]} e^{-2Lt}\|w(t)\|\,.$$

Hence Ψ has a unique fixed point, which by definition provides a mild solution of (7.40).

(ii) Assume that the generator A is a bounded linear operator. Then we can differentiate (7.41) w.r.t. time, and obtain

$$\begin{aligned}
\frac{d^+}{dt}u(t) \;&=\; \lim_{h\to 0+} \frac{S_{t+h}\bar u - S_t\bar u}{h} + \lim_{h\to 0+}\frac{1}{h}\int_t^{t+h} S_{t-s} f(s,u(s))\,ds \\
&\qquad + \lim_{h\to 0+}\int_0^t \frac{S_{t+h-s}f(s,u(s)) - S_{t-s}f(s,u(s))}{h}\,ds \\
&=\; AS_t\bar u + f(t,u(t)) + \textstyle\int_0^t AS_{t-s}f(s,u(s))\,ds \\
&=\; Au(t) + f(t,u(t))\,.
\end{aligned}$$

Observing that the forward derivative (obtained by taking the limit as $h \to 0+$) is a continuous function of time, we conclude that it coincides with the backward derivative (obtained by letting $h \to 0-$). This completes the proof.

10. For $u \in \mathbf{L}^1([0,1])$, define

$$(S_t u)(x) \;=\; \begin{cases} 0 & \text{if } \; x \le t/T\,, \\ u\Big(x - \dfrac{t}{T}\Big) & \text{if } \; x > t/T\,. \end{cases} \tag{27}$$

This corresponds to solving the PDE $\;u_t + u_x \;=\; 0$ with boundary condition $u(t,0) = 0$.

11. Assume S_τ is compact. If $t > \tau$ then $S_t = S_{t-\tau} S_\tau$ is the composition of the continuous operator $S_t - \tau$ with the compact operator S_τ, hence it is compact.

To see that the converse may not hold in general, observe that the semigroup (27) is trivially compact for $t \ge T$ but not compact for $0 \le t < T$.

12. (i) $(S_t u)(x) \;=\; u(x+t)$. Not that this is well defined also for $t < 0$. This is a group of isometries: $\|S_t u\|_{\mathbf{L}^1} = \|u\|_{\mathbf{L}^1}$.

(ii) If $v = E_h^- u \doteq (I - hA)^{-1} u$, then $v = u + h v_x$. We are thus looking for a function $v \in \mathbf{L}^1(\mathbb{R})$ such that $v - hv' = u$, hence

$$v'(x) \;=\; \frac{v(x)}{h} - \frac{u(x)}{h} \qquad\qquad x \in \mathbb{R}\,.$$

The explicit solution is provided by

$$v(x) \;=\; \int_x^\infty \frac{e^{(x-y)/h}}{h}\, u(y)\, dy.$$

Setting $t = y - x$, it is interesting to observe that the above formula is equivalent to

$$v(x) \;=\; \int_0^\infty \frac{e^{-t/h}}{h} u(x+t)\, dt \;=\; \int_0^\infty \frac{e^{-t/h}}{h} (S_t u)(x)\, dt\,.$$

(iii) If $u \in \mathcal{C}_c^\infty$ with support contained in the interval $[a,b]$, then the same is true of the derivative u_x. Hence $(E_h^+ u)(x) = u(x) + h u_x(h)$ is a smooth function supported inside $[a,b]$. By induction on n we see that the same holds for $(E_h^+)^n u$.

(iv) Take $T \ge b - a$. If u vanishes outside $[a,b]$, then $S_T u$ is supported inside $[a-T,\, b-T]$. In particular, $(S_T u)(x) = u(x+T) = 0$ for every $x \in [a, \infty[\,$.

On the other hand, every forward Euler approximation vanishes outside the interval $[a,b]$, hence the same must be true for the limit (if it exists). We conclude that, if the function u is not identically zero, the forward Euler approximations cannot converge to $S_T u$.

Chapter 8

1. (i) This is a distribution of infinite order.

(ii)-(iii) These linear functionals are not distributions on $\mathbb{R}$. They are both distributions on the open half line $]0,\infty[$.

(iv) This is a distribution of order zero on $\Omega = \ ]0,\infty[$.

2. By assumption, f has a weak derivative $Df \in \mathbf{L}^p(]a,b[) \subset \mathbf{L}^1(]a,b[)$. By Corollary 8.17, f coincides a.e. with an absolutely continuous function.

If f is continuously differentiable, for any $a < x < y < b$, using Hölder's inequality on the interval $[x,y]$ we obtain

$$|f(x) - f(y)| \ \leq \ \int_x^y 1\cdot |f'(s)|\,ds \ \leq \ \|1\|_{\mathbf{L}^q([x,y])}\cdot \|f'\|_{\mathbf{L}^p([x,y])} \qquad\qquad \frac{1}{q}+\frac{1}{p} = 1\,.$$

$$\|1\|_{\mathbf{L}^q([x,y])} \ = \ |x-y|^{1/q} \ = \ |x-y|^{1-\frac{1}{p}}, \qquad\qquad \|f'\|_{\mathbf{L}^p([x,y])} \ \leq \ \|f'\|_{\mathbf{L}^p([x,y])} \ \leq \ \|f\|_{W^{1,p}(]a,b[)}\,.$$

By approximation, the same inequality holds for every $f\in W^{1,p}$.

3. Fix any $\delta>0$ and consider the smaller square

$$Q_\delta \ \doteq \ \Big\{(x_1,x_2)\,;\quad \delta<x_1<1-\delta,\quad \delta<x_2<1-\delta\Big\}.$$

For any $\varepsilon\in]0,\delta]$, the mollified functions $f_\varepsilon \doteq J_\varepsilon * f$ are well defined on Q_δ. Moreover $D_{x_1}f_\varepsilon = J_\varepsilon * D_{x_1}f = 0$. Hence there exist a smooth function $g_\varepsilon : [\varepsilon,\, 1-\varepsilon]\mapsto \mathbb{R}$ such that $f_\varepsilon(x_1,x_2) = g_\varepsilon(x_2)$ for all $(x_1,x_2)\in Q_\delta$.

Letting $\varepsilon_n\to 0$, we obtain $\|f_\varepsilon - f\|_{\mathbf{L}^1(Q_\delta)}\to 0$. Hence, for a suitable sequence $\varepsilon_n\to 0$ we achieve the pointwise convergence

$$f(x_1,x_2) \ = \ \lim_{\varepsilon\to 0} f_\varepsilon(x_1,x_2) \ = \ \lim_{\varepsilon\to 0} g_\varepsilon(x_2) \qquad\qquad \text{for a.e. } (x_1,x_2)\in Q_\delta\,.$$

Since $\delta>0$ is arbitrary, this proves the result.

4. For every test function $\varphi\in\mathcal{C}_c^\infty(\Omega')$ we have

$$\int_{\Omega'} g\,\varphi\,dx \ = \ -\int_{\Omega'} f\,D_{x_1}\varphi\,dx \ = \ \int_{\Omega'}\frac{\partial f}{\partial x_1}\,\varphi\,dx\,.$$

By the uniqueness of the weak derivative, proved in Lemma 8.12, this implies $g(x) = \frac{\partial f}{\partial x_1}(x)$ for a.e. $x\in\Omega'$.

5. (i) Consider the mollifications $u_\varepsilon \doteq J_\varepsilon * u$. The assumption $u\in W^{1,\infty}(\Omega)$ implies that

$$|\nabla u_\varepsilon(x)| \ \leq \ \|u\|_{W^{1,\infty}} \qquad\qquad \text{for all } \ x\in\Omega_\varepsilon \ \doteq \ \{x\in\Omega\,;\quad B(x,\varepsilon)\subset\Omega\}\,.$$

Since Ω_ε is convex, for any $x,y\in\Omega_\varepsilon$ we have

$$|u_\varepsilon(x)-u_\varepsilon(y)| \ \leq \ \int_0^1 |\nabla u_\varepsilon(\theta x+(1-\theta)y)\cdot(x-y)|\,d\theta \ \leq \ \left(\max_{z\in\Omega_\varepsilon}|\nabla u_\varepsilon(z)|\right)\,|x-y| \ \leq \ \|u\|_{W^{1,\infty}}\cdot|x-y|\,.$$

Therefore, each u_ε is Lipschitz continuous with Lipschitz constant $L = \|u\|_{W^{1,\infty}}$. Taking the limit as $\varepsilon \to 0$ we obtain the result.

(ii) Consider the open set

$$\Omega \doteq \{(x_1, x_2);\ \ 1 < x_1^2 + x_2^2 < 4\} \setminus \{(x_1, x_2);\ \ x_2 = 0\,,\ \ x_1 > 0\}.$$

Let u be the angle function, in polar coordinates. In other words,

$$u(x_1, x_2) \doteq \theta\,, \qquad \text{if} \quad (x_1, x_2) = r(\cos\theta, \sin\theta)\,, \quad r = \sqrt{x_1^2 + x_2^2}, \quad 0 < \theta < 2\pi\,.$$

6. (i) Consider first the case where Ω is a bounded, open, convex set. Let u be Lipschitz continuous with constant C. For every fixed $(n-1)$-tuple $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, the function

$$s \mapsto u(x_1, x_2, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)$$

is defined for s in some open interval $]a, b[$ (possibly empty). Moreover, it is Lipschitz continuous of the same constant C. Being absolutely continuous, it is differentiable almost everywhere. We thus conclude that the partial derivative

$$u_{x_i}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h}\,.$$

exists for a.e. $x \in \Omega$. An integration by parts shows that this function provides the weak derivative $D_{x_i}u$. Since u is bounded and $|u_{x_i}(x)| \leq C$ at every point x where the partial derivative exists, it is clear that $u \in W^{1,\infty}(\Omega)$.

The result can be easily extended to a general open set Ω, observing that

$$\|u\|_{W^{1,\infty}}(\Omega) = \sup_{B(x,r) \subset \Omega} \|u\|_{W^{1,\infty}(B(x,r))}\,,$$

where the supremum is taken over all open balls contained in Ω.

(ii) Consider any bounded open subset $\Omega' \subset \Omega$. Then $W^{1,\infty}(\Omega') \subset W^{1,n+1}(\Omega')$. Hence by Theorem 8.41 the function u is differentiable a.e. on Ω'. By varying the set Ω' we conclude that u is differentiable a.e. on Ω.

7. A direct computation shows that

$$\|f\|^n_{\mathbf{L}^n(\Omega)} = c_n \cdot \int_0^1 r^{n-1} \left[\ln \ln \left(1 + \frac{1}{r}\right)\right]^n dr < \infty$$

because the integrand on the right hand side is bounded. Moreover, for $n \geq 2$ we have

$$\|\nabla f\|^n_{\mathbf{L}^n(\Omega)} = c_n \cdot \int_0^1 r^{n-1} \left[\frac{1}{\ln\left(1 + \frac{1}{r}\right)} \cdot \frac{1}{1 + \frac{1}{r}} \cdot \frac{1}{r^2}\, dr\right]^n dr$$

$$\leq c_n \cdot \int_0^1 \frac{1}{r \ln^n \left(1 + \frac{1}{r}\right)}\, dr < \infty\,.$$

8. (i) Let f be continuously differentiable. Then

$$|f(0)| \leq |f(x)| + \int_{[0,x]} |f'(y)|\,dy.$$

integrating both sides over the interval $[-1,1]$ we obtain

$$2|f(0)| \leq \int_{-1}^{1} |f(x)|\,dx + \int_{-1}^{1} (1-|x|)|f'(x)|\,dx \leq \|f\|_{W^{1,1}}.$$

Since $\mathcal{C}^1(\Omega)$ is dense in $W^{1,1}(\Omega)$, this functional can be extended by continuity to a bounded linear functional on the whole space $W^{1,1}(\Omega)$.

(ii) Next, assume $\Omega = B(0,1) \subset \mathbb{R}^2$. For $p > 2$, setting $\gamma = 1 - \frac{2}{p}$, Morrey's inequality yields

$$|f(0)| \leq \|f\|_{C^{0,\gamma}(\Omega)} \leq C \cdot \|f\|_{W^{1,p}(\Omega)},$$

for some constant C and every $f \in W^{1,p} \cap \mathcal{C}^1$. Hence the map $T : f \mapsto f(0)$ can be uniquely extended to a bounded linear functional on the entire space $W^{1,p}(\Omega)$.

To see that this functional is not continuous on $W^{1,p}(\Omega)$ for $p \in [1,2]$, we prove that

$$\sup_{f \in W^{1,2}(\Omega)} \frac{|f(0)|}{\|f\|_{W^{1,2}}} = \infty. \tag{28}$$

Observing that, for every $p \in [1,2]$, one has $\|f\|_{W^{1,p}} \leq C_p \|f\|_{W^{1,2}}$, from (28) we conclude that the functional $f \mapsto f(0)$ is unbounded in the space $\|f\|_{W^{1,p}}$ as well.

To prove (28), consider the function

$$f(x) \doteq \ln \ln \left(1 + \frac{1}{|x|}\right)$$

and the decreasing sequence of continuous functions

$$f_n(x) \doteq \min \left\{1, \frac{f(x)}{n}\right\}.$$

As shown in Problem **7**, one has $f \in W^{1,2}(\Omega)$. Hence

$$|f_n(0)| = 1, \qquad \|f_n\|_{W^{1,2}} \leq \frac{1}{n}\|f\|_{W^{1,2}} \to 0.$$

9. Assume $p > 2$ and set $\gamma = 1 - \frac{p}{2}$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. Then for every $t \in \mathbb{R}$ the restriction of f to the plane

$$\Sigma_t \doteq \{(t, x_2, x_3)\,;\ \ x_2, x_3 \in \mathbb{R}\}$$

lies in $\mathcal{C}_c^\infty(\Sigma_t)$. Morrey's inequality yields

$$|f(t,0,0)| \leq \|f\|_{\mathcal{C}^{0,\gamma}(\Sigma_t)} \leq C\,\|f\|_{W^{1,p}(\Sigma_t)}.$$

we have

$$\int_{\mathbb{R}^2} \Big(|f(\bar{x}_1, x_2, x_3)|^p + |\nabla f(\bar{x}_1, x_2, x_3)|^p\Big)\,dx_2 dx_3 < \infty.$$

If p is suitably large, then f coincides a.e. with a Hölder continuous function on the plane $\{(x_1, x_2, x_3)\,;\quad x_1 = \bar x_1\}$.

10. Construct a sequence of smooth approximations $u_\nu \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\|u_\nu - u\|_{W^{1,p}(\mathbb{R}^n)} \leq 2^{-\nu-1}$ for every $\nu \geq 1$. Setting $u_0 = 0$, $v_\nu \doteq u_{\nu+1} - u_\nu$, we have

$$u \;=\; w_0 + \sum_{\nu=1}^\infty w_\nu\,, \qquad\qquad \|w_\nu\|_{W^{1,p}} \;\leq\; 2^{-\nu} \quad \text{forall } \nu \geq 1\,.$$

We shall use the notation $x = v + y$, with $v \in V$, $y \in V^\perp$. Observing that

$$\int_{\mathbb{R}}^n \sum_{\nu=0}^\infty (|w_\nu|^p + |\nabla w_\nu|^p)\,dx \;=\; \int_{V^\perp} \sum_{\nu=0}^\infty \left(\int_{y+V} (|w_\nu|^p + |\nabla w_\nu|^p)\,dv\right) dy \;<\; \infty\,,$$

by Fubini's theorem we conclude that

$$\int_{y+V} \sum_{\nu=0}^\infty (|w_\nu|^p + |\nabla w_\nu|^p)\,dv \;<\; \infty \tag{29}$$

for a.e. $z \in V^\perp$. If (29) holds, then the sequence of partial sums is absolutely convergent in $W^{1,p}(y+V)$, hence also in $\mathcal{C}^{0,\gamma}(y+V)$, by Morrey's embedding theorem, with $\gamma = 1 - \frac{m}{p}$. Hence, restricted to the affine subspace $y+V$, the function u coincides a.e. with a Hölder continuous function.

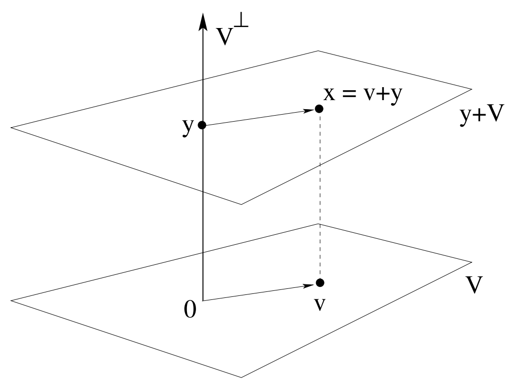

Figure 2: Each affine subspace $y+V$ has dimension $m < p$, hence Morrey's inequality can be applied.

(ii) For any smooth function $w \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with compact support, one has

$$\int_{V^\perp} |w(y)|^p dy \;\leq\; \int_{V^\perp} \|w\|^p_{\mathcal{C}^{0,\gamma}(y+V)}\,dy \;\leq\; C\int_{V^\perp} \|w\|^p_{W^{1,p}(y+V)}\,dy \;=\; C\|w\|^p_{W^{1,p}(\mathbb{R}^n)}\,.$$

By approximation, we conclude that $\|u\|_{\mathbf{L}^p(V^\perp)} \;\leq\; C\|w\|^p_{W^{1,p}(\mathbb{R}^n)}$ for every $u \in W^{1,p}(\mathbb{R}^n)$.

11. Since the inequality

$$\|g\|_{\mathcal{C}^{0,\gamma}(\Omega)} \;\leq\; C\,\|g\|_{W^{1,p}(\Omega)}\,, \qquad\qquad \gamma = 1 - \frac{n}{p}$$

is valid for every $g \in \mathcal{C}^\infty_c(\Omega)$, by approximation the same holds for every $f \in W^{1,p}_0(\Omega)$, after a modification on a set of measure zero.

12. Consider any open set $\Omega \subset \mathbb{R}^n$. Consider the open subsets

$$\Omega_{1/n} \doteq \Big\{ x \in \Omega\,; \quad |x| < n\,, \quad \overline{B}(x, 1/n) \subset \Omega \Big\}. \tag{30}$$

By the dominated convergence theorem, for any $f \in \mathbf{L}^p(\Omega)$, $1 \leq p < \infty$, the approximations

$$f_n(x) \doteq \begin{cases} f(x) & \text{if } x \in \Omega_{1/n} \\ 0 & \text{otherwise} \end{cases}$$

converge to f in $\mathbf{L}^p$. In turn, for $\varepsilon < 1/n$ the mollifications $J_\varepsilon * f_n$ lie in $\mathcal{C}_c^\infty(\Omega)$ and converge to f_n as $\varepsilon \to 0$. Hence f can be approximated in $\mathbf{L}^p(\Omega)$ by smooth functions with compact support.

Next, consider any sequence of functions $f_n \in C_c^\infty(\Omega)$. and assume $\|f_n - f\|_{\mathbf{L}^\infty} \to 0$. This implies that f is the uniform limit of a sequence of continuous functions which vanish on the boundary of Ω. Hence f is a bounded continuous function which satisfies

$$\lim_{n\to\infty} \sup \,\{|f(x)|\,; \quad x \in \Omega \setminus \Omega_{1/n}\} \;=\; 0\,. \tag{31}$$

Viceversa, every bounded continuous function f satisfying (31) lies in $W_0^{0,\infty}(\Omega)$. Indeed, if this holds, then we can approximate f with a sequence of functions $f_{n,\varepsilon_n} \in C_c^\infty(\Omega)$, choosing

$$f_{n,\varepsilon_n} \;=\; J_{\varepsilon_n} * f_n \;=\; J_{\varepsilon_n} * (f \cdot \chi_{\Omega_{1/n}})\,, \qquad\qquad \varepsilon_n << \frac{1}{n}\,.$$

13. Let $f_k(x) \doteq f(x)\varphi(k - |x|)$. Then f_k is supported on the set where $|x| \leq k$ and coincides with f for $|x| \leq k - 1$. By the dominated convergence theorem

$$\|f_k - f\|^p_{W^{1,p}} \;\leq\; C \int_{|x|>k-1} \left(|f|^p + \sum_i \left| \frac{\partial f}{\partial x_i} \right|^p \right) dx \;\to\; 0 \qquad\qquad \text{as} \quad k \to \infty.$$

Performing the mollifications $f_{k,\varepsilon_k} = J_{\varepsilon_k} * f_k$ with $\varepsilon_k << 1/k$ we obtain a sequence of smooth functions with compact support which converge to f in $W^{1,p}(\mathbb{R}^n)$.

14. If $u \in W^{2,p}(\mathbb{R}^+)$ then u is absolutely continuous, hence the limit $u(0) =\doteq \lim_{x\to 0+} u(x)$ is well defined. The even extension $Eu(x) = u(|x|)$ is absolutely continuous, with derivative $(Eu)_x(x) = (\text{sign } x)\, u_x(|x|)$ for a.e. $x \in \mathbb{R}$. Our assumptions imply

$$\int_{-\infty}^{\infty} \Big(|Eu|^p + |(Eu)_x|^p \Big)\, dx \;=\; 2 \int_0^\infty \Big(u|^p + |u_x|^p \Big)\, dx \;<\; \infty.$$

Hence $Eu \in W^{1,p}(\mathbb{R})$.

In general, $Eu \notin W^{2,p}(\mathbb{R})$. For example, take $u(x) = x/2$. Then $Eu(x) = |x|/2$ is a function whose first derivative is $(Eu)_x = \frac{1}{2}\text{sign } x$. However, the function $x \mapsto \frac{1}{2}\text{sign } x$ has a jump at $x = 0$. Its distributional derivative is a Dirac distribution, concentrating a unit mass at $x = 0$. Hence Eu does not have a weak second derivative.

15. (i) A direct computation yields

$$\|u_\lambda\|_{\mathbf{L}^q} \;=\; \left(\int |u(\lambda x)|^q\,dx\right)^{1/q} \;=\; \left(\int |u(y)|^q\,\lambda^{-n}dy\right)^{1/q} \;=\; \lambda^{-n/q}\|u\|_{\mathbf{L}^q}\,.$$

Hence the conclusion holds with $\alpha = -n/q$.

(ii) A similar computation yields

$$\|\nabla u_\lambda\|_{\mathbf{L}^p} \;=\; \left(\int |\lambda\,\nabla u(\lambda x)|^p\,dx\right)^{1/p} \;=\; \left(\int |u(y)|^p\,\lambda^{p-n}dy\right)^{1/p} \;=\; \lambda^{(p-n)/p}\|u\|_{\mathbf{L}^p}\,.$$

Hence the conclusion holds with $\beta = (p-n)/p$.

(iii) We have $\alpha = \beta$ if $-\frac{n}{q} = \frac{p-n}{p}$. This is the case if $\frac{1}{q} \;=\; \frac{1}{p} - \frac{1}{n}$, i.e. if $q = p^*$ is the Sobolev conjugate exponent to p.

16. Since $H^1(\Omega)$ is a Hilbert space and the sequence $(u_m)_{m\geq 1}$ is bounded, we can extract a convergent subsequence $u_{m_k} \rightharpoonup \tilde{u}$ in $H^1(\Omega)$. Moreover, since the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact, by possibly extracting a further subsequence we obtain the convergence $\|u_{m_k} - u\|_{\mathbf{L}^2} \to 0$. This clearly implies $\tilde{u} = u$, hence $u \in H^1(\Omega)$.

To prove a bound on $\|u\|_{H^1}$, set

$$M \doteq \liminf_{m\to\infty} \|u_m\|_{H^1}\,.$$

By possibly extracting a subsequence and relabeling, we can assume

$$M = \lim_{m\to\infty} \|u_m\|_{H^1}\,.$$

Writing

$$0 \;\leq\; (u_m - u,\, u_m - u)_{H^1} \;=\; (u_m,\, u_m)_{H^1} + (u,u)_{H^1} - 2(u_m,u)_{H^1}\,,$$

and letting $m \to \infty$, we obtain

$$M^2 + \|u\|^2_{H^1} - 2\|u\|^2_{H^1} \;\geq\; 0\,.$$

Hence $\|u\|_{H^1} \leq M$.

17. (i) Consider the functions

$$f_n(x) \doteq \begin{cases} 1 - \frac{n+1}{n}|x| & \text{if } \ |x| \leq \frac{n}{n+1}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Show that each f_n has compact support in Ω. Moreover, $\|f_n - f\|_{W^{1,p}} \to 0$. Choose a sequence $\varepsilon_n \to 0$ so that the mollified functions $J_{\varepsilon_n} * f_n$ have compact support and converge to f in H^1.

(ii) To show that $f \in W^{1,2}_0(\Omega_0)$, let $\varphi(x) = \log\log\left(1 + \frac{1}{|x|}\right)$. Notice that $\varphi \in W^{1,2}(\Omega)$ as claimed in problem **6.** For each integer n, define the function

$$g_n(x) \;=\; \max\left\{0\,,\ \min\left\{1 - \frac{n+1}{n}|x|\ ,\ n - \varphi(x)\right\}\right\}.$$

Show that each g_n has compact support in Ω_0. Moreover, $\|g_n - f\|_{W^{1,p}} \to 0$. Choose a sequence $\varepsilon_n \to 0$ so that the mollified functions $J_{\varepsilon_n} * f_n$ have compact support contained in Ω_0 and converge to f in H^1.

(iii) If the functions $\varphi_n \in \mathcal{C}_c^\infty(\Omega_0)$ form a Cauchy sequence w.r.t. the $W^{1,p}$ norm, with $p > 2$, then by Morrey's inequality the sequence $(\varphi_n)_{n\geq 1}$ converges to some continuous function g, uniformly on the closed disc $\overline{\Omega}$. In particular, $g = 0$ at the origin. Derive a contradiction, showing that f cannot coincide a.e. with a continuous function that vanishes at the origin.

18. (i) Since Ω is connected, by Corollary 8.16 the assumption implies that u coincides a.e. with a constant. Since the set where $u = 0$ has positive measure, we conclude that $u(x) = 0$ for a.e. $x \in \Omega$.

(ii) If (8.82) fails, consider a sequence of functions u_n such that

$$\|u_n\|_{\mathbf{L}^2} \;=\; 1\,, \qquad\qquad \|\nabla u_n\|_{\mathbf{L}^2} < \frac{1}{n}\,,$$

and $\text{meas}\big(\{x \in \Omega\,;\; u_n(x) = 0\}\big) \geq \alpha$ for every n.

Since the embedding $H^1(\Omega) \subset \mathbf{L}^2(\Omega)$ is compact, we can extract a subsequence $(u_{n_k})_{k\geq 1}$ such that $\|u_{n_k} - u\|_{\mathbf{L}^2} \to 0$ for some limit function $u \in \mathbf{L}^2(\Omega)$, and moreover $u_{n_k} \rightharpoonup u$ in $H^1(\Omega)$. By taking a further subsequence, we can assume the pointwise convergence $u_{n_k}(x) \to u(x)$ for a.e. $x \in \Omega$.

By assumption, the weak gradients ∇u_{n_k} converge to zero in $\mathbf{L}^2(\Omega)$. Hence $\nabla u \equiv 0$ and $u(x) = c$ for some constant c and a.e. $x \in \Omega$. We claim that $c = 0$, because otherwise

$$\|u_{n_k} - u\|^2_{\mathbf{L}^2} \;=\; \int_\Omega |u_{n_k}(x) - c|^2\,dx \;\geq\; \int_{\{x\,;\; u_{n_k}(x)=0\}} c^2\,dx \;\geq\; \alpha c^2$$

for every k.

On the other hand, the strong convergence in $\mathbf{L}^2$ implies $\|u\|_{\mathbf{L}^2} = \lim_{k\to\infty} \|u_{n_k}\|_{\mathbf{L}^2} = 1$, providing a contradiction.

19. Fix $i \in \{1,\ldots,n\}$ We claim that the function $v = u_{x_i}$, defined a.e. as the classical derivative of u, provides the weak derivative of u on $\mathbb{R}^n$.

In the following, without loss of generality we assume $i = 1$ and use the notation $x = (x_1, x')$. Moreover, we consider the set

$$\mathcal{N} \;\doteq\; \Big\{x' = (x_2,\ldots,x_n) \in \mathbb{R}^{n-1}\,;\;\; (t,x') \in K \;\text{ for some } t \in \mathbb{R}\Big\}\,.$$

By assumption, this set has zero $(n-1)$-dimensional measure.

For any test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we now have

$$\int_{\mathbb{R}^n} u\varphi_{x_1}\,dx \;=\; \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\infty} u\varphi_{x_1}\,dx_1\,dx' \;=\; \int_{\mathbb{R}^{n-1}\setminus\mathcal{N}} \left(-\int_{-\infty}^{\infty} u_{x_1}\varphi\,dx_1\right)dx' \;=\; -\int_{\mathbb{R}^n} u_{x_1}\varphi\,dx\,.$$

Hence $u_{x_1} = D_{x_1}u$, proving our claim.

In turn, the assumptions $u, \nabla u \in \mathbf{L}^p(\Omega)$ imply $u \in W^{1,p}(\Omega)$.

20. (i) Take $f(x) = g(x) = |x|^{-1/2}$

(iii) Take $f(x) = g(x) = |x|^{\frac{3}{2}-n}$.

21. Assume $1 \leq p < \infty$. If the result were not true, we could find a sequence $u_n \in W^{1,p}(\Omega)$ such that

$$\|u_n\|_{\mathbf{L}^p(\Omega)} \;=\; 1\,, \qquad \|u_n\|_{\mathbf{L}^p(\Omega')} \;\to\; 0\,, \qquad \|\nabla u_n\|_{\mathbf{L}^p(\Omega)} \;\to\; 0\,.$$

By the compact embedding $W^{1,p}(\Omega) \subset\subset \mathbf{L}^p(\Omega)$ there exists a subsequence u_{n_k} such that $\|u_{n_k} - u\|_{\mathbf{L}^p} \;\to\; 0$ for some $u \in \mathbf{L}^p(\Omega)$

The above conditions imply $\|u\|_{\mathbf{L}^p(\Omega')} = 0$, $\|\nabla u\|_{\mathbf{L}^p(\Omega)} = 0$. Since Ω is connected, we conclude that $u \equiv 0$. However, this is in contradiction with $\|u\|_{\mathbf{L}^p(\Omega)} = \lim_{n\to\infty} \|u\|_{\mathbf{L}^p(\Omega)} = 1$.

22. The set S can be represented as

$$S \;=\; \Big\{f :]0,1[\mapsto \mathbb{R}\,;\;\; \|f\|_{C^1} \leq M\,,\;\; f' \;\text{ is Lipschitz continuous with constant } M\Big\}.$$

Assume $\|f_n - f\|_{\mathcal{C}^0} \to 0$, with $f_n \in S$ for every n. By assumption, all functions $f_n, \partial_x f_n$ are uniformly bounded and Lipschitz continuous with constant M. Hence they can be extended by continuity to the closed interval $[0,1]$.

Since $f_n(x) \to f(x)$ uniformly on $[0,1]$, this implies that f is also Lipschitz continuous with constant M. By Ascoli's theorem, by taking a subsequence we can assume the uniform convergence $\partial_x f_{n_k} \to v$ for some Lipschitz continuous function v. Together, the limits

$$f_{n_k}(u) \;\to\; f(x)\,, \qquad \partial_x f_{n_k}(x) \;\to\; v(x) \qquad \text{uniformly for } x \in]0,1[$$

imply that $v(x) = f'(x)$. Since this limit does not depend on the particular subsequence, we conclude that the entire sequence $\partial_x f_n$ converges to $\partial_x f(x)$, uniformly on $]0,1[$.

Notice that this implies that $\partial_x f$ is also Lipschitz continuous with constant M. The above shows that $f \in S$, hence S is a closed subset of $\mathcal{C}^0(]0,1[)$. The above arguments prove both (ii) and (i).

23. Let $Eu \in W^{1,p}(\mathbb{R}^n)$ be an extension of u to the entire space $\mathbb{R}^n$, with $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}\Omega)}$. Consider the mollifications $u_k \doteq J_{1/k} * Eu$.

24. The assumption implies that f is absolutely continuous on any bounded interval. Hence f differentiable at a.e. point. Writing

$$f(x) \;=\; f(a) + \int_a^x g(s)\,ds,$$

we have

$$g_n(x) \;=\; \frac{1}{n}\int_x^{x+1/n} g(s)\,ds\,.$$

This implies the pointwise convergence $g_n(x) \to g(x)$ at every Lebesgue point x of g, hence almost everywhere.

To prove that $\|g_n - g\|_{\mathbf{L}^1([a,b])} \to 0$, let $\varepsilon > 0$ be given. Choose a continuous function φ such that $\|g - \varphi\|_{\mathbf{L}^1([a,\, b+1])} < \varepsilon$ and define

$$\varphi_n(x) \;=\; \frac{1}{n}\int_x^{x+1/n} \varphi(s)\, ds\,.$$

Then

$$\begin{aligned}
&\limsup_{n\to\infty} \|g_n - g\|_{\mathbf{L}^1([a,b])} \\
&\quad\le\; \limsup_{n\to\infty} \|g_n - \varphi_n\|_{\mathbf{L}^1([a,b])} + \limsup_{n\to\infty} \|\varphi_n - \varphi|_{\mathbf{L}^1([a,b])} + \limsup_{n\to\infty} \|\varphi - g\|_{\mathbf{L}^1([a,b])} \\
&\quad\le\; \varepsilon + 0 + \varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves the result.

25. The net smoothness of a function $u \in W^{2,2}(\mathbb{R}^3)$ is $k - \frac{n}{p} = 2 - \frac{3}{2} = \frac{1}{2}$. Therefore, the Sobolev embedding theorem implies that, after a modification on a set of measure zero, $u_n \in C^{0,1/2}$, with $\|u_n\|_{C^{0,1/2}} \le C$, for some constant C and every $n \ge 1$. Being the pointwise limit of a sequence of uniformly Hölder continuous functions, u is Hölder continuous as well.

Chapter 9

1. The PDE can be rewritten as $Lu = -f$, where $Lu = -\Big((u_x)_x + (xu_x)_y + (u_y)_y\Big)$. To prove that the operator L is uniformly elliptic, it suffices to check that the quadratic form $(\xi_1, \xi_2) \mapsto \xi_1^2 + x\xi_1\xi_2 + \xi_2^2$ is strictly positive definite for $(x, y) \in \Omega$. The conclusion is then obtained by applying Theorem 9.8.

2. (i) The fact that $B[u, v] = \langle u, v\rangle_\diamond$ is a continuous bilinear map on H_0^1 is clear. We need to show that it is strictly positive definite. For $|y| \le 1$ one has

$$a^2 + 2b^2 + 2yab \;\ge\; (2 - \sqrt{2})(a^2 + 2b^2) + \sqrt{2}(a^2 + 2b^2) + 2yab \;\ge\; (2 - \sqrt{2})(a^2 + 2b^2)\,.$$

Therefore, by Poincare's inequality,

$$B[u, u] \;=\; \int_\Omega (u_x^2 + 2u_y^2 + 2yu_xu_y)\, dxdy \;\ge\; (2 - \sqrt{2})\int_\Omega (u_x^2 + u_y^2)\, dxdy \;\ge\; \beta\, \|u\|_{H^1}^2$$

for some $\beta > 0$ and all $u \in H_0^1(\Omega)$.

(ii) We can now use the Lax-Milgram theorem and conclude that for every $f \in \mathbf{L}^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that $B[u, v] = (f, v)_{\mathbf{L}^2}$ for every $v \in H_0^1(\Omega)$. By a formal integration by parts, we find that this function u provides a weak solution to the elliptic boundary value problem

$$\begin{cases} -u_{xx} - 2u_{yy} - (2+y)u_{xy} &= f \qquad \text{on } \Omega\,, \\ u &= 0 \qquad \text{on } \partial\Omega\,. \end{cases}$$

3. The operator L is elliptic in a neighborhood of the point $(x,y) \in \mathbb{R}^2$ if and only if the coefficient matrix $\begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}$ is strictly positive definite. This is the case if and only if

$$(x,y) \;\in\; \Omega^+ \doteq \{(x,y) \in \mathbb{R}^2\,;\quad x > 0, y > 0, xy > 1\}\,.$$

The operator L is uniformly elliptic on a bounded domain Ω if and only if the closure $\overline{\Omega}$ is entirely contained in the open set Ω^+.

4. If $\phi \in H^1_0$ is a weak solution, then

$$\int_\Omega \nabla\phi \cdot \nabla v\, dx \;=\; \mu \int_\Omega \phi\, v\, dx \qquad\qquad \text{for all} \;\; v \in H^1_0(\Omega).$$

In particular, taking $v = \phi$ we obtain

$$\|\phi\|^2_{\mathbf{L}^2(\Omega)} \;=\; \int_\Omega |\nabla\phi|^2\, dx \;=\; \mu \int_\Omega |\phi|^2\, dx \;=\; \mu\|\phi\|^2_{\mathbf{L}^2(\Omega)}\,.$$

By assumption,

$$\beta^2 \;=\; \sup_{u \in H^1_0, u \neq 0} \frac{\|\phi\|^2_{\mathbf{L}^2(\Omega)}}{\|\nabla\phi\|^2_{\mathbf{L}^2(\Omega)}},$$

hence $\mu \geq 1/\beta^2$.

(ii) Using the representation formula (9.57), the solution of the parabolic equation with initial data $u(0) = g$ is provided by

$$u(t) \;=\; \sum_{k=1}^{\infty} e^{-\mu_k t}(g, \phi_k)_{\mathbf{L}^2}\phi_k\,, \tag{32}$$

where $\phi_k \in H^1_0(\Omega)$, the set $\{\phi_k\,;\quad k \geq 1\}$ is an orthonormal basis of $\mathbf{L}^2(\Omega)$ consisting of eigenfunctions of the Laplace operator, and μ_k are the corresponding eigenvalues. Since $\mu_k \geq \beta^{-2}$ for every $k \geq 1$, from (32) it follows

$$\|u(t)\|^2_{\mathbf{L}^2} \;=\; \sum_{k=1}^{\infty} \left| e^{-\mu_k t}(g, \phi_k)_{\mathbf{L}^2} \right|^2 \;\leq\; \sum_{k=1}^{\infty} \left| e^{-t/\beta^2}(g, \phi_k)_{\mathbf{L}^2} \right|^2 \;=\; e^{-2t/\beta^2}\, \|g\|^2_{\mathbf{L}^2}\,.$$

Taking square roots of both sides we obtain the result.

5. As in Theorem 9.9, let $\{\phi_k\,;\quad k \geq 1\}$ be an orthonormal basis of $\mathbf{L}^2(\Omega)$, consisting of eigenfunctions of the operator $-\Delta$. For every $v \in \mathcal{C}^\infty_c(\Omega)$ we have

$$B[v,v] \;\doteq\; \int_\Omega |\nabla v|^2\, dx \;=\; -\int_\Omega v \cdot \Delta v\, dx \;=\; \sum_{k=1}^{\infty} \mu_k (v, \phi_k)^2 \;\geq\; \mu_1\, \|v\|^2_{\mathbf{L}^2}\,.$$

By continuity, the same inequality remains valid for every $v \in H_0^1(\Omega)$. On the other hand, the eigenfunction $\phi_1 \in H_0^1(\Omega)$ satisfies

$$B[\phi_1, \phi_1] ~=~ \int_\Omega |\nabla \phi_1|^2 \, dx ~=~ -\int_\Omega \phi_1 \cdot \Delta \phi_1 \, dx ~=~ \mu_1 ~=~ \mu_1 \, \|\phi_1\|^2_{\mathbf{L}^2} .$$

We thus obtain a representation for the first eigenvalue:

$$\mu_1 ~=~ \inf_{0 \neq v \in H_0^1(\Omega)} \frac{B[v,v]}{\|v\|^2_{\mathbf{L}^2}} ~=~ \inf_{0 \neq v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega |v|^2 \, dx} . \tag{33}$$

Similarly,

$$\tilde\mu_1 ~=~ \inf_{0 \neq v \in H_0^1(\Omega)} \frac{B[v,v]}{\|v\|^2_{\mathbf{L}^2}} ~=~ \inf_{0 \neq v \in H_0^1(\Omega)} \frac{\int_{\widetilde\Omega} |\nabla v|^2 \, dx}{\int_{\widetilde\Omega} |v|^2 \, dx} .$$

The inclusion $\Omega \subseteq \widetilde\Omega$ implies $H_0^1(\Omega) \subset H_0^1(\widetilde\Omega)$. Hence the above representation formulas yield $\tilde\mu_1 \leq \mu_1$.

6. Choose $\gamma \doteq \max_{t \in [0,T]} q(t)$ and consider the operator

$$L_\gamma u ~\doteq~ -(p(t)u')' + (\gamma - q(t))u .$$

Since $p(t) \geq \theta > 0$, the operator L_γ is uniformly elliptic. We claim that the bilinear form

$$B_\gamma[u,v] ~=~ \int_0^T \Big(p(t)u'(t)v'(t) + (\gamma - q(t))u(t)v(t) \Big) \, dt$$

is strictly positive definite on $H_0^1(]0,T[)$. Indeed,

$$B_\gamma[u,u] ~\geq~ \int_0^T p(t)|u'(t)|^2 \, dt ~\geq~ \beta \|u\|_{H^1}$$

for some $\beta > 0$ and all $u \in H_0^1(]0,T[)$.

As in the Theorem 9.9, the inverse operator L_γ^{-1} is a linear, compact self-adjoint operator from $\mathbf{L}^2(\Omega)$ into itself. By the Hilbert-Schmidt theorem, the space $\mathbf{L}^2(\Omega)$ admits an orthonormal basis $\{\phi_k \, ; ~~ k \geq 1\}$ consisting of eigenfunctions of L_γ^{-1}. The corresponding eigenvalues satisfy $\lambda_k > 0$, $\lambda_k \to 0$ as $k \to \infty$. This implies

$$\left(\frac{1}{\lambda_k} - \gamma \right) \phi_k ~=~ L \phi_k .$$

The above analysis shows that the eigenvalues of the Sturm-Liuville problem (9.85) are given by $\mu_k = \gamma - \frac{1}{\lambda_k}$. Hence $\lim_{k \to \infty} \mu_k = -\infty$.

7. By assumption, for each $k \geq 1$ we have

$$\int_\Omega \nabla \phi_k \cdot \nabla v \, dx ~=~ \mu_k \int_\Omega \phi_k v \, dx$$

for some eigenvalue μ_k and all $v \in H_0^1(\Omega)$. Assume $j \neq k$ and choose $v = \phi_j$. Then

$$\int_\Omega \nabla \phi_k \cdot \nabla \phi_j \, dx ~=~ \mu_k \int_\Omega \phi_k \phi_j \, dx ~=~ 0 .$$

Hence $(\phi_k, \phi_j)_{H^1} = 0$.

8. (i) Let $\{\varphi_k\,;\quad k \geq 1\}$ be an orthonormal basis of $\mathbf{L}^2(\Omega)$ consisting of eigenfunctions of the Laplace operator Δ. We claim that, for every $f \in \mathbf{L}^2(\Omega)$ and $m \geq 1$, the linear system of algebraic equations

$$B[u_m, \varphi_j] \;=\; \sum_{k=1}^{m} c_k B[\varphi_k, \varphi_j] \;=\; (f, \varphi_j)_{\mathbf{L}^2} \qquad\qquad j = 1, \ldots, m$$

has a unique solution $c_1, \ldots, c_m$.

For this purpose, it suffices to show that the $m \times m$ matrix of coefficients $A = (a_{ij})$, with $a_{ij} = B[\varphi_k, \varphi_j]$, is invertible. Assume, on the contrary, that there exists a nonzero vector $\xi = (\xi_1, \ldots, \xi_n)$ such that

$$\sum_{j=1}^{n} B[\varphi_k, \varphi_j]\xi_j \;=\; 0 \qquad\qquad k = 1, \ldots, m\,.$$

Then, setting $v = \sum_j \xi_j \varphi_j \neq 0$ we obtain

$$B[v, v] \;=\; \sum_{j,k=1}^{m} B[\varphi_k, \varphi_j]\xi_j\xi_k \;=\; 0,$$

contradicting the assumption that the bilinear form $B[\cdot,\cdot]$ is strictly positive definite.

(ii) If the identity

$$B[u_m, v] \;=\; (f, v)_{\mathbf{L}^2}$$

holds for $v = \varphi_1, \ldots, \varphi_m$, then by linearity it remains valid for every $v \in span\{\varphi_1, \ldots, \varphi_m\}$. In particular, since B is strictly positive definite, this implies

$$\beta\|u_m\|_{H^1}^2 \;\leq\; B[u_m, u_m] \;=\; (f, u_m)_{\mathbf{L}^2} \;\leq\; \|f\|_{\mathbf{L}^2}\,\|u_m\|_{\mathbf{L}^2} \;\leq\; \|f\|_{\mathbf{L}^2}\,\|u_m\|_{H^1}\,,$$

for some constant $\beta > 0$. Therefore

$$\|u_m\|_{H^1} \leq \beta^{-1}\|f\|_{\mathbf{L}^2}\,.$$

Thanks to the uniform bound, we can extract a subsequence $(u_{n_j})_{j\geq 1}$ and achieve the weak convergence $u_{n_j} \rightharpoonup u$ in $H_0^1(\Omega)$. We now have

$$B[u, \varphi_k] \;=\; \lim_{m\to\infty} B[u_m, \varphi_k] \;=\; (f, \varphi_k).$$

By linearity

$$B[u, v] \;=\; \lim_{m\to\infty} B[u_m, v] \;=\; (f, v) \tag{34}$$

for every $v \in span\{\varphi_k\,;\quad k \geq 1\}$. By approximation, (34) remains valid for every $v \in \overline{span\{\varphi_k\,;\quad k \geq 1\}} = H_0^1(\Omega)$. This proves that u is a weak solution to the elliptic problem.

9. According to (9.36), the solution u of the boundary value problem satisfies an abstract equation of the form $u - Ku = h$, where $K : \mathbf{L}^2(\Omega) \mapsto \mathbf{L}^2(\Omega)$ is a compact operator and

$h = L_\gamma^{-1} f$, where L_γ^{-1} is another compact operator. Moreover, we are assuming that the operator $I-K$ is a continuous bijection. By the open mapping theorem, the inverse operator $(I-K)^{-1}$ is continuous. Being the composition of a compact operator and a continuous one, the map $f \mapsto u = (I-K)^{-1} L_\gamma^{-1} f$ is compact operator.

10. (i) By Problem 9 in Chapter 2, every function $\varphi \in \mathcal{C}_c^\infty(Q)$ can be uniformly approximated by a finite linear combination of the functions $\phi_{m,n}$. Since $\mathcal{C}_c^\infty(Q)$ is dense in $\mathbf{L}^2(Q)$, we conclude that $\mathrm{span}\{\phi_{m,n}\,;\quad m,n \geq 1\}$ is dense in $\mathbf{L}^2(Q)$.

A straightforward computation shows that

$$-\Delta \phi_{m,n} \;=\; \mu_{m,n}\,\phi_{m,n}\,, \qquad\qquad \mu_{m,n} \;=\; \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$

(ii) If μ_1 is the smallest eigenvalue of $-\Delta$ on Ω, then by the result proved in problem **5.** the assumption $\Omega \subseteq Q$ implies $\mu_1 \geq \mu_{1,1} = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}$. The conclusion now follows from (33).

11. This is a special case of problem 8. Consider the 2×2 matrix $A = (a_{ij})$ and the vector $b = (b_1, b_2)$ with entries

$$a_{ij} \;=\; B[\varphi_i, \varphi_j] \;=\; \int_0^3 \varphi_i'(x)\varphi_j'(x)\,dx\,, \qquad\qquad b_i \;=\; \int_0^3 1\cdot\varphi_i\,dx\,.$$

An explicit calculation yields

$$A \;=\; \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad\qquad b \;=\; \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solving the system of two algebraic equations

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we find $\xi_1 = \xi_2 = 1/3$. The Galerkin approximation thus yields

$$u_2(x) \;=\; \frac{\varphi_1(x) + \varphi_2(x)}{3} \;=\; \begin{cases} x/3 & \text{if } \; x \in [0,1]\,, \\ 1/3 & \text{if } \; x \in [1,2]\,, \\ (3-x)/3 & \text{if } \; x \in [2,3]\,. \end{cases}$$

The exact solution of the boundary value problem is $u(x) = \frac{3}{2}x - \frac{1}{2}x^2$.

12. (i) Here L is the symmetric operator defined by

$$Lu = -(2u_x)_x - \frac{1}{2}(yu_y)_x - \frac{1}{2}(yu_x)_y - (3u_y)_y\,.$$

The corresponding symmetric bilinear form is

$$B[u,v] \;=\; \int_\Omega \left(2u_xv_x + \frac{y}{2}(u_xv_y + u_yv_x) + 3u_yv_y\right)\,dx\,.$$

On the open domain $\Omega = \{(x,y)\,;\ \ x^2+y^2<1\}$, the operator L is uniformly elliptic. Indeed, for $|y|\leq 1$ we have

$$2\xi_1^2 + 2y\xi_1\xi_2 + 3\xi_2^2 \ \geq\ \xi_1^2+\xi_2^2 \qquad\qquad \text{for every}\ \ \xi=(\xi_1,\xi_2)\in\mathbb{R}^2.$$

(ii) Define the energy as

$$E(t) \ =\ \frac{1}{2}\|u_t\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}B[u,u] \ =\ \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{2}\int_\Omega \Big(2u_x^2 + yu_xu_y + 3u_y^2\Big)\,dx\,.$$

Differentiating w.r.t. time, and using the boundary condition $u = u_t = 0$ on $\partial\Omega$, performing an integration by parts we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &= (u_t, u_{tt})_{\mathbf{L}^2} + B[u,u_t]\\ &= \int_\Omega u_tu_{tt}\,dx + \int_\Omega \Big(2u_xu_{xt} + \frac{y}{2}(u_yu_{xt}+u_xu_{yt}) + 3u_yu_{yt}\Big)\,dx\\ &= \int_\Omega u_tu_{tt}\,dx - \int_\Omega u_t\left(2u_{xx}+yu_{xy}+2u_{yy}+\frac{u_x}{2}\right)\,dx \ =\ 0\,.\end{aligned}$$

13. (i) The fact that on $H_0^1(\Omega)$ the norm $\|\cdot\|_\diamond$ is equivalent to the H^1 norm is a straightforward consequence of the fact that the bilinear form $B[\cdot,\cdot]$ is strictly positive definite.

(ii) Repeat the arguments used in the proof of Lemma 9.7.

14. For any $g\in\mathbf{L}^2(\Omega)$ the formula

$$S_tg \ =\ \sum_{k=1}^\infty e^{-\mu_kt}(g,\phi_k)_{\mathbf{L}^2}\phi_k\,. \tag{35}$$

defines a trajectory $t\mapsto S_tg\in\mathbf{L}^2(\Omega)$. Here $\{\phi_k\,;\ \ k\geq 1\}$ is an orthonormal basis of $\mathbf{L}^2(\Omega)$. We need to show that this map is n times continuously differentiable, for every $n\geq 1$. Toward this goal we observe that, for a given $N\geq 1$, the partial sum is continuously differentiable:

$$\frac{d^n}{dt^n}\left(\sum_{k=1}^N e^{-\mu_kt}(g,\phi_k)_{\mathbf{L}^2}\phi_k\right) \ =\ \sum_{k=1}^N(-\mu_k)^ne^{-\mu_kt}(g,\phi_k)_{\mathbf{L}^2}\phi_k\,.$$

We claim that, as $N\to\infty$, the above sum converges to a well defined limit, uniformly for t in compact subsets of $]0,\infty[$. Since the ϕ_k form an orthonormal sequence, it suffices to prove that

$$\lim_{N\to\infty}\sum_{n>N}\left|(-\mu_k)^ne^{-\mu_kt}(g,\phi_k)_{\mathbf{L}^2}\right|^2 \ =\ 0$$

uniformly for t in compact subsets of $]0,\infty[$.

Consider any interval $[a,b]$, with $0<a<b<\infty$. Then

$$M \ \doteq\ \sup_{\mu\geq 0,t\in[a,b]}\mu^ne^{-\mu t} \ <\ \infty\,.$$

Given any $\varepsilon > 0$, choose N so large that

$$M \sum_{n>N} |(g, \phi_k)|^2 \;<\; \varepsilon\,.$$

This is certainly possible because $\sum_{n\geq 1} |(g,\phi_k)|^2 = \|g\|^2_{\mathbf{L}^2} < \infty$. Then, for every $t \in [a, b]$,

$$\sum_{n>N} \left|(-\mu_k)^n e^{-\mu_k t} (g, \phi_k)_{\mathbf{L}^2}\right|^2 \;\leq\; M \sum_{n>N} |(g, \phi_k)|^2 \;<\; \varepsilon\,.$$

15. For $\gamma > 0$, the bilinear form

$$B_\gamma[u, v] \;=\; \int_\Omega \nabla u \cdot \nabla v + \gamma uv\, dx$$

satisfies

$$B_\gamma[u, u] \;\geq\; \beta\, \|u\|^2_{H^1}\,, \qquad\qquad \beta \doteq \min\{1, \gamma\}.$$

Hence for every $f \in \mathbf{L}^2(\Omega)$ there exists a unique $u \in H^1(\Omega)$ such that

$$B_\gamma[u, v] \;=\; (f, v)_{\mathbf{L}^2} \qquad\qquad \text{for every} \;\; v \in H^1(\Omega).$$

The map $f \mapsto u = L_\gamma^{-1} f$ is a self-adjoint, compact linear operator from $\mathbf{L}^2(\Omega)$ into itself.

A function $u \in \mathbf{L}^2(\Omega)$ is a weak solution to the Neumann problem if and only if

$$u - L_\gamma^{-1} \gamma u \;=\; L_\gamma^{-1} f\,.$$

This can be written in the abstract form

$$(I - K)u \;=\; h\,, \qquad\qquad K \doteq \gamma L_\gamma^{-1}\,, \quad h \doteq L_\gamma^{-1} f\,. \tag{36}$$

The equation (36) has a solution if and only if

$$h \;\in\; Range(I - K) \;=\; [Ker(I - K)]^\perp.$$

This is the case if and only if $(h, v)_{\mathbf{L}^2} \;=\; 0$ for every solution v of $v - Kv = 0$, i.e. for every weak solution of the homogeneous Neumann problem

$$\begin{cases} -\Delta u \;=\; 0 & \text{on} \;\; \Omega\,, \\[2ex] \dfrac{\partial u}{\partial \nu} \;=\; 0 & \text{on} \;\; \partial\Omega\,. \end{cases} \tag{37}$$

By definition, $v \in H^1(\Omega)$ is a weak solution of the above problem if

$$\int_\Omega \nabla v \cdot \nabla w\, dx \;=\; 0 \qquad\qquad \text{for every} \;\; w \in H^1(\Omega)\,.$$

Choosing $w = v$, this implies

$$\int_\Omega |\nabla v|^2\, dx \;=\; 0\,.$$

Since the open set Ω is connected, we conclude that the solutions of (37) are precisely the constant functions.

Going back to the original problem, by the previous analysis the Neumann boundary value problem has a solution if and only if $\int_\Omega f\cdot v\,dx = 0$ for every constant function v. Of course, this holds if and only if $\int_\Omega f\,dx = 0$.

16. On the space $H_0^2(\Omega)$ consider the continuous bilinear form

$$B[u,v] \doteq \int_\Omega \Delta u\,\Delta v\,dx\,.$$

We claim that there exists $\beta>0$ such that $B[u,u] \geq \beta\,\|u\|_{H^2}^2$, namely

$$\int_\Omega \Big(\sum_i u_{x_ix_i}\Big)\Big(\sum_j u_{x_jx_j}\Big)\,dx \;\geq\; \int_\Omega \sum_{ij} |u_{x_ix_j}|^2\,dx + \int_\Omega \sum_i |u_{x_i}|^2\,dx + \int_\Omega |u|^2\,dx\,. \qquad (38)$$

If $u\in\mathcal{C}_c^\infty(\Omega)$, integrating twice by parts we obtain

$$\int_\Omega u_{x_ix_i}u_{x_jx_j}\,dx \;=\; -\int_\Omega u_{x_ix_ix_j}u_{x_j}\,dx \;=\; \int_\Omega u_{x_ix_j}u_{x_ix_j}\,dx\,.$$

Hence

$$\int_\Omega \Big(\sum_i u_{x_ix_i}\Big)\Big(\sum_j u_{x_jx_j}\Big)\,dx \;=\; \int_\Omega \sum_{ij} |u_{x_ix_j}|^2\,dx\,. \qquad (39)$$

Applying Poincare's inequality to each function u_{x_i} we obtain

$$\int_\Omega |u_{x_i}|^2\,dx \;\leq\; C\cdot\int_\Omega |\nabla u_{x_i}|^2\,dx \;=\; C\cdot\int_\Omega \sum_j |u_{x_ix_j}|^2\,dx \qquad (40)$$

for some constant C. Similarly,

$$\int_\Omega |u|^2\,dx \;\leq\; C\cdot\int_\Omega \sum_j |u_{x_i}|^2\,dx\,. \qquad (41)$$

Together, (39)–(41) yield (38), whenever $u\in\mathcal{C}_c^\infty(\Omega)$. By an approximation argument, the same estimate is valid for every $u\in H_0^2(\Omega)$.

Since $B[\cdot,\cdot]$ is strictly positive definite, the Lax-Milgram theorem yields the existence of a unique $u\in H_0^2(\Omega)$ such that

$$B[u,v] \;=\; (f,v)_{\mathbf{L}^2} \qquad \text{for every } \; v\in H_0^2(\Omega)\,.$$

By definition, u is the unique weak solution of the boundary value problem

$$\begin{cases} \Delta^2 u = f & x\in\Omega\,,\\[2mm] u = \dfrac{\partial u}{\partial\nu} = 0 & x\in\partial\Omega\,, \end{cases}$$

Chapter 10

1. Since K is compact, it can be covered by finitely many of the sets A_i, say $K \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$. Consider the functions $\rho_i(x) = d(x, A_i^c)$, where $A_i^c = K \setminus A_i$ denotes the complement of the set A_i. These functions are all Lipschitz continuous. Namely $|\rho_i(x) - \rho_i(y)| \leq d(x, y)$. Hence the function $r(x) = \max_i \rho_i(x)$ is also Lipschitz continuous.

We now observe that, if $x \in A_i$ then $\rho(x) \geq \rho_i(x) > 0$. Hence the minimum of the continuous function $r(\cdot)$ on the compact set K is strictly positive. By construction, it is now clear that the constant $\rho \doteq \min_{x \in K} r(x) > 0$ satisfies the requirement.

2. (i) If $x_n \to \bar{x}$ then every subsequence converges to $\bar{x}$ as well. To prove the converse implication, assume that the sequence does not converge to $\bar{x}$. Hence there exists $\varepsilon > 0$ such that for every N one can find $n > N$ such that $d(x_n, \bar{x}) > \varepsilon$. If this holds, then we can construct a subsequence $(x_{n_k})_{k \geq 1}$ such that $d(x_{n_k}, \bar{x}) > \varepsilon$ for every k. Clearly, from this subsequence one cannot extract any further subsequence converging to $\bar{x}$.

(ii) If a convergent subsequence $(x_{n_k})_{k \geq 1}$ exists, then it would be Cauchy. In particular, $\lim_{j,k \to \infty} d(x_{n_j}, x_{n_k}) = 0$. but this is impossible if $d(x_{n_j}, x_{n_k}) \geq \delta$ for $j \neq k$.

(iii) The assumption implies that, given $\varepsilon > 0$, from any subsequence $(x_n)_{n \in I}$ one can extract a further subsequence $(x_n)_{n \in I'}$ such that

$$d(x_m, x_n) \;<\; 2\varepsilon \qquad \text{for all} \;\; m, n \in I'.$$

Here $I' \subset I \subset \mathbb{N}$ are infinite sets of natural numbers.

We argue by induction on $k = 1, 2, \ldots$ Let $I_1 \subset \mathbb{N}$ be a infinite set of indices such that

$$d(x_m, x_n) \;<\; 2^{-1} \qquad \text{for all} \;\; m, n \in I_1 .$$

After $I_{k-1} \subset \mathbb{N}$ has been constructed, we choose an infinite set $I_k \subset I_{k-1}$ such that

$$d(x_m, x_n) \;<\; 2^{-k} \qquad \text{for all} \;\; m, n \in I_k .$$

After all the sets I_k have been constructed, we choose a sequence $n_1 < n_2 < n_3 < \cdots$ such that $n_k \in I_k$ for every k. We claim that the sequence $(x_{n_k})_{k \geq 1}$ is Cauchy. Indeed, if $m < n$, then $d(x_m, x_n) \;\leq\; 2^{-m}$. Since the space E is complete, this subsequence has a limit.

3. This is an elementary integral:

$$\|f\|_{\mathbf{L}^1} \;=\; \int_0^{1/2} \frac{1}{x(\ln x)^2}\, dx \;=\; \int_{-\infty}^{-\ln 2} \frac{1}{y^2}\, dy \;=\; \frac{1}{\ln 2}\, .$$

Next, observe that the mollifier J_ε is supported on the interval $-\varepsilon, \varepsilon]$ and strictly positive on the subinterval $[-\varepsilon/2,\, \varepsilon/2]$. Namely, it satisfies

$$J_\varepsilon(x) \;\geq\; \phi_\varepsilon(x) \;\doteq\; \begin{cases} \dfrac{\delta}{\varepsilon} & \text{if } \; 0 < x < \dfrac{\varepsilon}{2}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

for some constant $\delta > 0$. For $0 < x < 1/6$, choosing $\varepsilon = 2x$ we obtain

$$F(x) \ \geq\ (J_{2x}*f)(x) \ \geq\ (\phi_{2x}*f)(x) \ =\ \int_0^{2x} \frac{\delta}{2x}\frac{1}{s(\ln s)^2}\,ds \ =\ \frac{\delta}{2x}\int_{-\infty}^{\ln 2x}\frac{1}{y^2}\,dy \ =\ \frac{\delta}{2x|\ln(2x)|}\,.$$

Hence $F \notin \mathbf{L}^1(\mathbb{R})$.

4. All functions f_n are uniformly bounded, because

$$|f_n(x)| \ \leq\ |f_n(0)| + \|g\|_{\mathbf{L}^1} \qquad\qquad \text{for all}\ \ n \geq 1,\ x \in \mathbb{R}$$

and by assumption the values $f_n(0)$ range in a bounded set.

Using the previous Problem 2 (iii), given $\varepsilon > 0$ it suffices to prove that from any subsequence one can extract a further subsequence $(f_{n_k})_{k\geq 1}$ such that

$$\limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}(\mathbb{R})} \ \leq\ \varepsilon\,. \tag{42}$$

Toward this goal, choose a constant M large enough so that

$$\int_{-\infty}^{-M} g(x)\,dx + \int_M^{\infty} g(x)\,dx \ <\ \frac{\varepsilon}{2}\,.$$

We claim that, on the compact interval $[-M, M]$, the functions f_n are equicontinuous. Indeed, Fix $x \in [-M, M]$ and $\epsilon > 0$. Choose $\delta > 0$ such that $\int_{x-\delta}^{x+\delta} g(s)\,ds < \epsilon$. Then $|f_n(y) - f_n(x)| < \epsilon$ for all $y \in [x-\delta\,,\ x+\delta]$ and $n \geq 1$.

Given any subsequence, we can thus apply Ascoli's theorem and extract a further subsequence $(f_{n_k})_{k\geq 1}$ which converges uniformly on the interval $[-M, M]$. We now estimate (42) by writing

$$\begin{aligned}\limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}(\mathbb{R})} \ &=\ \max\Big\{\limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}(]-\infty,-M])} \\ &\quad + \limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}([-M,M])} + \limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}([-M,M])}\end{aligned} \tag{43}$$

We now have

$$\limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}([-M,M])} \ =\ 0\,.$$

Moreover

$$\limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}(]-\infty,-M])} \ \leq\ \limsup_{j,k\to\infty}|f_{n_j}(-M) - f_{n_k}(-M)| + 2\int_{-\infty}^{-M} g(x)\,dx \ \leq\ 0 + \varepsilon\,,$$

$$\limsup_{j,k\to\infty}\|f_{n_j} - f_{n_k}\|_{\mathcal{C}([M,\infty[)} \ \leq\ \limsup_{j,k\to\infty}|f_{n_j}(M) - f_{n_k}(M)| + 2\int_M^{\infty} g(x)\,dx \ \leq\ 0 + \varepsilon\,.$$

Hence (42) holds.

5. Set $g_n(x) \doteq \arctan f_n(x)$ and consider the functions

$$a(x) \ \doteq\ \liminf_{n\to\infty} g_n(x) \ \leq\ \limsup_{n\to\infty} g_n(x) \ \doteq\ b(x)\,.$$

The functions a, b are Lebesgue measurable, and so is the set $A \doteq \{x \in \mathbb{R}\,; \quad -\pi < a(x) = b(x) < \pi\}$. By replacing each function f_n with $\tilde{f}_n \doteq \chi_A \cdot f_n$, it is not restrictive to assume that the sequence $(f_n)_{n\geq 1}$ converges pointwise at every point $x \in \mathbb{R}$. By Fatou's lemma,

$$\|f\|_{\mathbf{L}^1(\mathbb{R})} \;=\; \int \Big(\lim_{n\to\infty} |f_n(x)|\Big)\,dx \;\leq\; \liminf_{n\to\infty} \int |f_n(x)|\,dx \;\leq\; C\,.$$

6. Let $f : [a,b] \mapsto \mathbb{R}$ be absolutely continuous, and let $A \subset [a,b]$ be a set with measure zero. Fix any $\varepsilon > 0$. We need to show that $meas(f(A)) \leq \varepsilon$.

Toward this goal, let $\delta > 0$ be such that $\sum_{i=1}^N |f(b_i) - f(a_i)| < \varepsilon$ for any finite family of disjoint intervals $[a_i, b_i] \subset [a,b]$, $i = 1, \ldots, N$, with total length $\sum_{i=1}^n (b_i - a_i) < \delta$. Since $meas(A) = 0$, there exists an open set $V \supseteq A$ such that $meas(V) < \delta$. Since $V \subset \mathbb{R}$ is open, we can write V as a countable union of its connected components, which are disjoint open intervals: $V \;=\; \bigcup_{k\geq 1}]c_k, d_k[\,$. We now have

$$meas(f(A)) \;\leq\; meas(f(V)) \;\leq\; \sup_n \sum_{k=1}^n meas\Big(f([c_k, d_k])\Big) \;=\; \sup_n \sum_{k=1}^n \Big|f(M_k) - f(m_k)\Big|. \quad (44)$$

Here $m_k, M_k \in [c_k, d_k]$ are points where the continuous function f attains respectively its minimum and maximum, restricted to the interval $[c_k, d_k]$. Observing that

$$\sum_{k=1}^n |m_k - M_k| \;\leq\; \sum_{k=1}^n (d_k - c_k) \;<\; \delta \qquad\qquad \text{for all} \quad n \geq 1\,,$$

we conclude that the right hand side of (44) is $\leq \varepsilon$.

7. (i) Choose a subsequence such that $\|f_{n_k}\|_{\mathbf{L}^1} \;\leq\; 2^{-k}$ for every $k \geq 1$. Then for any $N \geq 1$ we have

$$\int_0^1 \Big(\sup_{n>N} |f_n(x)|\Big)\,dx \;\leq\; \int_0^1 \sum_{n>N} |f_n(x)|\,dx \;\leq\; 2^{-N}.$$

Calling

$$A_\varepsilon \;\doteq\; \Big\{x \in [0,1]\,; \quad \limsup_{n\to\infty} |f_n(x)| \geq \varepsilon\Big\},$$

for every integer N we have

$$meas(A_\varepsilon) \;\leq\; meas\Big(\{x \in [0,1]\,; \quad \sup_{n>N} |f_n(x)| \geq \varepsilon\}\Big) \;\leq\; \int_0^1 \Big(\sup_{n>N} |f_n(x)|\Big)\,dx \;\leq\; \frac{2^{-N}}{\varepsilon}\,.$$

Since N is arbitrary, this yields $meas(A_\varepsilon) = 0$. The above shows that $\limsup_{n\to\infty} |f_n(x)| = 0$ for a.e. $x \in [0,1]$. Hence $\lim_{n\to\infty} f_n(x) = 0$ for a.e. x.

(ii) For $2^k < n \leq 2^{k+1}$, define

$$f_n(x) \;=\; \begin{cases} 1 & \text{if} \quad x \in \left[\dfrac{n - 2^k - 1}{2^k}\,,\; \dfrac{n - 2^k}{2^k}\right], \\[2ex] 0 & \text{otherwise}\,. \end{cases}$$

8. Let $B(x_j, r_j)$, $j \geq 1$ be a countable family of disjoint open balls contained in Ω. For each j, let g_j be the characteristic function of the set $B(x_j, r_j)$. Clearly, all these functions lie in $\mathbf{L}^p(\Omega)$ and are linearly independent, hence the space $\mathbf{L}^p(\Omega)$ is infinite dimensional.

Moreover, $\|g_j\|_{\mathbf{L}^\infty} = 1$, $\|g_i - g_j\|_{\mathbf{L}^\infty} = 1$ for $i \neq j$.

To cover the case $1 \leq p < \infty$, define

$$f_j(x) \doteq c_j g_j(x), \qquad c_j = \left(\frac{1}{meas(B(x_j, r_j))}\right)^{1/p}.$$

Then

$$\|f_j\|_{\mathbf{L}^p(\Omega)} = 1, \qquad \|f_i - f_j\|_{\mathbf{L}^p(\Omega)} = 2^{1/p} \quad \text{for all} \quad i \neq j.$$

9. It is clear that the set S is totally ordered: if $(t_1, f(t_1))$ and $(t_2, f(t_2))$ lie in S, with $t_1 \leq t_2$, then $(t_1, f(t_1)) \preceq (t_2, f(t_2))$. To prove that S is maximal, assume $(x, y) \notin S$. To fix the ideas, assume $f(x) < y$, the other case being similar. Then there exists $t > x$ such that $f(t) < y$. The set $S \cup \{(x, y)\}$ is not totally ordered, because the two relations

$$(x, y) \preceq (t, f(t)), \qquad (t, f(t)) \preceq (x, y)$$

are both false.

A maximal totally ordered subset of different kind is

$$S \doteq \{(-1, y); \quad y \leq 0\} \cup \{(x, 0); \quad x \in [-1, 1]\} \cup \{(1, y); \quad y \geq 0\}.$$

10. The proof is by induction. When $m = 2$, this is the standard Hölder's inequality. Assume that the result is true for $m = 2, 3, \ldots, N-1$. Let $f_k \in \mathbf{L}^{p_k}(\Omega)$, with $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_N} = 1$.

If $p_N = \infty$, then by the inductive hypothesis we immediately get

$$\int_\Omega |f_1 \cdots f_{N-1} \cdot f_N| \, dx \leq \int_\Omega |f_1 \cdots f_{N-1}| \, dx \cdot \|f_N\|_{\mathbf{L}^\infty} \leq \prod_{k=1}^N \|f_k\|_{\mathbf{L}^{p_k}}.$$

If $1 \leq p_N < \infty$, choose q so that

$$\frac{1}{q} \doteq 1 - \frac{1}{p_N} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_{N-1}}.$$

The standard Hölder inequality yields

$$\int_\Omega |f_1 \cdots f_{N-1} \cdot f_N| \, dx \leq \|f_1 \cdots f_{N-1}\|_{\mathbf{L}^q} \|f_N\|_{\mathbf{L}^{p_N}}. \tag{45}$$

We now observe that $\frac{q}{p_1} + \cdots + \frac{q}{p_N} = 1$, while $|f_k|^q \in \mathbf{L}^{p_k/q}(\Omega)$ for $k = 1, \ldots, N-1$. The inductive assumption yields

$$\|f_1 \cdots f_{N-1}\|_{\mathbf{L}^q} = \left(\int_\Omega |f_1|^q \cdots |f_{N-1}|^q \, dx\right)^{1/q} \leq \left(\prod_{k=1}^{N-1} \|f_k^q\|_{\mathbf{L}^{p_k/q}}\right)^{1/q} = \prod_{k=1}^{N-1} \|f_k\|_{\mathbf{L}^{p_k}}. \tag{46}$$

Together, (45)-(46) yield the result for $m = N$. By induction, the generalized Hölder inequality is valid for every $m \geq 2$.