Errata

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Errata
Changes appear in yellow. Line $k+$ (resp., line $k-$) denotes the $k$th line from the top (resp., the bottom) of a page. My thanks go to the following individuals who have contributed to this list: Tobias Wöhrer, Simon Becker, Dennis Cutraro, Mateusz Piorkowski, Laura Kanzler, Mateus Sampaio, Laura Shou, Noema Nicolussi, Andreas Geyer-Schulz, Rene Allerstorfer, Manuel Culqui Rodriguez, Fritz Gesztesy, Marcel Griesemer, Michael Hofacker, Maxim Zinchenko, Jannik Pitt.

Page 16. First line: for $a \in \ell^p(N)$, $b \in \ell^q(N)$.

Page 25. Proof of Theorem 0.25: and we can choose $m_2 = \sqrt{\sum_j \|u_j\|_1^2}$.

Page 34. Proof of Lemma 0.36: (if $K_2(x,.) f(.) \not\in L^p(Y, d\nu)$, the inequality is trivially true).

Page 36. Add the following at the end of Lemma 0.39: Moreover, if $u$ and $f$ both have compact support, then $f_k \in C_c^\infty(\mathbb{R}^n)$.

Page 36. Proof of Lemma 0.41: ... $\varphi_n \in C_c^\infty(\mathbb{R}^n)$ with support inside some open ball $X$ which converges ... continuous functions $\varphi_n$ with support in $X$ which converges to $g$ ...
Proof of Lemma 1.11: (ii) follows from $\langle \varphi, A^{**} \psi \rangle = \langle A^{*} \varphi, \psi \rangle = \langle \varphi, A \psi \rangle$.

Last sentence in the proof of Theorem 1.16: Since $f - \varepsilon < f_{z_i}$ for all $z_i$ we have $f - \varepsilon < f_{z}$ and we have found a required function.

Problem 1.23: Show that the span of $\{(t - z)^{-1}|z \in U\}$ is dense in $C_{\text{loc}}(\mathbb{R})$.

Line after (2.15): measurable function $A : \mathbb{R}^d \to \mathbb{C}$.

Clearly we have $\alpha A = \alpha A$ for $\alpha \in \mathbb{C} \setminus \{0\}$ and $A + B = A + B$ provided $A$ is closable and $B$ is bounded (Problem 2.8).

Problem 2.8: Suppose that if $A$ is closable and $B$ is bounded. Show that $\alpha A = \alpha A$ for $\alpha \in \mathbb{C} \setminus \{0\}$ and $A + B = A + B$.

Proof of Lemma 2.7: $\| (A - z) \psi \|^2 = \| (A - x) \psi - iy \psi \|^2$

(2.46)

Problem 2.8: Suppose that if $A$ is closable and $B$ is bounded. Show that $\alpha A = \alpha A$ for $\alpha \in \mathbb{C} \setminus \{0\}$ and $A + B = A + B$.

Proof of Lemma 2.11: $\mathcal{D}(\tilde{A}) = \{ \psi \in \mathcal{H}_A | \exists \tilde{\psi} \in \mathcal{H} : \langle \varphi, \psi \rangle_A = \langle \varphi, \tilde{\psi} \rangle, \forall \varphi \in \mathcal{D}(A) \} = \mathcal{H}_A \cap \mathcal{D}(A^*)$

as $\mathcal{D}(A) \subset \mathcal{H}_A$ is dense and $\langle \varphi, \psi \rangle_A = \langle (A + 1) \varphi, \psi \rangle$ for $\varphi \in \mathcal{D}(A), \psi \in \mathcal{H}_A$.

Proof of Lemma 2.15:

$2 |\text{Re} \langle \varphi, A \psi \rangle| \leq \frac{1}{2} \left| q(\psi + \varphi) - q(\psi - \varphi) \right| \leq \frac{\|q\|}{2} \left( \|\psi + \varphi\|^2 + \|\psi - \varphi\|^2 \right)$

$= \|q\| \left( \|\psi\|^2 + \|\varphi\|^2 \right)$

Proof of Theorem 2.19:

$f'(\lambda) = -\|(A - E + \lambda)^{-1} \varphi\|^2 \leq -f(\lambda)^2$

Problem 2.18: Then so does $A + B$ if $\|B\| < \|A^{-1}\|^{-1}$.

Paragraph after Lemma 2.28: A conjugate linear map $C : \mathcal{H} \to \mathcal{H}$ is called a conjugation if it satisfies $C^2 = \mathbb{I}$ and $\langle C \psi, C \varphi \rangle = \langle \varphi, \psi \rangle$.

Problem 4.11: $\chi_{\Omega}(A) = \frac{1}{2\pi i} \int_{\Gamma} R_A(z) \, dz$.

(4.31) $\|A\| = \langle \psi, A^2 \psi \rangle = \langle \psi, A^* A \psi \rangle = \|A \psi\|^2$, $\psi \in \mathcal{D}(\|A\|) = \mathcal{D}(A)$,
(4.34) \[ U^*U = P_{\text{Ker}(A)} \mathbb{B}, \quad \text{and} \quad \text{UU}^* = P_{\text{Ker}(A^*)} \mathbb{B}, \]

Page 139: Last line of Theorem 4.10: \( \text{Ker}(U) \mathbb{B} = \text{Ker}(A) \mathbb{B} \)

Page 141: (ii) We have

(4.40) \[ \inf_{\psi \in U(\varphi_1, \ldots, \varphi_{n-1})} \langle \psi, A\psi \rangle \geq E_n, \]

since \( A \) restricted to \( \text{span}\{\varphi_1, \ldots, \varphi_{n-1}\} \) is bounded from below by \( E_n \) (which is immediate from the spectral theorem).

Page 141: Corollary 4.13: Suppose \( A \) and \( B \) are self-adjoint operators with \( \mathcal{D}(A) = \mathcal{D}(B) \) and \( A \geq B \) (i.e., \( A - B \geq 0 \)).

Page 146: Proof of Theorem 5.7: Since \( K(A - i)^{-1} \) is compact by assumption,

Page 154: Proof of Theorem 5.9: We will assume that \( K \) is compact.

Page 155: Problem 5.7:

(5.27) \[ \mathcal{S}_{rc} = \{ \psi \in \mathcal{S} | \lim_{t \to \infty} \langle \psi, e^{-itA} \psi \rangle = 0 \} \supseteq \mathcal{S}_{ac}, \]

Page 159. Theorem 6.4:

(6.4) \[ \gamma - \max \left( \frac{b}{1-a}, \frac{b}{1-a} \right). \]
Page 159. Proof of Theorem 6.4; last sentence: The explicit bound (6.4) follows since this condition implies $\|BR_A(-\lambda)\| < 1$ by virtue of (6.2) from the proof of the previous lemma.

Page 161. Lemma 6.8:

(6.9) \[ s_n(K) = \min_{\psi_1, \ldots, \psi_{n-1}} \sup_{\psi \in U(\psi_1, \ldots, \psi_{n-1})} \|K\psi\|, \]

Page 162. Proof of Lemma 6.9: last formula

\[ \gamma_n = \|K - K_n\| = \sup_{\|\psi\|=1} \|K(\psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle \varphi_j)\| \]

Page 164. Proof of Lemma 6.10: Conversely, choose $\varphi_i = \hat{\varphi}_i$.

Page 176. Theorem 6.25: add for $\lambda > \frac{b}{a} - \gamma$, after (6.44)

Page 176. Line before equation (6.45): Furthermore, we can define $C_q(\lambda)$ for all $\frac{b}{a} z \in \rho(A)$, using

Page 179. Problem 6.18: Suppose $A$ is self-adjoint, $\lambda \in \mathbb{R}$, and $R$ is bounded. Show that $R = R_A(\lambda)$ if and only if $\langle (A - \lambda)\varphi, R\psi \rangle = \langle \varphi, \psi \rangle$ for all $\varphi \in \mathcal{D}(A)$, $\psi \in \mathcal{B}$.

Page 180. Corollary 6.32: Then this holds for all $z$ in the interior of $\Gamma$.

Page 195. Line 2+: Clearly $H^{\frac{1+4}{2}}(\mathbb{R}^n) \subset H^{\frac{1}{2}}(\mathbb{R}^n)$

Page 200. Discussion after Lemma 7.20: $|\psi(x, t)|^2 d^n x = |\psi(\frac{x}{2t})|^2 d^n x$

Page 209. Last line of the proof of Theorem 8.2: $0 = (\frac{b}{a} z^*)\|A\psi\|^2$

Page 209.

(8.13) \[ \psi(x) = \left(\frac{\lambda}{\pi}\right)^{n/4} e^{-\frac{1}{2}|x-x_0|^2} e^{\lambda x_0 x}, \]

Page 211. First equation in the proof of Lemma 8.3:

\[ \frac{1}{\sqrt{2\pi}} \int \hat{\varphi}(x) e^{-\frac{x^2}{2}} \sum_{j=0}^k \frac{(it)^j}{j!} dx = 0 \]

Page 212. Theorem 8.4: There exists an orthonormal basis of simultaneous eigenvectors for the operators $L^2$ and $L^*_A$.

Page 222. Proof of Lemma 9.5: Choosing $f_1 = v$, $f_2 = f$, $f_3 = v^*$, $f_4 = f^*$, we infer (9.15).
Page 222. Problem 9.1: and \( f(d) = \gamma \), \( pf'(d) = \delta \).

Page 222. Problem 9.3: Let \( \phi \in L^1_{loc}(I) \) be real-valued.

Page 222. Problem 9.4: Add the assumption that \( a \) is regular. Otherwise one can also start the integration at an arbitrary point in \((a,b)\).

Page 223. Replace the last sentence by: Moreover, the following set is a core for \( A \)

\[(9.21) \quad D_1 = \{ f \in D(\tau) \mid \exists x_0 \in I : \forall x \in (a,x_0), V_x(f) = 0, \exists x_1 \in I : \forall x \in (x_1,b), W_x(f) = 0 \}, \]

where we set \( V_x(f) = W_x(v,f) \), \( W_x(f) = W_x(w,f) \) if \( \tau \) is l.c. at \( a \) and \( b \) and \( V_x(f) = f(x) \), \( W_x(f) = f(x) \) if \( \tau \) is l.p. at \( a \), respectively.

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\[(9.23) \quad W_a(v,f) = 0 \iff \cos(\alpha)BC^2_a(f) + \sin(\alpha)BC^3_a(f) = 0, \]

where \( \tan(\alpha) = \frac{BC^2_a(v)}{BC^3_a(v)} \).

Page 228. Theorem 9.10: Delete ”(which are simple)” . And the following claim about simplicity of eigenvalues only applies to separated boundary conditions as in Theorem 9.6.

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\[(9.37) \quad (Uf)(\lambda) = \frac{1}{\sqrt{2\pi}} \left( \int_R e^{i\sqrt{\lambda}x} f(x) \, dx \right), \quad \lambda \in \sigma(H_0) = [0, \infty). \]

Page 233. Proof of Lemma 9.13:

\[ \sum \int_R F_j(\lambda)^* \int_a^b u_j(\lambda,x)g(x)r(x) \, dx \, d\mu_j(\lambda) = \int_a^b (U^{-1}F)(x)^*g(x)r(x) \, dx. \]

Interchanging integrals on the left-hand side

Page 233. Delete the last sentence: Note that since we can replace \( u_j(\lambda,x) \) by \( \gamma_j(\lambda)u_j(\lambda,x) \) where \( |\gamma_j(\lambda)| = 1 \), it is no restriction to assume that \( u_j(\lambda,x) \) is real valued.

Page 250. Second line in Section 9.7: on \((a,b) = \mathbb{R}\).

Page 252. Proof of Lemma 9.35: where \( M_n = \sup_{|m| \geq n} \int_m^{m+1} |q(x)| \, dx \).

Page 255. First line: the zeros of \( \psi_n \) interlace the zeros of \( \psi_{n+1} \).

Page 256. Problem 9.18: Change the hint according to:

(Hint: Let \( \varphi_\varepsilon(x) = \exp(-\varepsilon^2 x^2) \) and investigate \( \langle \varphi_\varepsilon, H\varphi_\varepsilon \rangle \).)
Page 261.
(10.23) \[ A \Phi = \tau \Phi, \quad \mathcal{D}(A) = \{ \Phi \in L^2(0, 2\pi) \mid \Phi \in AC^1(0, 2\pi), \Phi'' \in L^2(0, 2\pi), \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi) \}. \]

Page 268. Line 3+: Note that the \( L^k_j(r) \) are polynomials of degree \( j \) which

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(A.55) \[ F(z) = \int_{\mathbb{X}} f(z, y) d\mu(y) \]

Page 330. Proof of Lemma A.35:

\[ \mu(x-) \leq \liminf \mu_n(x) \leq \limsup \mu_n(x) \leq \mu(x+) \]

Page 330. Problem A.32 can be deleted as the claim is part of Lemma A.36.

Page 333. Problem A.34. This claim is clearly wrong (take a function which is constant on an interval). It should be deleted.

Addendum

Page 81. Proof of Theorem 2.14: Since the rest is not so straightforward, here is a complete proof:

Proof. Since \( \mathcal{H}_q \) is dense, \( \hat{\psi} \) and hence \( A \) is a well-defined operator. Moreover, replacing \( q \) by \( q(\cdot) - \gamma \| \cdot \|^2 \) and \( A \) by \( A - \gamma \), it is no restriction to assume \( \gamma = 0 \). Next it will be convenient to look at the definition from a somewhat more abstract point of view: We have a conjugate linear continuous embedding \( j : \mathcal{H} \to \mathcal{H}_q^*, \psi \mapsto \langle \psi, \cdot \rangle \) (here \( \mathcal{H}_q \) is equipped with \( \| \cdot \|_q \)) with Ran(\( j \)) dense. Indeed, if Ran(\( j \)) were not dense, there would be some nonzero \( \varphi \in \mathcal{H}_q^* \cong \mathcal{H}_q \) (the identification given by the Riesz lemma via evaluation) such that \( \varphi(j(\psi)) = \langle j(\psi), \varphi \rangle = 0 \) for all \( \psi \in \mathcal{H} \) implying the contradiction \( \varphi = 0 \).

Next, there is a conjugate linear isometric isomorphism \( \hat{A} : \mathcal{H}_q \to \mathcal{H}_q^* \), \( \psi \mapsto s(\psi, \cdot) + \langle \psi, \cdot \rangle \) (Riesz lemma) and our operator \( A \) is given by \( j^{-1} \hat{A} - \mathbb{1} \). Moreover, \( \mathcal{D}(A) = \hat{A}^{-1} \text{Ran}(j) \) is dense in \( \mathcal{H}_q \) and hence also in \( \mathcal{H} \). By construction, \( q_A(\psi) = q(\psi) \) for \( \psi \in \mathcal{D}(A) \), which shows that \( A \) is nonnegative and as in the proof of Lemma 2.11, it follows that Ran(\( A + 1 \)) = \( \mathcal{H} \). Thus \( A \) is self-adjoint. Finally, note that the fact that \( \mathcal{D}(A) \) is dense in \( \mathcal{H}_q \) implies \( \mathcal{H}_A = \mathcal{H}_q \).

Concerning uniqueness let \( \hat{A} \) be another self-adjoint operator with the same properties. Then equality of the associated quadratic forms (and hence of the sesquilinear forms) on \( \Omega \) implies \( \langle A\psi, \varphi \rangle = \langle \psi, A\varphi \rangle \) for \( \psi \in \mathcal{D}(A), \varphi \in \mathcal{D}(\hat{A}) \). But this shows \( \psi \in \mathcal{D}(\hat{A}^*) = \mathcal{D}(\hat{A}) \) and \( A\psi = \hat{A}^*\psi = A\psi \) and vice versa. \( \square \)
Page 118. Here is an amplification of Theorem 3.16:

**Theorem 3.16.** For every self-adjoint operator $A$ there is an ordered spectral basis $\{\psi_j\}_{j=1}^{N}$. Moreover, it can be chosen such that $d\mu_\psi = \chi_{\Omega_j} \, d\mu$, where $\mu$ is a maximal spectral measure and $\Omega_{j+1} \subseteq \Omega_j$. The dimension $N$ is the spectral multiplicity of $A$.

**Proof.** First of all observe that for every $\varphi$ there is a maximal spectral vector $\psi$ such that $\varphi \in \mathcal{H}_\psi$. To see this start with a maximal spectral vector $\tilde{\psi}$. Then $d\mu_\varphi = f d\mu_{\tilde{\psi}}$ and we set $\Omega = \{\lambda|f(\lambda) > 0\}$. Then $P_\Lambda(\Omega)\varphi = \varphi$ since $||P_\Lambda(\Omega)\varphi||^2 = \int_\Omega \mu_\varphi = \int_\Omega f d\mu_{\tilde{\psi}} = ||\varphi||^2$. Now set $\psi = \varphi + P(\mathbb{R}\setminus\Omega) \tilde{\psi}$ and observe $d\mu_\psi = d\mu_\varphi + \chi_{\mathbb{R}\setminus\Omega} d\mu_{\tilde{\psi}} = (f + \chi_{\mathbb{R}\setminus\Omega}) d\mu_{\tilde{\psi}}$. Since $f + \chi_{\mathbb{R}\setminus\Omega} > 0$ we see that $d\mu_\psi$ is absolutely continuous with respect to $d\mu_\tilde{\psi}$ and hence $\psi$ is a maximal spectral vector with $\varphi = P_\Lambda(\Omega) \psi \in \mathcal{H}_\psi$ as required.

Now start with some total set $\{\tilde{\psi}_j\}$ and proceed as in Lemma 3.4 to obtain an ordered spectral basis $\{\psi_j\}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to $\mu_{\tilde{\psi}_j}$ all spectral measures are absolutely continuous with respect to $\mu = \mu_{\psi_1}$, that is, $d\mu_{\psi_j} = f_j d\mu$. Choosing $\Omega_j = \{\lambda|f_j(\lambda) > 0\}$ we can replace $\psi_j \to \chi_{\Omega_j}(A) f_j(A)^{-1/2} \psi_j$ such that $f_j \to \chi_{\Omega_j}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to $\mu_{\psi_j}$ we can even assume $\Omega_{j+1} \subseteq \Omega_j$.

Finally, we show that the spectral multiplicity of $A$ is $N$. By the first part we can assume that $A$ is multiplication by $\lambda$ in $\bigoplus_{j=1}^{N} L^2(\mathbb{R}, \chi_{\Omega_j} \, d\mu)$. Let $\{\psi_j\}_{j=1}^{n}$ be a spectral basis with $n < N$. We will show that there is some vector in the orthogonal complement of $\bigoplus_{j} \mathcal{H}_{\psi_j}$. Of course such a vector exists pointwise for every $\lambda$ but it is not clear that the components can be chosen measurable. To see this we use a Gauss-type elimination: For this note that we can multiply every vector $\psi_j$ with a non-vanishing function or add multiples of the other vectors to a given one without changing $\bigoplus_{j} \mathcal{H}_{\psi_j}$. Hence we can first normalize the first component of every $\psi_j$ to be a characteristic function. Moreover, by adding all other vectors to $\psi_1$ we can assume that its first component is positive on a maximal set $\tilde{\Omega}_1$. In fact, after another normalization we can assume that $\psi_{1,1} = \chi_{\tilde{\Omega}_1}$ and after subtracting multiples of $\psi_1$ from the remaining vectors we can assume $\psi_{j,1} = 0$ for $j \geq 2$. If $\mu_1(\mathbb{R}\setminus\Omega_1) > 0$ then $\varphi = (\chi_{\mathbb{R}\setminus\Omega_1}, 0, \ldots)$ would be in the orthogonal complement and we are done. So assume $\chi_{\Omega_1} = 1$ and continue with the other components until they satisfy $\psi_{j,k} = \delta_{j,k}$ for $1 \leq j, k \leq n$. Then $\varphi = (-\psi_{1,n+1}, \ldots, -\psi_{n,n+1}, 1, 0, \ldots)$ is in the orthogonal complement contradicting our assumption that $\{\psi_j\}_{j=1}^{n}$ is a spectral basis. \qed