Errata

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Errata

Changes appear in yellow. Line $k+$ (resp., line $k-$) denotes the kth line from the top (resp., the bottom) of a page. My thanks go to the following individuals who have contributed to this list: Tobias Wöhrer, Simon Becker, Dennis Cutraro, Mateusz Piorkowski, Laura Kanzler, Mateus Sampaio, Laura Shou, Noema Nicolussi, Andreas Geyer-Schulz, Rene Allerstorfer.

Page 16. First line: for $a \in \ell^p(\mathbb{N})$, $b \in \ell^q(\mathbb{N})$.

Page 25. Proof of Theorem 0.25: and we can choose $m_2 = \sqrt{\sum_j \|u_j\|_1^2}$.

Page 36. Add the following at the end of Lemma 0.39: Moreover, if u and f both have compact support, then $f_k \in C_c^\infty(\mathbb{R}^n)$.

Page 36. Proof of Lemma 0.41: ... $\varphi_n \in C_c^\infty(\mathbb{R}^n)$ with support inside some open ball X which converges ... continuous functions φ_n with support in X which converges to g ...

Page 54. Proof of Lemma 1.11: (ii) follows from $\langle \varphi, A^{**}\psi\rangle = \langle A^*\varphi, \psi\rangle = \langle \varphi, A\psi\rangle$.

Page 60. Last sentence in the proof of Theorem 1.16: Since $f - \varepsilon < f_{z_l}$ for all z_l we have $f - \varepsilon < f_\varepsilon$ and we have found a required function.

Page 61. Problem 1.23: Show that the span of $\{(t-z)^{-1} | z \in U\}$ is dense in $C_\infty(\mathbb{R})$.

Page 66. Line after (2.15): measurable function $A : \mathbb{R}^d \to \mathbb{C}$.

Page 72. Line 18+: Clearly we have $\overline{\alpha A} = \alpha \overline{A}$ for $\alpha \in \mathbb{C}\backslash\{0\}$ and $\overline{A+B} = \overline{A} + B$ provided A is closable and B is bounded (Problem 2.8).

Page 75. Proof of Lemma 2.7:

$$\begin{aligned} \|(A-z)\psi\|^2 &= \|(A-x)\psi - \mathrm{i}y\psi\|^2 \\ &= \|(A-x)\psi\|^2 + y^2\|\psi\|^2 \geq y^2\|\psi\|^2, \end{aligned} \tag{2.46}$$

Page 76. Problem 2.8: Suppose that if A is closable and B is bounded. Show that $\overline{\alpha A} = \alpha \overline{A}$ for $\alpha \in \mathbb{C}\backslash\{0\}$ and $\overline{A+B} = \overline{A} + B$.

Page 78. Proof of Lemma 2.11:

$$\mathfrak{D}(\tilde{A}) = \{\psi \in \mathfrak{H}_A | \exists \tilde{\psi} \in \mathfrak{H} : \langle \varphi, \psi \rangle_A = \langle \varphi, \tilde{\psi} \rangle, \forall \varphi \in \mathfrak{D}(A)\} = \mathfrak{H}_A \cap \mathfrak{D}(A^*)$$

as $\mathfrak{D}(A) \subset \mathfrak{H}_A$ is dense and $\langle \varphi, \psi \rangle_A = \langle (A+1)\varphi, \psi \rangle$ for $\varphi \in \mathfrak{D}(A)$, $\psi \in \mathfrak{H}_A$.

Page 82. Proof of Lemma 2.15:

$$\begin{aligned} 2|\operatorname{Re}\langle \varphi, A\psi \rangle| &= \frac{1}{2}|q(\psi+\varphi) - q(\psi-\varphi)| \leq \frac{\|q\|}{2}(\|\psi+\varphi\|^2 + \|\psi-\varphi\|^2) \\ &= \|q\|(\|\psi\|^2 + \|\varphi\|^2) \end{aligned}$$

Page 88. Problem 2.18: Then so does $A + B$ if $\|B\| < \|A^{-1}\|^{-1}$.

Page 93. Paragraph after Lemma 2.28: A conjugate linear map $C : \mathfrak{H} \to \mathfrak{H}$ is called a **conjugation** if it satisfies $C^2 = \mathbb{I}$ and $\langle C\psi, C\varphi \rangle = \langle \varphi, \psi \rangle$.

Page 135. Problem 4.11:

$$\chi_\Omega(A) = -\frac{1}{2\pi\mathrm{i}} \int_\Gamma R_A(z) dz,$$

Page 138.

$$\||A|\psi\|^2 = \langle \psi, |A|^2\psi \rangle = \langle \psi, A^*A\psi \rangle = \|A\psi\|^2, \quad \psi \in \mathfrak{D}(|A|) = \mathfrak{D}(A), \tag{4.31}$$

Page 139:

$$U^*U = P_{\operatorname{Ker}(A)^\perp}, \qquad UU^* = P_{\operatorname{Ker}(A^*)^\perp}, \tag{4.34}$$

Page 139: Last line of Theorem 4.10: $\mathrm{Ker}(U)^\perp = \mathrm{Ker}(A)^\perp$

Page 155. Problem 5.9:

$$\mathfrak{H}_{rc} = \{\psi \in \mathfrak{H} | \lim_{t\to\infty} \langle \psi, \mathrm{e}^{-\mathrm{i}tA}\psi\rangle = 0\} \supseteq \mathfrak{H}_{ac}, \tag{5.27}$$

Page 159. Theorem 6.4:

$$\gamma - \max\Big(a|\gamma| + b, \frac{b}{1-a}\Big). \tag{6.4}$$

Page 161. Lemma 6.8:

$$s_n(K) = \min_{\psi_1,\dots,\psi_{n-1}} \sup_{\psi\in U(\psi_1,\dots,\psi_{n-1})} \|K\psi\|, \tag{6.9}$$

Page 162. Proof of Lemma 6.9: last formula

$$\gamma_n = \|K - K_n\| = \sup_{\|\psi\|=1} \|K(\psi - \sum_{j=1}^{n} \langle\varphi_j, \psi\rangle \varphi_j)\|$$

Page 164. Proof of Lemma 6.10: Conversely, choose $\varphi_i = \hat{\phi}_i$

Page 176. Line before equation (6.45): Furthermore, we can define $C_q(\lambda)$ for all $-z \in \rho(A)$, using

Page 195. Line 2+: Clearly $H^{r+1}(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$

Page 200. Discussion after Lemma 7.20: $|\psi(x,t)|^2 d^n x = |\hat{\psi}(\frac{x}{2t})|^2 \frac{d^n x}{(2t)^n}$

Page 209. Last line of the proof of Theorem 8.2: $0 = (z+z^*)\|\hat{A}\psi\|^2$

Page 211. First equation in the proof of Lemma 8.3:

$$\frac{1}{\sqrt{2\pi}} \int \overline{\varphi(x)} \mathrm{e}^{-\frac{x^2}{2}} \sum_{j=0}^{k} \frac{(\mathrm{i}tx)^j}{j!} dx = 0$$

Page 222. Problem 9.1: and $f(d) = \gamma$, $(pf')(d) = \delta$.

Page 222. Problem 9.3: Let $\phi \in L^1_{loc}(I)$ be real-valued.

Page 222. Problem 9.4: Add the assumption that a is regular. Otherwise one can also start the integration at an arbitrary point in (a,b).

Page 223. Replace the last sentence by: Moreover, the following set is a core for A

$$\begin{aligned}\mathfrak{D}_1 = \{f \in \mathfrak{D}(\tau)|\ & \exists x_0 \in I: \forall x \in (a,x_0),\ V_x(f) = 0, \\ & \exists x_1 \in I: \forall x \in (x_1,b),\ W_x(f) = 0\},\end{aligned} \tag{9.21}$$

where we set $V_x(f) = W_x(v, f)$, $W_x(f) = W_x(w, f)$ if τ is l.c. at a, b and $V_x(f) = f(x)$, $W_x(f) = f(x)$ if τ is l.p. at a, b, respectively.

Page 228. Theorem 9.10: Delete "(which are simple)". And the following claim about simplicity of eigenvalues only applies to seperated boundary conditions as in Theorem 9.6.

Page 250. Second line in Section 9.7: on $(a, b) = \mathbb{R}$.

Page 256. Problem 9.18: Change the hint according to:
(Hint: Let $\varphi_\varepsilon(x) = \exp(-\varepsilon^2 x^2)$ and investigate $\langle\varphi_\varepsilon, H\varphi_\varepsilon\rangle$.)

Page 261.

$$A\Phi = \tau\Phi, \quad \mathfrak{D}(A) = \{\Phi \in L^2(0, 2\pi)|\ \Phi \in AC^1[0, 2\pi], \Phi'' \in L^2(0, 2\pi), \\ \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi)\}. \tag{10.23}$$

Page 268. Line 3+: Note that the $L_j^{(k)}(r)$ are polynomials of degree j which

Page 322.

$$F(z) = \int_Y f(z, y)\, d\mu(y) \tag{A.55}$$

Page 330. Proof of Lemma A.35:

$$\mu(x-) \le \liminf \mu_n(x) \le \limsup \mu_n(x) \le \mu(x+)$$

Page 330. Problem A.32 can be deleted as the claim is part of Lemma A.36.

Page 333. Problem A.34. This claim is clearly wrong (take a function which is constant on an interval). It should be deleted.

Addendum

Here is an amplification of Theorem 3.16:

Theorem 3.16. *For every self-adjoint operator A there is an ordered spectral basis $\{\psi_j\}_{j=1}^N$. Moreover, it can be chosen such that $d\mu_{\psi_j} = \chi_{\Omega_j} d\mu$, where μ is a maximal spectral measure and $\Omega_{j+1} \subseteq \Omega_j$. The dimension N is the spectral multiplicity of A.*

Proof. First of all observe that for every φ there is a maximal spectral vector ψ such that $\varphi \in \mathfrak{H}_\psi$. To see this start with a maximal spectral vector $\tilde{\psi}$. Then $d\mu_\varphi = f\, d\mu_{\tilde{\psi}}$ and we set $\Omega = \{\lambda | f(\lambda) > 0\}$. Then $P_A(\Omega)\varphi = \varphi$ since $\|P_A(\Omega)\varphi\|^2 = \int_\Omega d\mu_\varphi = \int_{\mathbb{R}} f\, d\mu_{\tilde{\psi}} = \|\varphi\|^2$. Now set $\psi = \varphi + P(\mathbb{R}\backslash\Omega)\tilde{\psi}$ and observe $d\mu_\psi = d\mu_\varphi + \chi_{\mathbb{R}\backslash\Omega} d\mu_{\tilde{\psi}} = (f + \chi_{\mathbb{R}\backslash\Omega}) d\mu_{\tilde{\psi}}$. Since $f + \chi_{\mathbb{R}\backslash\Omega} > 0$ we see that $d\mu_{\tilde{\psi}}$ is absolutely continuous with respect to $d\mu_\psi$ and hence ψ is a maximal spectral vector with $\varphi = P_A(\Omega)\psi \in \mathfrak{H}_\psi$ as required.

Now start with some total set $\{\tilde{\psi}_j\}$ and proceed as in Lemma 3.4 to obtain an ordered spectral basis $\{\psi_j\}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to μ_{ψ_j} all spectral measures are absolutely continuous with respect to $\mu = \mu_{\psi_1}$, that is, $d\mu_{\psi_j} = f_j d\mu$. Choosing $\Omega_j = \{\lambda | f_j(\lambda) > 0\}$ we can replace $\psi_j \to \chi_{\Omega_j}(A) f_j(A)^{-1/2} \psi_j$ such that $f_j \to \chi_{\Omega_j}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to μ_{ψ_j} we can even assume $\Omega_{j+1} \subseteq \Omega_j$.

Finally, we show that the spectral multiplicity of A is N. By the first part we can assume that A is multiplication by λ in $\bigoplus_{j=1}^N L^2(\mathbb{R}, \chi_{\Omega_j} d\mu)$. Let $\{\psi_j\}_{j=1}^n$ be a spectral basis with $n < N$. We will show that there is some vector in the orthogonal complement of $\bigoplus_j \mathfrak{H}_{\psi_j}$. Of course such a vector exists pointwise for every λ but it is not clear that the components can be chosen measurable. To see this we use a Gauss-type elimination: For this note that we can multiply every vector ψ_j with a non-vanishing function or add multiples of the other vectors to a given one without changing $\bigoplus_j \mathfrak{H}_{\psi_j}$. Hence we can first normalize the first component of every ψ_j to be a characteristic function. Moreover, by adding all other vectors to ψ_1 we can assume that its first component is positive on a maximal set $\tilde{\Omega}_1$. In fact, after another normalization we can assume that $\psi_{1,1} = \chi_{\tilde{\Omega}_1}$ and after subtracting multiples of ψ_1 from the remaining vectors we can assume $\psi_{j,1} = 0$ for $j \geq 2$. If $\mu_1(\mathbb{R} \backslash \tilde{\Omega}_1) > 0$ then $\varphi = (\chi_{\mathbb{R} \backslash \tilde{\Omega}_1}, 0, \dots)$ would be in the orthogonal complement and we are done. So assume $\chi_{\Omega_1} = 1$ and continue with the other components until they satisfy $\psi_{j,k} = \delta_{j,k}$ for $1 \leq j, k \leq n$. Then $\varphi = (-\psi_{1,n+1}, \dots, -\psi_{n,n+1}, 1, 0, \dots)$ is in the orthogonal complement contradicting our assumption that $\{\psi_j\}_{j=1}^n$ is a spectral basis. $\square$