Selected Solutions (Sketch)

Chapter 1

Exercise 1.2. Prove that $x \in X$ is periodic if and only if the orbit of x is compact.

Sketch: The "only if" part is immediate. We sketch the "if" part. Let $\mathrm{Orb}(x)$ be compact. If $\mathrm{Orb}(x)$ has no isolated points, then $\mathrm{Orb}(x)$ is uncountable (see the remark next), a contradiction. Thus $\mathrm{Orb}(x)$ has an isolated point. Then every point of $\mathrm{Orb}(x)$ is isolated. By compactness $\mathrm{Orb}(x)$ is finite; that is, x is periodic.

Remark. A compact set without isolated points is called *perfect*. Any perfect set is uncountable. The idea is this: Let Λ be a perfect set. Since there is no isolated point in Λ, there are two different points x_0 and x_1 in Λ. Take disjoint compact neighborhoods U_0 and U_1 of x_0 and x_1 in Λ, respectively. Again, since there is no isolated point in Λ, there are four different points, x_{00} and x_{01} in U_0, x_{10} and x_{11} in U_1, respectively. Take four disjoint compact neighborhoods U_{ij} of x_{ij} in Λ such that $U_{ij} \subset U_i$, $i, j = 0, 1$. Inductively, we get a tree of compact neighborhoods. At every fixed level the neighborhoods are finite and disjoint. The tree has uncountably many branches. Each branch is a nested sequence of compact neighborhoods in Λ hence there is a point in the intersection. This gives uncountably many points in Λ.

Exercise 1.5. Let $x \in X$. Prove that

(1) $\omega(x)$ can not be a union of two disjoint closed invariant subsets.

(2) If $\omega(x)$ is a union of finitely many periodic orbits, then $\omega(x)$ is in fact a periodic orbit.

(3) If $\omega(x)$ is a union of countably many periodic orbits, then $\omega(x)$ is in fact a periodic orbit.

How about if $\omega(x)$ is a union of uncountably many periodic orbits?

Sketch: (1) Suppose $\omega(x)$ is a union of two disjoint closed invariant subsets A and B. Take two small neighborhoods U and V of A and B, respectively, such that $fU \cap V = \emptyset$ and $fV \cap U = \emptyset$. This means no point can jump from U to V in one step and vise versa. Take N large so that $f^n(x) \in U \cup V$ for all $n \geq N$. We may assume $f^N(x) \in U$. Then $f^{n+i}(x)$ will remain in U for all $i > 0$, contradicting $B \subset \omega(x)$.

Item (2) is a corollary of (1). For item (3) note that there must be an isolated point z in $\omega(x)$ because otherwise $\omega(x)$ would be uncountable (see the remark in Exercise 1.2), contradicting that $\omega(x)$ is a countable union of periodic orbits. But z is periodic, so $\mathrm{Orb}(z)$

is finite and hence both open and closed in $\omega(x)$. Then $\mathrm{Orb}(z)$ and $\omega(x) - \mathrm{Orb}(z)$ are two disjoint closed invariant subsets of $\omega(x)$. By (1), $\omega(x) = \mathrm{Orb}(z)$.

It is possible for $\omega(x)$ to be a union of uncountably many periodic orbits, say fixed points. We describe such an example. Take a 2D flow ϕ_t with a circle C which is a periodic orbit. Assume every nearby point x spirals in, namely $\omega(x, \phi_t) = C$. Multiply the vector filed by a non-negative function that vanishes exactly on C. Then under the new flow ψ_t, every point of C becomes a singularity, while every nearby point x still spirals in along the same orbit, but slower and slower. Let ψ_1 be the time-1 map of the new flow. Then every point of C is a fixed point of ψ_1. Since $\mathrm{Orb}(x, \psi_t)$ spirals in slower and slower, $\omega(x, \psi_1) = C$.

Exercise 1.8. Prove if $\Omega(f) = X$, then $\{x : x \in \omega(x)\}$ is dense in X.

Sketch: Denote $R(f) = \{x : x \in \omega(x)\}$. By definition, $x \in \omega(x)$ means that for every m, there is n such that $d(f^n x, x) < 1/m$. That is,

$$R(f) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} R_{nm},$$

where

$$R_{nm} = \{x \in X : d(f^n x, x) < 1/m\}.$$

Clearly R_{nm} is open. Now $\Omega(f) = X$. Hence $\bigcup_{n=1}^{\infty} R_{nm}$ is dense in X. Thus, in this case, $R(f)$ is residual; hence dense in X.

Exercise 1.20. Prove if X is connected and $\mathrm{CR}(f) = X$, then X is a chain class.

Sketch: Take a Lyapunov function $\phi : X \to R$ of f gauranteed by Conley's Fundamental Theorem. Then $\phi(\mathrm{CR}(f))$ is nowhere dense. Since $\mathrm{CR}(f) = X$ is connected, $\phi(\mathrm{CR}(f))$ is a single point. But two chain recurrent points have the same ϕ-value if and only if they are in the same chain class. Thus $X = \mathrm{CR}(f)$ is a (single) chain class.

Exercise 1.21. Prove if $f : X \to X$ has a unique chain class C then $X = C$.

Sketch: We prove $X \subset C$. Take any $x \in X$. Then $\omega(x)$ and $\alpha(x)$ are both contained in C. Then x is chain equivalent to every point of C. Then $x \in C$.

Exercise 1.23. Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with $\mathrm{P}(f) \neq \emptyset$. Prove either $\mathrm{CR}(f) = \mathrm{P}(f)$, or $\mathrm{CR}(f) = S^1$.

Sketch: By Theorem 1.10, all periodic points of f have the same period, say m. Switching to f^m if necessary we may assume that $\mathrm{P}(f) = \mathrm{Fix}(f)$. Note that every non-fixed point moves within a connected component of $S^1 - \mathrm{Fix}(f)$. If all points of $S^1 - \mathrm{Fix}(f)$ move in the same direction (say clockwise), then it can be proved that $\mathrm{CR}(f) = S^1$. Otherwise there is a trapping interval I for f, and the problem reduces to f on I and f^{-1} on $S^1 - I$. Note that for interval homeomorphisms chain recurrent points are simply fixed points. Hence $\mathrm{CR}(f) = \mathrm{Fix}(f)$.

Exercise 1.24. Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with $\mathrm{P}(f) = \emptyset$. Prove $\mathrm{CR}(f) = S^1$.

Sketch: Take any $x \in S^1$. By Theorem 1.12, $\Omega(f)$ is a minimal set. Hence $\omega(x)$ and $\alpha(x)$ are both contained in (hence equal to) this minimal set. Then $x \in \mathrm{CR}(f)$.

Chapter 2

Exercise 2.2. Prove that all norms on E are equivalent.

Hint: Consult a book in functional analysis. (The idea of the proof is that the unit sphere of a finite dimensional normed space is compact hence any positive continuous function on the unit sphere has a positive minimum.)

Exercise 2.3. Let $A : E \to E$ be a linear isomorphism. Prove that A is hyperbolic if and only if A has no eigenvalue of absolute value 1.

Hint: The "only if" part is obvious. To prove the "if" part one may first work out the case when A is a Jordan block.

Exercise 2.4. Let $A : E \to E$ be a linear isomorphism. Prove the following three conditions are equivalent:

(1) A is hyperbolic;

(2) $B^s \cap B^u = \{0\}$;

(3) $D^s + D^u = E$.

Hint: Use the equivalent definition of hyperbolicity by eigenvalues.

Exercise 2.11. Let $A_\alpha : R \to R$ denote the linear map

$$A_\alpha(x) = \alpha x.$$

(a) Prove if $0 < \alpha < 1$ and $0 < \beta < 1$, then A_α and A_β are topologically conjugate.

(b) Prove if $\alpha \neq \beta$, then there is no lipeomorphism $h : R \to R$ such that $hA_\alpha = A_\beta h$.

Sketch: (b) Suppose $h : R \to R$ is a lipeomorphism such that $hA_\alpha = A_\beta h$, where $0 < \alpha < \beta < 1$. Note that 0 is the unique fixed point for both A_α and A_β. Hence $h(0) = 0$. Take any $0 < a < 1$ and let $h(a) = b$. Then $A_\alpha^n(a) = \alpha^n a$ and $A_\beta^n(b) = \beta^n b$. By conjugacy $h(\alpha^n a) = \beta^n b$. Thus

$$\frac{|h(\alpha^n a) - h(0)|}{|\alpha^n a - 0|} = \frac{\beta^n b}{\alpha^n a}$$

can be arbitrarily large, contradicting that h is Lipschitz.

Chapter 3

Exercise 3.5. Prove the shift map $\sigma : \Sigma_2 \to \Sigma_2$ has uncountably many disjoint minimal sets.

Sketch: We first describe an embedding process: replacing 0 by 00, 1 by 11, and taking the new "origin" to be the left spot of the two spots of the replacement of the old origin. This process embeds Σ_2 into a proper subset A of Σ_2, invariant under σ^2 such that $\sigma^2|_A$ is conjugate to σ. Briefly, A is a horseshoe for σ^2. Its σ-orbit $A \cup \sigma(A)$ is σ-invariant. If we can find two such horseshoes A and B for two iterates of σ, respectively, with disjoint orbits, then by induction, we will get a tree of horseshoes. At every level n the 2^n horseshoes are disjoint. Each branch of the tree is a nested sequence of compact invariant sets of σ, hence the intersection is a compact invariant set of σ and hence contains a minimal set of σ. This gives uncountably many disjoint minimal sets. Thus it suffices to prove the following

Claim. There are n and m together with a compact invariant set A of σ^n and a compact invariant set B of σ^m such that $\sigma^n|_A$ and $\sigma^m|_B$ are both conjugate to σ and such that $\mathrm{Orb(A,\sigma)} \cap \mathrm{Orb(B,\sigma)} = \emptyset$.

We simply take $n = 2$ and $m = 3$. To get A we replace 0 by 00 and 1 by 11. To get B we replace 0 by 010 and 1 by 101. It remains to prove $\mathrm{Orb(A,\sigma)} \cap \mathrm{Orb(B,\sigma)} = \emptyset$. It is enough to note that, for every point of $\mathrm{Orb(A,\sigma)}$, every 0 is in a couple 00 and every 1 is in a couple 11, while for every point of $\mathrm{Orb(B,\sigma)}$, there is at least one 0 that is right between 1 and 1, or at least one 1 that is right between 0 and 0.

Exercise 3.9. Let f_A be an Anosov toral automorphism on T^2 induced by an Anosov automorphism A on R^2. Show that the number of fixed points of f_A equals $|\det(A - I)|$.

Sketch: $f_A(x) = x$ if and only if there is $k \in \mathrm{Z}^2$ such that $Ax = x + k$, or $(A - I)x \in \mathrm{Z}^2$. Thus the number of fixed points of f_A equals the number of solutions of $(A - I)x = 0$, mod Z^2, or equivalenly, the number of g-preimages of 0, where $g : \mathrm{T}^2 \to \mathrm{T}^2$ is the induced map by $A - I$. Since 1 is not an eigenvalue of A and hence $A - I$ is an linear isomorphism of R^2, all points of T^2 have the same number of g-preimages. The number is just the area of $(A - I)([0, 1] \times [0, 1])$, which is $|\det(A - I)|$.

Chapter 4

Exercise 4.2. Prove if $x \in M$ is a transverse homoclinic point of a hyperbolic fixed point $p \in M$, then $\mathrm{Orb}(x)$ is a hyperbolic orbit.

Sketch: Take $T_x(W^s(p)) \oplus T_x(W^u(p))$ to be the direct sum at x. Iterating it by Tf gives a direct sum at every point of $\mathrm{Orb}(x)$, which form an invariant splitting on $\mathrm{Orb}(x)$. Since all but finitely many iterates are near p, the splitting will be hyperbolic.

Exercise 4.3. Let $\Lambda \subset M$ be a compact invariant set of f, $E \subset T_\Lambda M$ be a Tf-invariant C^0 subbundle. Prove the following three conditions are equivalent.

(a) There are $C \geq 1$ and $0 < \lambda < 1$ such that

$$|Tf^n(v)| \leq C\lambda^n |v|, \ \forall \ v \in E, n \geq 0.$$

(b) There are $0 < \mu < 1$ and $N \geq 0$ such that

$$|Tf^n(v)| \leq \mu |v|, \ \forall \ v \in E, n \geq N.$$

(c) For any $0 \neq v \in E$, there is $n = n(v) \geq 0$ such that

$$|Tf^n(v)| < |v|.$$

Sketch: That (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are straightforward. To prove (c) $\Rightarrow$ (a) note that the unit sphere bundle $\{v \in E : |v| = 1\}$ of E is compact as Λ is compact. By taking a finite cover one gets an integer $N \geq 1$ and a number $0 < \mu < 1$ such that every $v \in E$ has a suitable time $0 < m = m(v) \leq N$ such that $|Tf^m(v)| \leq \mu|v|$. Then letting $\lambda = \mu^{1/N}$ gives the desired estimate for every iterate n that is a sum of consecutive suitable times. For every other n, there is a remainder of length at most N. This can be handled by letting C bound all possible errors in ratio that can occur within N iterates.

Exercise 4.5. Let $\Lambda \subset M$ be a set. For any $x \in \Lambda$, let $E(x) \subset T_x M$ be an m-dimensional linear subspace. Prove $E = \bigcup_{x\in\Lambda} E(x)$ is an m-dimensional C^0 subbundle of $T_\Lambda M$ if and only if $E(x)$ varies continuously in $x \in \Lambda$.

Sketch: The "only if" part is straightforward. To prove the "if" part, assume $E(x)$ varies continuously in $x \in \Lambda$. Fix $x \in \Lambda$. We prove there is a neighborhood U of x in Λ together with m linearly independent C^0 vector fields e_1, ..., e_m on U such that, for every $y \in U$, the vectors $e_1(y)$, ..., $e_m(y)$ span $E(y)$. Taking local coordinates we may assume that the problem is in (an open set of) a d-dimensional Euclidean space E^d with a basis α_1, ... α_d. Denote E_1 the linear subspace of E^d spanned by α_1, ... α_m, and E_2 the one spanned by α_{m+1}, ... α_d. Thus $E^d = E_1 \oplus E_2$. We may assume that x is the origin of E^d, and that $E(x)$ is E_1. By translation we have the (constant) m-frame $\alpha_1(y)$, ... $\alpha_m(y)$ at every point $y \in E^d$. Let $\pi : E^d \to E_1$ be the projection with respect to the direct sum. It is straightforward (with many details) to prove that, there is a small neighborhood U of x in Λ such that, for every $y \in U$, π maps $E(y)$ isomorphically onto the hyperplane $E_1 + y$, hence maps a (unique) C^0 m-frame $e_1(y)$, ..., $e_m(y)$ that span $E(y)$ onto the (constant) m-frame $\alpha_1(y)$, ... $\alpha_m(y)$ that span $E_1 + y$.

Exercise 4.7. Let $\Lambda \subset M$ be a compact invariant set of f. Assume that, with respect to a C^0 direct sum

$$T_\Lambda M = E_1 \oplus E_2,$$

$Tf = A$ is represented as

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

such that

$$\max\{|A_{11}^{-1}|, |A_{22}|\} < 1.$$

Prove that Λ is a hyperbolic set of f.

Hint: Use the graph transform method as in the proof of Lemma 4.5.

Exercise 4.10. Let $T_\Lambda M = G_1^s \oplus G_1^u$ and $T_\Lambda M = G_2^s \oplus G_2^u$ be two dominated splittings on Λ. Show that for any $x \in \Lambda$, either $G_1^s(x) \subseteq G_2^s(x)$, or $G_1^u(x) \subseteq G_2^u(x)$. In particular, if $\dim G_1^s(x) = \dim G_2^s(x)$, then $G_1^s(x) = G_2^s(x)$ and $G_1^u(x) = G_2^u(x)$ (that is, for fixed index, dominated splitting is unique).

Sketch: Note that either $G_1^s(x) \subseteq G_2^s(x)$ or $G_2^s(x) \subseteq G_1^s(x)$. In fact, if there are $u \in G_1^s(x) - G_2^s(x)$ and $v \in G_2^s(x) - G_1^s(x)$ with $|u| = |v| = 1$, then

$$|Tf^n(u)| < |Tf^n(v)|, \qquad |Tf^n(v)| < |Tf^n(u)|$$

for all large n, a contradiction. Likewise, either $G_1^u(x) \subseteq G_2^u(x)$ or $G_2^u(x) \subseteq G_1^u(x)$. In particular, if $\dim G_1^s(x) = \dim G_2^s(x)$, then $G_1^s(x) = G_2^s(x)$ and $G_1^u(x) = G_2^u(x)$.

Now suppose it is not true that either $G_1^s(x) \subseteq G_2^s(x)$ or $G_1^u(x) \subseteq G_2^u(x)$. Then $G_2^s(x) \subseteq G_1^s(x)$ with $G_2^s(x) \neq G_1^s(x)$ and $G_2^u(x) \subseteq G_1^u(x)$ with $G_2^u(x) \neq G_1^u(x)$. This contradicts $T_\Lambda M = G_2^s \oplus G_2^u$.

Exercise 4.12. Let $\Lambda \subset M$ be a compact invariant set of f with a dominated splitting. Prove there is a C^1 neighborhood $\mathcal{U}$ of f and a number $a > 0$ such that for any $g \in \mathcal{U}$, any compact invariant set $\Delta \subset B(\Lambda, a)$ of g has a dominated splitting with respect to g.

Hint: Consult the proofs of Lemma 4.5 and Theorem 4.6.

Exercise 4.15. Let $TM = G^s \oplus G^u$ be a dominated splitting of f. Let $p \in M$ and $q \in M$ be two hyperbolic periodic saddles of f with $\dim W^s(p) = \dim W^s(q) = \dim G^s$. Prove $W^u(p)$ intersects $W^s(q)$ transversely.

Sketch: Let $x \in W^u(p) \cap W^s(q)$. Then

$$T_x(W^s(q)) = \{v \in T_xM \mid |Tf^n(v)| \to 0,\ n \to \infty\},$$

$$T_x(W^u(p)) = \{v \in T_xM \mid |Tf^{-n}(v)| \to 0,\ n \to \infty\}.$$

An argument similar to Exercise 4.10 shows that either $G^s(x) \subset T_x(W^s(x))$ or $T_x(W^s(x)) \subset G^s(x)$. In fact if there are $u \in G^s(x) - T_x(W^s(x))$ and $v \in T_x(W^s(x)) - G^s(x)$ with $|u| = |v| = 1$, then $|Tf^n(u)| < |Tf^n(v)|$ for all large n, and $|Tf^n(v)| < |Tf^n(u)|$ for some arbitrarily large n, a contradiction. Likewise for $G^u(x)$ and $T_x(W^u(p))$.

Now the two splittings have the same index, so $G^s(x) = T_x(W^s(q))$ and $G^u(x) = T_x(W^u(p))$, solving Exercise 4.15.

Exercise 4.22. Let $T_xM = E(x) \oplus F(x),\ x \in M$, be a continuous invariant splitting of f. Prove if this splitting restricted to the nonwandering set $\Omega(f)$ is hyperbolic, then the whole splitting is hyperbolic (f is Anosov).

Sketch: By continuity the dimension $\dim(E(x))$ is constant. For every $x \notin \Omega(f)$,

$$\lim_{n\to\infty} d(f^n(x), \Omega(f)) = 0,$$

$$\lim_{n\to\infty} d(f^{-n}(x), \Omega(f)) = 0.$$

Thus the proof goes like Exercise 4.2.

Exercise 4.27. Let $f : M \to M$ be an Anosov diffeomorphism with $\Omega(f) = M$. We also assume M is connected. Prove that

(1) For every periodic point p of f, $W^s(p)$ is dense in M.

(2) For every point $x \in M$, $W^s(x)$ is dense in M.

Sketch: (1) Since f is Anosov, by the shadowing lemma, $\Omega(f) = \overline{\mathrm{P}(f)}$. By the spectral decomposition theorem, $\overline{\mathrm{P}(f)}$ decomposes into a disjoint union of finitely many basic sets. Since $M = \Omega(f)$ and M is connected, M itself is a basic set, say

$$M = \overline{P}_1 \cup \cdots \cup \overline{P}_r,$$

where P_i are equivalent classes of periodic points such that $\overline{P}_i \cap \overline{P}_j = \emptyset,\quad i \neq j$, and such that $f(\overline{P}_1) = \overline{P}_2,\ \ldots,\ f(\overline{P}_r) = \overline{P}_1$. (Here two periodic points p and q are called *equivalent* if $W^u(p)$ and $W^s(q)$ have a transverse intersection and $W^u(q)$ and $W^s(p)$ have a transverse intersection, as defined in the proof of the spectral decomposition theorem.) Again, since M is connected, $r = 1$. Thus all periodic points of f (which are dense in M) are mutually equivalent. Let p be a periodic point of f. For any open set U in M, there is a periodic point $q \in U$ such that $q \sim p$. Let k be the product of the periods of p and q. Applying the λ-lemma to f^k shows that $W^s(p)$ accumulates on q, proving (1).

(2) Since f is Anosov, every point of M has a "product neighborhood" W, meaning W is foliated by stable discs and also foliated by unstable discs such that any pair of stable and unstable discs meet transversely at a point of W. One may require that every product neighborhood has size $\leq a$ for some fixed $a > 0$. Let $\delta > 0$ be the Lebesque number associated with the cover of M by product neighborhoods. Then any two points $x, y \in M$ with $d(x, y) < \delta$ are contained in a product neighborhood and hence $W^s(x)$ intersects $W^u(y)$ transversely.

Let $x \in M$ and let $U \subset M$ be an open set. We prove $W^s(x) \cap U \neq \emptyset$. Take a periodic point $p \in U$. Let $r > 0$ be small so that $W_r = W^u_r(p) \subset U$. Let m be the period of p. Then

$$W^u(p) = \bigcup_{n=1}^{\infty} f^{mn}(W_{r/2})$$

is a nested union. Now $W^u(p)$ is dense in M (applying (1) to f^{-1}). Hence

$$M = \bigcup_{n=1}^{\infty} B_\delta(f^{mn}(W_{r/2})),$$

where δ is the Lebesque number as above and $B_\delta(f^{mn}(W_{r/2}))$ denotes the δ-neighborhood of $f^{mn}(W_{r/2})$ in M. Since M is compact, there is N such that

$$M = B_\delta(f^{mN}(W_{r/2})).$$

We may have chosen N large so that $f^{mN}(W_r)$ contains an a-neighborhood of $f^{mn}(W_{r/2})$ inside $W^u(p)$. Then for every $z \in M$, there is $y \in f^{mN}(W_{r/2})$ such that $d(z, y) < \delta$, and hence z and y are contained in a product neighborhood. Then $f^{mN}(W_r)$ intersects $W^s(z)$ transversely. Letting $z = f^{mN}(x)$ gives $f^{mN}(W_r) \cap W^s(f^{mN}(x)) \neq \emptyset$. Applying f^{-mN} gives $W_r \cap W^s(x) \neq \emptyset$, proving (2).

Chapter 5

Exercise 5.2. Let p and q be two hyperbolic periodic points of f. We say p and q are *homoclinically related* if $W^s(\mathrm{Orb}(p))$ and $W^u(\mathrm{Orb}(q))$ have a transverse intersection, and $W^s(\mathrm{Orb}(q))$ and $W^u(\mathrm{Orb}(p))$ have a transverse intersection. The closure of the set of hyperbolic periodic points that are homoclinically related to p is called the *homoclinic class* of p, denoted $H(p, f)$. Prove that

(1) $H(p, f)$ is transitive.

(2) If $H(p, f)$ is not a single (periodic) orbit, it coincides with the closure of the set of transverse homoclinic points of p.

Sketch: Note that the relation of being "homoclinically related" is similar but different from the equivalent relation defined in the proof of the spectral decomposition theorem. Here it talks about $\mathrm{Orb}(p)$ and $\mathrm{Orb}(q)$, while there it talks about p and q. In particular, the relation of being "homoclinically related" will lead to equivalence classes that are f-invariant, but the equivalent classes defined in the proof of the spectral decomposition theorem are generally not f-invariant. Another important difference is that here a homoclinic class is not assumed to be hyperbolic. In fact, generally a homiclinic class may not be hyperbolic, and two homoclinic classes may not be disjoint, and a diffeomorphism may have infinitely many homoclinic classes. There is a brilliant study of research on homoclinic classes, see Bonatti-Diaz-Viana (2005).

(1) It is easy to prove that being homoclinically related is an (invariant) equivalent relation on the set of hyperbolic periodic points, and the homoclinic class $H(p, f)$ of p is a closed invariant set of f. The arguments are similar to that used in the proof of the spectral decomposition theorem.

To prove that $H(p, f)$ is transitive, let U and V be any two open sets in $H(p, f)$. By Birkhoff theorem, it suffices to prove there is a point $z \in H(p, f)$ whose orbit travels from U to V. Take a periodic point $q \in U$ and a periodic point $r \in V$ that are homoclinically related to p. Then there are $q' \in \mathrm{Orb}(q)$ and $r' \in \mathrm{Orb}(r)$ such that $q' \to r'$, where the (temporary) notation $q' \to r'$ is used to denote that $W^u(q')$ and $W^s(r')$ have a transverse intersection. Then for every $x \in \mathrm{Orb}(q)$ there is $y \in \mathrm{Orb}(r)$ such that $x \to y$. Likewise, for every $y \in \mathrm{Orb}(r)$ there is $x \in \mathrm{Orb}(q)$ such that $y \to x$. Then there is a cycle

$$q = x_0 \to x_1 \to \cdots \to x_k = q$$

from q to itself. Note that the k points of the cycle together with the k points of transverse intersections may not form an invariant set of f. Nevertheless their orbits form a hyperbolic invariant set Λ of f. Also note that for any $\delta > 0$, there is a periodic δ-chain C of f in Λ that contains q and x_1. But $x_1 \in \mathrm{Orb}(r)$. Hence for any $\delta > 0$, there is a periodic δ-chain C' in Λ that contains q and r. (C' can be obtained by inserting the orbit of r into C right at x_1.) By the shadowing lemma, for any $\varepsilon > 0$, there is a periodic point z whose orbit stays in the ε-neighborhood of Λ and passes the ε-neighborhoods of q and r. If ε is small enough, z will be homoclinically related to q hence will belong to $H(p, f)$.

(2) Let $H^*(p, f)$ be the closure of the transverse homoclinic points of p, which by definition is the set of transverse intersection points of $W^s(\mathrm{Orb}(p))$ and $W^u(\mathrm{Orb}(p))$. To prove $H^*(p, f) \subset H(p, f)$, it suffices to prove that every transverse homoclinic point x of p is accumulated by hyperbolic periodic points that are homoclinically related to p. This is by a similar argument of shadowing as in (1). Now assume $H(p, f)$ is not a single (periodic) orbit. To prove $H(p, f) \subset H^*(p, f)$, let $q \neq p$ be homoclinically related to p. It

suffices to prove $q \in H^*(p, f)$. By the λ-lemma, $W^s(\mathrm{Orb}(p))$ and $W^u(\mathrm{Orb}(q))$ both pile on q, hence produce transverse homoclinic points of p arbitrarily close to q.

Exercise 5.17. Prove if a chain class is hyperbolic, then it is isolated, transitive, and with periodic points dense.

Sketch: Let C be a hyperbolic chain class. For each n, there is a $1/n$-periodic chain A_n such that the Hausdorff distance between A_n and C is $\leq 1/n$, meaning by definition that $A_n \subset B_{1/n}(C)$ and $C \subset B_{1/n}(A_n)$. By shadowing, there is a sequence of periodic orbits P_n of f such that $P_n \to C$ in the Hausdorff metric. Thus there is $N > 0$ such that if $n, m \geq N$ then, having a uniform size of stable and unstable manifolds, P_n and P_m are sufficiently Hausdorff close. Hence P_n and P_m are homoclinically related (definition in Exercise 5.2). Thus $C \subset H(P_N, f)$, the homoclinic class of P_N (definition in Exercise 5.2). By Exercise 5.2, $H(P_N, f)$ is transitive. But C is a chain class; hence $C \supset H(P_N, f)$. Thus $C = H(P_N, f)$.

It remains to prove that C is isolated. Since $C = H(P_N, f)$ is hyperbolic, there is a neighborhood U of C such that the maximal invariant set in $\overline{U}$ is hyperbolic. Shrinking U if necessary we assume that any periodic orbit Q contained in $\overline{U}$ is homoclinically related to P_N. We prove U is an isolating neighborhood of C. Let $x \in C$ with $\mathrm{Orb}(x) \subset \overline{U}$. Then $\omega(x)$ and $\alpha(x)$ are both in $\overline{U}$. By shadowing, $\omega(x)$ is a Hausdorff limit of a sequence Q_n of hyperbolic periodic orbits that are contained in U. Thus $\omega(x) \subset H(P_N, f) = C$. Likewise $\alpha(x) \subset C$. Thus $x \in C$. This proves that C is isolated.

Exercise 5.22. Let $f : M \to M$ be Anosov. If $\mathrm{CR}(f) = M$, prove that f is transitive (we always assume M is connected).

Hint: Use Theorem 4.32 and the spectral decomposition theorem.

Exercise 5.23. Let $f : M \to M$ be Anosov. Prove f satisfies Axiom A.

Hint: Use the shadowing lemma. Compare with Exercise 4.23, which also uses the shadowing lemma.

Exercise 5.24. Let $f : M \to M$ satisfy Axiom A. Prove if f satisfies the strong transversality then f satisfies the no-cycle condition.

Sketch: Suppose there is a cycle

$$z_1 \in W^u(B_{i_1}) \cap W^s(B_{i_2}) - \Omega(f),\ z_2 \in W^u(B_{i_2}) \cap W^s(B_{i_3}) - \Omega(f),$$
$$..., z_m \in W^u(B_{i_m}) \cap W^s(B_{i_1}) - \Omega(f).$$

By the In Phase Theorem, there are $x_1, x_1' \in B_{i_1}, ..., x_m, x_m' \in B_{i_m}$ such that

$$z_1 \in W^u(x_1') \cap W^s(x_2) - \Omega(f),\ z_2 \in W^u(x_2') \cap W^s(x_3) - \Omega(f),$$

$$..., z_m \in W^u(x'_m) \cap W^s(x_1) - \Omega(f).$$

By the strong transversality condition, the intersection $W^u(x'_1) \cap W^s(x_2)$ at z_1 is transverse. Likewise for the other z_j. Replacing x_j and x'_j respectively by nearby periodic points p_j and p'_j in B_{i_j}, we get another transverse cycle

$$y_1 \in W^u(p'_1) \cap W^s(p_2) - \Omega(f),\ \ y_2 \in W^u(p'_2) \cap W^s(p_3) - \Omega(f),$$

$$..., y_m \in W^u(p'_m) \cap W^s(p_1) - \Omega(f).$$

Hence

$$\dim W^u(p'_j) + \dim W^s(p_{j+1}) \geq \dim M.$$

Since p'_j and p_j are in the same basic set, they have the same index and the stable and unstable manifolds of their orbits intersect transversely. In particular,

$$\dim W^u(p'_j) + \dim W^s(p_j) = \dim M.$$

Then the indices form a monotone sequence:

$$\dim W^s(p_{j+1}) \geq \dim W^s(p_j).$$

But the indices are cyclic due to the cycle. Thus $\dim W^s(p_j)$ are constant for all j. By the λ-lemma, y_1 is non-wandering, a contradiction.

Chapter 6

Exercise 6.5. (C^r closing lemma) Let $f : S^1 \to S^1$ be a C^r diffeomorphism with $\mathrm{P}(f) = \emptyset$, $r \geq 1$. Prove that for any C^r neighborhood $\mathcal{U}$ of f, there is $g \in \mathcal{U}$ such that $\mathrm{P}(g) \neq \emptyset$.

Sketch: Compose a small (rigid) rotation θ_t with f. More precisely, take $x \in S^1$ such that $x \in \omega(x)$. We may assume that a subsequence $f^{n_i}(x)$ converges to x, say from the "left". In the universal cover this means that, for any $\varepsilon > 0$, there are i and m such that $F^{n_i}(\overline{x}) \in (\overline{x} + m - \varepsilon, \overline{x} + m)$. Since F is monotone, the intermediate value theorem gives $t \in (0, \varepsilon)$ such that $(F + t)^{n_i}(\overline{x}) = \overline{x} + m$.