# Supplementary bibliography for "DGTRH"

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The book (hereafter nicknamed GSM177) having the character of a course, I did not try to be exhaustive in giving the references. Because of some afterthoughts and also because of more recent publications, I feel compelled to add the following[1] to the original bibliography.

## Origins of differential algebra and differential Galois theory

We mention only the algebraic side, for the sources in Liouville, Picard, Vessiot, Garnier, ..., see the bibliographies.

A concise introduction to general differential algebra with some applications is the great classical [Kap96].

The use of algebraic groups in this area was mainly the work of Kolchin. See [Kol99]; and, for a more historical approach, the last chapter of [Bor01].

## General introduction to differential Galois theory

Along with [Sin90a, Sin90b], which were already mentioned, consider [Sin99, Sin09]. These include a discussion of the direct and inverse problem, and also some applications (of which GSM177 is poor).

Also, in french, a nice initiation: "Raconte-moi... La théorie de Galois différentielle", Gazette des Mathématiciens - n. 152, Avril 2017. It can be found at the following URL:
`http://math.univ-lyon1.fr/~roques/gazette.pdf`

## Riemann-Hilbert correspondance and Galois theory

Along with the fuchsian case tackled in [MS16], which was already mentioned, consider, for the irregular case, the second volume of the collection [LR16]. (The third

[1] Note that the weight of "pure" differential Galois theory, as compared with the transcendental approach of GSM177, is slightly bigger in this supplementary bibliography.

volume of the collection [Del16] is also extremely interesting but more out of scope.)

## Stokes phenomenon

The Stokes phenomenon is central in studying irregular equations. A preliminary tool is summation theory (backed with asymptotic theories). See [LR16] (and also [Bal94]).

More information on Stokes phenomenon proper can be found in [BJL79, Mal79]. (Also see [Sib77].)

Its application to Riemann-Hilbert correspondence and to differential Galois theory, is largely due to Ramis: along with [JC09], which was already quoted in GSM177's bibliography, see [Ram96, Ram85].

It has been developped into a powerful tool in [LR03, LR90, LR94].

These references include as well expository as detailed technical presentations.

## Direct problem, effective computations

Effective computation of a differential Galois group is a non trivial problem. A classical text using tannakian methods is [Kat87].

Using Ramis' generalization of Schlesinger's theorem to irregular equations, that is, using the Stokes phenomenon, was notably done in [DM89, Mit96].

A more algebraic method, relying on so-called "Kovacic's algorithm" for equations of order 2 (which by the way covers most applications to special functions) is developped in [DLR92].

Here are more recent works in this direction: [BCWD16, BCDW20]; they are of a more algebraic and algorithmic nature.

## Non linear differential equations and Malgrange groupoid

It seems that there are three theories; the most developped and used is that of Malgrange, followed by that of Umemura[2], then by that of Cartier. (For the latter, there unfortunately seems to exist no published text.)

[2] Hiroshi Umemura, a world expert in algebraic geometry and differential equations, passed away in 2019. A special issue of the "Annales de la Faculté de Sciences de Toulouse" will be devoted to him in 2020. This issue should interest the readers of GSM177.

Along with [Mal02, Ume08], which were already mentioned, consider [Ume09, Ume11, Mal12].

A rather different point of view can be found in a talk of Daniel Bertrand, to be found at the following URL:
`http://divizio.perso.math.cnrs.fr/CIRM2009/notes/bertrand.pdf`

### Some recent applications

GSM177 gives few applications of differential Galois theory. Some are quoted in the very first references at the beginning of the present supplementary bibliography. Here are some more recent ones.

The Galois theory for *linear* differential equations was applied to integrability problems for *non linear* equations in [MR10].

The Galois theory for *non linear* differential equations (Malgrange style) was applied to Painlevé equations, for instance in [Cas08].

The linear theory was given a recent spectacular application to difficult problems in combinatorics in [DHRS20].

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