Errata, Changes, and Addenda for the Book
"Modular Forms: a Classical Approach"

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Last modified: December 2, 2017

Errata: errors, mathematical or otherwise, which *must* be corrected.

Attribution: omissions of references and attribution of original results that *must* be inserted.

Changes: modifications which improve the presentation.

Addenda: additions of relevant, useful, and/or interesting material.

Errata. p. 4 line 13: replace "$\tau \in H$" by "$\tau \in \mathfrak{H}$".

p. 489: In Proposition 12.4.30, replace " If $f \in S_k(\Gamma)$, then" by

"$f \in S_k(\Gamma)$ and $a < \Re(s) < k/2 - 1$ then"

also insert a factor $e^{-i\bar{s}\pi/2}$ on the right-hand side of the inequality in the statement of the proposition.

p. 496-497: In Theorem 12.5.11 replace upper case "I" by lower case "i" at 4 instances.

p. 522 line 6: replace "If p is any prime" by "If q is any prime", and

lines 8-9, replace "vanish when $p \nmid n$" by "vanish when $q \nmid n$ (when χ is defined modulo N/q if $q \mid N$)"

p. 582 line 2: replace "$\mathfrak{g}(D;a)$" by "$\mathfrak{g}(\underline{D};a)$".

p. 597 line 3: in the formula for $E_k^0(\tau)$ replace "ε_d^{-k}" by "ε_d^{-2k}".

p. 605 line 11, replace the formula for $T(p^4)$ by

$$T(p^4) = T(p^2)^2 - \chi(p^2)(p^{2k-2} + p^{2k-3}) .$$

p. 605 middle: in the displayed formula for $T_{k,\chi}(p^2)f(\tau)$ there are two misprints:

first, in the second summand, replace "$p^{k-2}\varepsilon_p^{2k}$" by "$p^{k-2}\varepsilon_p^{2k}\chi(p)$";

second, in the third summand on the next line replace "p^k" by "p^{2k-2}".

p. 605 line -8: in the displayed formula for $b_p(m)$ replace "n" by "m" (4 times).

p. 607 line 12: in the first summation index, replace "$m \geq 0s$" by "$m \geq 0$".

Attribution. p. 441: Add the following sentence at the end of the first paragraph in the chapter introduction: "To obtain this formula we use the method of Diamantis and O'Sullivan [**DO10**]. Many of the definitions and results in this Section are taken from their paper with few modifications except for providing more details at certain places."

p. 450: Before Prop. 12.2.16, add the following sentence: "The following proposition is given in [**DO10**] as Theorem 3.1. Since the authors of [**DO10**] only sketch the proof we include a full proof here."

p. 458: At the end of the closing paragraph of section 12.3.2 (i.e. just before section 12.3.3) add: "Note that (b) above is Prop. 9.3 of [**JO05**]."

p. 458: Insert the following sentence between Definition 12.3.5 and Proposition 12.3.6 : "The following result is given in [**DO10**] as Lemma 2.2 and (2.9)."

p. 460: Insert the following sentence before Corollary 12.3.8: "The following result is Proposition 2.1 of [**DO10**]."

p. 462: Insert the following sentence before Lemma 12.3.11: "The following result is Lemma 3.2 of [**DO10**]."

p. 463: In Definition 12.3.12, replace "For $n \in \mathbb{Z}$ we set" by: "For $n \in \mathbb{Z}$, following [**DO10**], we set"

p. 481, line 6: Replace "We have the following result:" by "The following result is Theorem 1.2 of [**DO10**] and we also include the proof for completeness."

p. 481: Before Corollary 12.4.18 insert the following paragraph: "The following important result and its consequences was first obtained by Diamantis and O'Sullivan, see Theorem 1.3 of [**DO10**]. There it was also used to provide simplified proofs of the classical results by Manin and Kohnen and Zagier which we also give below in Theorems 12.4.20 and 12.4.22."

p. 483: before Theorem 12.4.20, replace ",which we restate as" by "which we restate below. The proof we give here first appeared in [**DO10**]."

p. 483: Insert the following sentence before Corollary 12.4.22: "Using the notation of [**DO10**] Theorem 4.1 we have the following theorem by Kohnen and Zagier [**KZ84**]."

p. 488: Insert the following paragraph immediately before Proposition 12.4.29: "The series $R_k(s)$ was originally introduced by the first author [**Coh81**] for integer values of s and later used for complex values by, for instance Kohnen [**Koh97**]. However, the first published proof of the normal convergence of this, and an even more general, series was given in Proposition 5.1 of [**DO10**]. We include most of the proof here but refer the reader to [**DO10**] for the complete proof. Note that $R_k(s)$ is obtained as a special case of the series $\Omega_{\mathfrak{a}}(z, \tau; s, k)$ in [**DO10**] with $\mathfrak{a} = \infty$, $z = \tau$ and $\tau = 0$."

p. 489: Insert the following sentence before Proposition 12.4.30: "The following result is a special case of Proposition 5.4 in [**DO10**]."

Changes. p. 48 line -1, replace "Proposition 2.3.2" by "Propositions 2.3.2 and 2.3.3"

p. 67 line -12, replace "as in the text" by "as in Propositions 2.3.2 and 2.3.3"

p. 163 line 5: the formula for $[f,g]_2$: remove the large outside parentheses

and put on the same line as the equal sign.

p. 358 lines 9-10: remove "(we have, in fact, ...10.6)"

p. 359 line 5: after "conductor of χ" add "; see Exercise 10.6"

p. 382 replace the statement of Exercise 10.6 by the following:

"Show that if instead of using d^2x in the proof of the first part of Proposition 10.3.18 one uses $da^{-1}x$, where a^{-1} is the inverse of a modulo m, one obtains the stronger statement due to Atkin–Li that $f_\psi \in M_k(\Gamma_0(N'), \psi^2\chi)$ with $N' = \text{lcm}(N, qm, m^2)$, where q is the conductor of χ."

p. 519 line -14: in the displayed formula, replace "$T(q)f =$" by

"$T(q)f = U(q)f =$"

p. 520, the proof of Proposition 13.2.5, while correct, should be improved:

line 7, after "$\gamma_i = T^i$ for $i \bmod q$" add "(so in particular $\chi_1(\gamma_i) = 1$)"

line 9, replace the three lines "It follows that $h =$ ... ... assume that $q \nmid N/q$." by

"If $q \mid N/q$, it follows that $h = T(q)f$, proving (a), so we now assume

that $q \nmid N/q$, hence $h = T(q)f + \chi_1(q)q^{k/2-1}f|_k V_q$ for any

integral matrix of the form $V_q = \left(\begin{smallmatrix} q & x \\ Ny & qz \end{smallmatrix}\right)$ with determinant q."

p. 531 middle: replace the ending period of Corollary 13.3.13 by a comma, and on the next line add "where χ must be considered as a character modulo N."

p. 545 line 9: remove the word "Remarkable" from the title of subsection 13.3.3.

p. 549 at the end of line 9: add a footnote: "B. Edixhoven has informed us that this result is in fact immediate by considering the natural pairing $(f, T) \mapsto a_1(T(f))$ between the space of modular forms and the Hecke algebra."

Addenda:

p. 400, line -1, replace the displayed formula by

$$P_f(X) = \frac{(-2\pi i)^{k-1}}{(k-2)!}\int_0^{i\infty}(\tau-X)^{k-2}f(\tau)\,d\tau = \sum_{0\le j\le k-2}\frac{(2\pi i)^j}{j!}L(f,k-1-j)X^j\ .$$

Note that the equality between the two formulas follows immediately by a direct computation or from Theorem 11.1.1.

p. 598 line -12: after "Joris [**Jor77**]" add "; see Exercise 15.X1"

In the exercises to Chapter 15, add the following two exercises:

15.X1. Let $d > 0$ be an odd integer, and let

$$S = \sum_{r \pmod d}\left(\frac{r}{d}\right)e^{2\pi i n(r/d)}$$

be the Gauss sum occurring in Remark 15.1.6. Show that if we write uniquely $d = Df^2$ with D squarefree we have $S = 0$ if $f \nmid n$, and otherwise

$$S = d^{1/2}\left(\frac{-4}{d}\right)\sum_{a\mid\gcd(f,n/f)} a\mu(f/a)\left(\frac{n/a^2}{D}\right)\ .$$

15.X2. Let $f \in M_k(\Gamma_0(N),\chi)$. If $d \ge 1$ and k is integral, we know that $B(d)f \in M_k(\Gamma_0(dN),\chi)$. Show that if $k \in 1/2+\mathbb{Z}$ is half-integral we have instead $B(d)f \in M_k(\Gamma_0(dN),\chi\chi_{4d})$, where χ_{4d} is the Kronecker symbol $\chi_{4d}(n) = \left(\frac{4d}{n}\right)$.

In Chapter 15, add the following subsections to the end of section 15 on half-integral weight modular forms:

15.1.7. The Serre–Stark Theorem. We have seen since the beginning of this book that the the elementary Jacobi theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is a modular forms of weight 1/2, more precisely $\theta \in M_{1/2}(\Gamma_0(4))$. More generally, by Corollary 2.3.21 for any even primitive character ψ of conductor f we have $\theta(\psi) = \sum_{n\in\mathbb{Z}}\psi(N)q^{n^2} \in M_{1/2}(\Gamma_0(4f^2),\psi)$, hence by Exercise 15.X1

$$\theta(d,\psi) = B(d)\theta(\psi) = \sum_{n\in\mathbb{Z}}\psi(n)q^{dn^2} \in M_{1/2}(\Gamma_0(4df^2),\chi_{4d}\psi)\ .$$

An important theorem of Serre–Stark [**SS77**] states that these forms form a basis for all modular forms of weight 1/2:

Theorem 1.26 (Serre–Stark)**.** *Let $4 \mid N$ and χ be an even Dirichlet character.*

(1) *A* basis *for the space $M_{1/2}(\Gamma_0(N),\chi)$ is given by the $\theta(d,\psi)$ defined above with ψ even primitive, such that $4df^2 \mid N$ and $\chi \sim \chi_{4d}\psi$.*

(2) *A basis for the space $S_{1/2}(\Gamma_0(N),\chi)$ is given by the $\theta(d,\psi)$ above which do* not *satisfy the following additional condition: ψ is the square of a character of conductor f if f is odd, or of conductor $2f$ if f is even.*

Remarks.

(1) Note that $\chi \sim \chi'$ means that the corresponding primitive characters are equal. In particular ψ is determined by d: it is the primitive character associated with $\chi\chi_{4d}$, and the condition on d is that the square of its conductor divides $N/(4d)$.

(2) It is of course easy from this theorem to deduce the dimensions of $M_{1/2}(\Gamma_0(N),\chi)$ and of $S_{1/2}(\Gamma_0(N),\chi)$ for any given N and χ.

p. 51 line -8: add "(see Theorem XXX in Chapter 15)", where XXX refers to the above theorem.

15.1.8. Dimension Formulas for Half-Integral Weight. Using the Riemann–Roch theorem, it is possible to compute $\dim(S_k(\Gamma_0(N),\chi)) - \dim(M_{2-k}(\Gamma_0(N),\overline{\chi}))$, both for integral and half-integral weight. Since for $k \le 0$ we have $\dim(M_k(\Gamma_0(N),\chi)) = \dim(S_k(\Gamma_0(N),\chi)) = 0$ for all $k \le 0$ except that $\dim(M_0(\Gamma_0(N))) = 1$, this gives us $\dim(M_k)$ and $\dim(S_k)$ for all $k \in \mathbb{Z}$ *except* for $k = 1$ (see Section 7.4). In the present subsection we will give the formulas for $\dim(S_k(\Gamma_0(N),\chi)) - \dim(M_{2-k}(\Gamma_0(N),\overline{\chi})$ for $k \in 1/2 + \mathbb{Z}$, and thanks to the Serre–Stark theorem this will give us $\dim(S_k(\Gamma_0(N),\chi))$ and $\dim(M_k(\Gamma_0(N),\chi))$ for $k \in 1/2 + \mathbb{Z}$ with no exception.

The following is directly taken from Cohen–Oesterlé [**CO77**] with an evident misprint corrected. That paper does not contain any proof, but complete proofs can be found in [**WP12**]. We need to define constants λ_p for all primes $p \mid N$:

Definition 1.27. Let F be the conductor of χ. Write

$$N = \prod_{p|N} p^{r_p} \quad \text{and} \quad F = \prod_{p|N} p^{s_p}\ ,$$

so that $s_p \le r_p$ and $s_2 \ne 1$. We define constants λ_p as follows:

(1) For $p \ge 3$ or $p = 2$ and $r_2 \ge 4$ we set

$$\lambda_p = \begin{cases} p^{\lfloor r_p/2 \rfloor} + p^{\lfloor (r_p-1)/2 \rfloor} & \text{if } 2s_p \le r_p\ , \\ 2p^{r_p - s_p} & \text{if } 2s_p > r_p\ . \end{cases}$$

(2) If $r_2 = 3$ we let $\lambda_2 = 3$. If $r_2 = 2$, we introduce the following condition (C): there exists a prime $p \mid N$ with $p \equiv 3 \pmod 4$ and either r_p odd or $0 < r_p < 2s_p$.

Then if (C) is satisfied we let $\lambda_2 = 2$, and otherwise we let

$$\lambda_2 = 2 + (-1)^{s_2/2+k+1/2}/2 \ .$$

Theorem 1.28 (Cohen–Oesterlé)**.** *For any $k \in 1/2 + \mathbb{Z}$ we have*

$$\dim(S_k(\Gamma_0(N),\chi)) - \dim(M_{2-k}(\Gamma_0(N),\overline{\chi})) = \frac{k-1}{12} N \prod_{p\mid N}\left(1+\frac{1}{p}\right) - \frac{1}{2}\prod_{p\mid N}\lambda_p \ .$$

If we denote the right-hand side by $R(N,k,\chi)$, for $k \geq 5/2$ we thus have

$$\dim(S_k(\Gamma_0(N),\chi)) = R(N,k,\chi) \ , \ \dim(M_k(\Gamma_0(N),\chi)) = -R(N,2-k,\chi) \ ,$$

while

$$\dim(S_{3/2}(\Gamma_0(N),\chi)) = R(N,3/2,\chi) + \dim(M_{1/2}(\Gamma_0(N),\chi) \quad \text{and}$$
$$\dim(M_{3/2}(\Gamma_0(N),\chi)) = -R(N,1/2,\chi) + \dim(S_{1/2}(\Gamma_0(N),\chi)) \ ,$$

where the dimensions in weight 1/2 are given by the Serre–Stark theorem.

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