I would like to thank Max Lipton and Xinrui Zhao for providing a simple solution to Exercise 3.32(ii), please see the file Lipton-Zhao-exercise-3-32.pdf.

Another example can be obtained by modifying the function in Theorem 1.15 in the second edition.

Theorem 1 *Let* $v(x) = |x|$ *for* $x \in [-1, 1]$ *and extend* v *to* $\mathbb{R}$ *as a periodic function of period* 2. *Then the function*

$$u(x) = \sum_{n=1}^{\infty} \tfrac{1}{4^{n/p}} v(4^n x), \quad x \in \mathbb{R},$$

has finite p-variation but it is not p-absolutely continuous.

We recall that if $1 < p < \infty$, then $AC_p([a, b])$ is given by all functions $v : [a, b] \to \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left(\sum_{i=1}^{n} |v(b_i) - v(a_i)|^p \right)^{1/p} \leq \varepsilon$$

for every finite number of nonoverlapping intervals (a_i, b_i), $i = 1, \ldots, n$, with $[a_i, b_i] \subseteq I$ and

$$\left(\sum_{i=1}^{n} (b_i - a_i)^p \right)^{1/p} \leq \delta.$$

Proof. To prove that $u \notin AC_p([0, 1])$, take $x \in [0, 1]$ and $h_m = \pm \frac{1}{2} \frac{1}{4^m}$, where the sign is chosen in such a way that in the open interval of endpoints $4^m x$ and $4^m(x + h_m)$ there is no integer. Then as in the proof of Theorem 1.15, we have that

$$\begin{aligned} v_n(x + h_m) - v_n(x) &= \tfrac{1}{4^{n/p}} v(4^n(x + h_m)) - \tfrac{1}{4^{n/p}} v(4^n x) \\ &= \tfrac{1}{4^{n/p}} [v(4^n x \pm \frac{1}{2} 4^{n-m}) - v(4^n x)]. \end{aligned}$$

If $n > m$, then by periodicity the right-hand side is zero. If $n = m$, then

$$|v_m(x + h_m) - v_m(x)| = \tfrac{1}{4^{m/p}} 4^m |h_m| = \tfrac{1}{4^{m/p}} \tfrac{1}{2}.$$

Finally, if $n < m$, then using the fact that v is Lipschitz continuous with Lipschitz constant 1 we get

$$|v_n(x + h_m) - v_n(x)| \leq \tfrac{1}{4^{n/p}} 4^n |h_m| = \tfrac{1}{2} \tfrac{1}{4^{m-n+n/p}}.$$

Hence,

$$u(x + h_m) - u(x) = \sum_{n=1}^{m} (v_n(x + h_m) - v_n(x))$$

and using the inequality $|a+b| \geq |b| - |a|$, we get

$$
\begin{aligned}
|u(x+h_m) - u(x)| &= \Big| v_m(x+h_m) - v_m(x) + \sum_{n=1}^{m-1} (v_n(x+h_m) - v_n(x)) \Big| \\
&\geq |v_m(x+h_m) - v_m(x)| - \sum_{n=1}^{m-1} |v_m(x+h_m) - v_m(x)| \\
&\geq \tfrac{1}{4^{m/p}} \tfrac{1}{2} - \tfrac{1}{2} \sum_{n=1}^{m-1} \tfrac{1}{4^{m-n+n/p}} = \tfrac{1}{4^{m/p}} \tfrac{1}{2} - \tfrac{1}{2} \tfrac{1}{4^m} \sum_{n=1}^{m-1} 4^{n-n/p} \\
&= \tfrac{1}{4^{m/p}} \tfrac{1}{2} \left(1 - \tfrac{1}{4^{1-1/p}-1}\right) + \tfrac{1}{2} \tfrac{1}{4^m} \tfrac{4}{4-4^{1/p}} \\
&\geq \tfrac{1}{4^{m/p}} \tfrac{1}{2} \left(1 - \tfrac{1}{4^{1-1/p}-1}\right) = c|h_m|^{1/p}.
\end{aligned} \tag{1}
$$

Take $\ell = 4^m$, $x = a_i = \frac{i}{4^m}$, $i = 0, \ldots, 4^m - 1$, and $h_m = \frac{1}{2}\frac{1}{4^m}$ and observe that in the open interval of endpoints $4^m a_i = i$ and $4^m(a_i + h_m) = 4^m(\frac{i}{4^m} + \frac{1}{2}\frac{1}{4^m}) = i + \frac{1}{2}$ there is no integer. Moreover,

$$b_i := a_i + h_{m,i} = \frac{i}{4^m} + \frac{1}{2}\frac{1}{4^m} < \frac{i+1}{4^m} = a_{i+1}.$$

Hence, the intervals (a_i, b_i) are pairwise disjoint. Using (1),

$$\sum_{i=1}^{4^m-1} |u(b_i) - u(a_i)|^p \geq c \sum_{i=1}^{4^m-1} |b_i - a_i| = c4^{m-1}|h_m| = \frac{c}{2}\frac{4^m-1}{4^m} \to \frac{c}{2} > 0,$$

while

$$\sum_{i=1}^{4^m-1} |b_i - a_i|^p = \frac{1}{2^p}\frac{4^m-1}{4^{mp}} \to 0 \quad \text{as } m \to \infty.$$

This shows that $u \notin AC_p([0,1])$.

To see that u has finite p-variation, let's prove that it is Hölder continuous of exponent $1/p$. Given, $0 \leq x < y \leq 1$, let $m = \lfloor \log_4[1/(y-x)] \rfloor$. Since v is Lipschitz continuous with Lipschitz constant 1, we can write

$$
\begin{aligned}
|u(x) - u(y)| &\leq \sum_{n=1}^{m} \tfrac{1}{4^{n/p}} |v(4^n x) - v(4^n y)| + 2 \sum_{n=m+1}^{\infty} \tfrac{1}{4^{n/p}} \\
&\leq \sum_{n=1}^{m} 4^{n(1-1/p)}(y-x) + 2 \sum_{n=m+1}^{\infty} \tfrac{1}{4^{n/p}} \\
&\leq C4^{m(1-1/p)}(y-x) + C4^{-m/p} \\
&\leq C(y-x)^{1/p}
\end{aligned}
$$

where we used the fact that $4^m \leq 1/(y-x) < 4^{m+1}$. In turn, for any partition $0 = x_0 < \cdots < x_\ell = 1$,

$$\sum_{i=1}^{\ell} |u(x_i) - u(x_{i-1})|^p \leq C^p \sum_{i=1}^{\ell} (x_i - x_{i-1}) = C^p,$$

and so $\mathrm{Var}_p\, u \leq C$. ■