Exercise 0.0.1. Let $p > 1$ and define $u(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k/p}} \cos(2^k \pi x)$ for $x \in [0,1]$. Show that u has finite p-variation but is not p-absolutely continuous. (see *A First Course in Sobolev Spaces* by Leoni, Exercise 3.32 (ii))

From Hardy Theorem 1.32, we get that u is $\frac{1}{p}$-Hölder continuous, which also means u has bounded p-variation.

For $N \in \mathbb{N}$, let $h = 2^{-N}$, and for j between 0 and $2^N - 1$, let $x_j = j2^{-N}$. We now claim that for certain j with conditions we will later derive, there is a constant $c > 0$ independent of N and j (but not p) such that $|u(x_j + h) - u(x_j)| \geq ch^{\frac{1}{p}}$ for all N sufficiently large. To deduce that u is not in AC_p, it turns out we won't need to prove the more general claim for arbitrary x and h.

Let $u_k(x) = \frac{1}{2^{k/p}} \cos(2^k \pi x)$, so $u(x) = \sum_{k=0}^{\infty} u_k(x)$. Note that if $k > N$, $u_k(x_j + h) - u_k(x_j) = 0$ by periodicity. Thus, for the infinite series of differences, we only need to examine the finite sum up to N. We will consider the bound

$$
\begin{aligned}
|u(x_j + h) - u(x_j)| &= \left| \sum_{k=N-s}^{N} [u_k(x_j + h) - u_k(x_j)] + \sum_{k=0}^{N-s-1} [u_k(x_j + h) - u_k(x_j)] \right| \\
&\geq \left| \sum_{k=N-s}^{N} [u_k(x_j + h) - u_k(x_j)] \right| - \left| \sum_{k=0}^{N-s-1} [u_k(x_j + h) - u_k(x_j)] \right| \\
&=: A_s - B_s,
\end{aligned}
$$

where s is an integer between 0 and N which we will see depends on p.

For the first term, observe that

$$
\begin{aligned}
A_s = \Bigg| &\frac{1}{2^{N/p}} [\cos((j+1)\pi) - \cos(j\pi)] \\
&+ \frac{1}{2^{(N-1)/p}} \left[\cos((j+1)\frac{\pi}{2}) - \cos(j\frac{\pi}{2}) \right] \\
&+ \cdots + \frac{1}{2^{(N-s)/p}} \left[\cos((j+1)\frac{\pi}{2^s}) - \cos(j\frac{\pi}{2^s}) \right] \Bigg| \\
= \frac{1}{2^{N/p}} \Bigg| &[\cos((j+1)\pi) - \cos(j\pi)] \\
&+ 2^{\frac{1}{p}} \left[\cos((j+1)\frac{\pi}{2}) - \cos(j\frac{\pi}{2}) \right] \\
&+ \cdots + 2^{\frac{s}{p}} \left[\cos((j+1)\frac{\pi}{2^s}) - \cos(j\frac{\pi}{2^s}) \right] \Bigg|.
\end{aligned}
$$

From here and onwards, assume $j \equiv -1 \pmod{2^{s+1}}$. As there is some $m \in \mathbb{N}$ such that $j = m2^{s+1} - 1$, for $0 \le k \le s$, we have

$$\begin{aligned}\cos((j+1)\frac{\pi}{2^k}) - \cos(j\frac{\pi}{2^k}) &= \cos(m2^{s-k+1}\pi) - \cos(m2^{s-k+1}\pi - \frac{\pi}{2^k}) \\ &= 1 - \cos(\frac{\pi}{2^k}).\end{aligned}$$

Hence, we can conclude that on these specially chosen j,

$$\begin{aligned}A_s &= 2^{-N/p}\sum_{k=0}^{s}\left(1-\cos(\frac{\pi}{2^k})\right)2^{k/p} \\ &\ge 2^{-N/p}(\left(1-\cos(\frac{\pi}{2^0})\right)2^{0/p} + \left(1-\cos(\frac{\pi}{2^1})\right)2^{1/p}) \\ &\ge 3\cdot 2^{-N/p}.\end{aligned}$$

Next, let q be the Hölder conjugate of p satisfying $\frac{1}{p}+\frac{1}{q}=1$. For the remaining $k = 0, \dots, N-s-1$ terms, we can apply the fact that $\cos(x)$ is 1-Lipschitz to get

$$\begin{aligned}B_s &\le \sum_{k=0}^{N-s-1}\frac{1}{2^{k/p}}\left|\cos(2^{k-N}\pi(j+1)) - \cos(2^{k-N}\pi j)\right| \\ &\le \sum_{k=0}^{N-s-1}\frac{2^{k-N}\pi}{2^{k/p}} \\ &= \pi 2^{-N}\sum_{k=0}^{N-s-1}\left(2^{\frac{1}{q}}\right)^k \\ &= \pi 2^{-N}\left(\frac{2^{(N-1)/q}-1}{2^{\frac{1}{q}}-1}\right) \\ &\le \frac{\pi 2^{-N+(N-s)/q}}{2^{\frac{1}{q}}-1} \\ &= 2^{-N/p}\frac{\pi 2^{-\frac{s}{q}}}{2^{\frac{1}{q}}-1} \\ &=: h^{\frac{1}{p}}d_s\end{aligned}$$

Applying the inequality $|a+b| \ge |a| - |b|$ yields

$$\begin{aligned}|u(x_j+h) - u(x_j)| &\ge A_s - B_s \\ &\ge (3-d_s)h^{\frac{1}{p}}.\end{aligned}$$

We will thus get the desired bound should we find an s such that

$$2^{-s/q}\frac{\pi}{2^{\frac{1}{q}}-1} < 3. \tag{1}$$

Finding an $s < N$ which satisfies (1) is always possible provided N is sufficiently large. Note that once a valid N is found, the chosen s is still valid for all larger N.

Finally, let S be the set of integers in $\{0, 1, \ldots, 2^N - 1\}$ congruent to -1 modulo 2^{s+1}. There are 2^{N-s-1} such integers. Then

$$\begin{aligned}\sum_{j\in S} |u(x_j+h) - u(x_j)|^p &\geq c^p|S|h \\ &= \frac{c^p}{2^{s+1}}.\end{aligned}$$

. However, we have

$$\begin{aligned}\sum_{j\in S} ((x_j+h) - x_j)^p &= (2^{N-s-1})(2^{-Np}) \\ &= \frac{2^{-N(p-1)}}{2^{s+1}}.\end{aligned}$$

Raise both sides to the power of $\frac{1}{p}$ and send $N \to \infty$, which tends to zero. This proves $u \notin AC_p$.