# 1 On Hardy's Inequality

The proof of the following theorems is inspired by the books of Dautray and Lions and by the paper "Some problems of vector analysis and generalized formulations of boundary-value problems for the Navier-Stokes equations" by Ladyzhenskaya and V. A. Solonnikov and was written in collaboration with Ian Tice.

In what follows, $B = B(0,1)$ and given a function $u \in L^1(B)$, we set

$$u_B := \frac{1}{\mathcal{L}^N(B)} \int_B u \, dx.$$

**Theorem 1** *Let $N \geq 2$ and $1 \leq p < \infty$ and let $u \in C_c^\infty(\mathbb{R}^N)$ be such that $u = 0$ inside a ball $B(0,r)$ for some $r > 1$. Then*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{(1+\|x\|^2)^{p/2}} dx \leq \left(\frac{p}{|N-p|}\right)^p \int_{\mathbb{R}^N} \|\nabla u(x)\|^p dx$$

*for $p \neq N$ and*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^N}{(1+\|x\|^2 \log^2 \|x\|)^{N/2}} dx \leq \left(\frac{N}{N-1}\right)^N \int_{\mathbb{R}^N} \|\nabla u(x)\|^N dx \qquad (1)$$

*for $p = N$.*

**Proof. Step 1:** Assume that $p \neq N$. Using spherical coordinates

$$\int_{\mathbb{R}^N \setminus \overline{B(0,r)}} \frac{|u(x)|^p}{(1+\|x\|^2)^{p/2}} dx \leq \int_{\mathbb{R}^N \setminus \overline{B(0,r)}} \frac{|u(x)|^p}{\|x\|^p} dx$$
$$= \int_{\partial B(0,1)} \int_r^\infty \rho^{N-1-p} |u(\rho y)|^p d\rho d\mathcal{H}^{N-1}(y).$$

Fix $y \in \partial B(0,1)$ and define $f(\rho) := u(\rho y)$, $\rho > 0$. Integrating by parts and using Hölder's inequality gives

$$\int_r^\infty \rho^{N-1-p} |f(\rho)|^p d\rho = -\frac{p}{N-p} \int_r^\infty \rho^{N-p} f(\rho) |f(\rho)|^{p-2} f'(\rho) \, d\rho$$
$$\leq \frac{p}{|N-p|} \left(\int_r^\infty \rho^{N-1-p} |f(\rho)|^p d\rho\right)^{1-1/p} \left(\int_r^\infty \rho^{N-1} |f'(\rho)|^p d\rho\right)^{1/p},$$

where we wrote $\rho^{N-p} = \rho^{\frac{N-1}{p} + \frac{N-1}{p'} + 1 - p}$ and used the fact that

$$\left(\frac{N-1}{p'} + 1 - p\right) p' = N - 1 - p.$$

Since $\int_r^\infty \rho^{N-1-p} |f(\rho)|^p d\rho < \infty$, this gives

$$\int_r^\infty \rho^{N-1-p} |f(\rho)|^p d\rho \leq \left(\frac{p}{|N-p|}\right)^p \int_r^\infty \rho^{N-1} |f'(\rho)|^p d\rho.$$

In turn,

$$\int_{\mathbb{R}^N\setminus\overline{B(0,r)}} \frac{|u(x)|^p}{(1+\|x\|^2)^{p/2}}dx \le \left(\frac{p}{|N-p|}\right)^p \int_{\partial B(0,1)}\int_r^\infty \rho^{N-1}|\partial_\rho u(\rho y)|^p d\rho d\mathcal{H}^{N-1}(y)$$
$$= \left(\frac{p}{|N-p|}\right)^p \int_{\mathbb{R}^N\setminus\overline{B(0,r)}} \|\nabla u(x)\|^p dx.$$

**Step 2:** Assume that $p = N$. Using spherical coordinates

$$\int_{\mathbb{R}^N\setminus\overline{B(0,r)}} \frac{|u(x)|^N}{(1+\|x\|^2\log^2\|x\|)^{N/2}}dx \le \int_{\mathbb{R}^N\setminus\overline{B(0,r)}} \frac{|u(x)|^N}{\|x\|^N\log^N\|x\|}dx$$
$$= \int_{\partial B(0,1)}\int_r^\infty \frac{1}{\rho\log^N\rho}|u(\rho y)|^N d\rho d\mathcal{H}^{N-1}(y).$$

Fix $y \in \partial B(0,1)$ and define $f(\rho) := u(\rho y)$, $\rho > 0$. Integrating by parts and using Hölder's inequality gives

$$\int_r^\infty \frac{1}{\rho\log^N\rho}|f(\rho)|^N d\rho = -\frac{N}{N-1}\int_r^\infty \frac{1}{\log^{N-1}\rho} f(\rho)|f(\rho)|^{N-2}f'(\rho)\, d\rho$$
$$\le \frac{N}{N-1}\left(\int_r^\infty \frac{1}{\rho\log^N\rho}|f(\rho)|^N d\rho\right)^{1-1/N}\left(\int_r^\infty \rho^{N-1}|f'(\rho)|^N d\rho\right)^{1/N},$$

where we wrote $1 = \rho^{N-1-(N-1)}$ and used the fact that $N' = \frac{N}{N-1}$. Since $\int_r^\infty \frac{1}{\rho\log^N\rho}|f(\rho)|^N d\rho$, this gives

$$\int_r^\infty \frac{1}{\rho\log^N\rho}|f(\rho)|^N d\rho \le \left(\frac{N}{N-1}\right)^N \int_r^\infty \rho^{N-1}|f'(\rho)|^N d\rho.$$

In turn,

$$\int_{\mathbb{R}^N\setminus\overline{B(0,r)}} \frac{|u(x)|^N}{(1+\|x\|^2\log^2\|x\|)^{N/2}}dx \le \left(\frac{N}{N-1}\right)^N \int_{\partial B(0,1)}\int_r^\infty \rho^{N-1}|\partial_\rho u(\rho y)|^N d\rho d\mathcal{H}^{N-1}(y)$$
$$= \left(\frac{N}{N-1}\right)^N \int_{\mathbb{R}^N\setminus\overline{B(0,r)}} \|\nabla u(x)\|^N dx.$$

This concludes the proof. ■

We recall that the *homogeneous Sobolev space* $\dot{W}^{1,p}(\mathbb{R}^N)$ is the space of all functions $u \in L^1_{\rm loc}(\mathbb{R}^N)$ whose first order weak partial derivatives $\partial_i u$ belongs to $L^p(\mathbb{R}^N)$ for every $i = 1, \ldots, N$. The space $\dot{W}^{1,p}(\mathbb{R}^N)$ is equipped with the semi-norm

$$|u|_{\dot{W}^{1,p}(\mathbb{R}^N)} := \|\nabla u\|_{L^p(\mathbb{R}^N)}. \tag{2}$$

**Theorem 2** *Let $1 \le p < N$ and let $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. Then there exists a constant $m_u$ (depending on $u$) such that*

$$\int_{\mathbb{R}^N} \frac{|u(x)-m_u|^p}{(1+\|x\|^2)^{p/2}}dx \le c\int_{\mathbb{R}^N} \|\nabla u(x)\|^p dx$$

*for some constant $c = c(N,p) > 0$.*

**Proof. Step 1:** We claim that there exists a constant $c > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{(1+\|x\|^2)^{p/2}} dx \le c \int_{\mathbb{R}^N} \|\nabla u(x)\|^p dx.$$

Indeed, if not then we can find a sequence of functions $u_n \in C_c^\infty(\mathbb{R}^N)$, such that

$$\int_{\mathbb{R}^N} \|\nabla u_n(x)\|^p dx \le \frac{1}{n}, \quad \int_{\mathbb{R}^N} \frac{|u_n(x)|^p}{(1+\|x\|^2)^{p/2}} dx = 1. \tag{3}$$

Then

$$\frac{1}{5^{p/2}} \int_{B(0.2)} |u_n(x)|^p dx \le 1, \quad \int_{B(0,2)} \|\nabla u_n(x)\|^p dx \le \frac{1}{n}.$$

and so we can find a subsequence, not relabelled, such that $u_n \to u$ in $W^{1,p}(B(0,2))$ for some constant function $u$.

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $\varphi = 1$ in $B(0,1)$ and $\varphi = 0$ outside $B(0,2)$. For $v \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \|\nabla(\varphi v)\|^p dx \le 2^{p-1}\|\varphi\|_\infty^p \int_{B(0,2)} \|\nabla v\|^p dx + 2^{p-1}\|\nabla\varphi\|_\infty^p \int_{B(0,2)} |v|^p dx,$$

$$\int_{\mathbb{R}^N} \|\nabla((1-\varphi)v)\|^p dx \le 2^{p-1}\|1-\varphi\|_\infty^p \int_{\mathbb{R}^N} \|\nabla v\|^p dx + 2^{p-1}\|\nabla\varphi\|_\infty^p \int_{B(0,2)} |v|^p dx.$$

Define $v_n := \varphi u_n$ and $w_n := (1-\varphi)u_n$. By applying the previous inequalities to $v_n - v_k$ and $w_n - w_k$ and using the fact that $u_n \to u$ in $W^{1,p}(B(0,2))$ and $\nabla u_n \to 0$ in $L^p(\mathbb{R}^N; \mathbb{R}^N)$ we get that $\{\nabla v_n\}_n$ and $\{\nabla w_n\}_n$ are Cauchy sequences in $L^p(\mathbb{R}^N; \mathbb{R}^N)$ and so they converge.

In turn, by the previous theorem applied to $w_n - w_k$ and the fact that $w_n = 0$ in $B(0,1)$,

$$\int_{\mathbb{R}^N} \frac{|w_n - w_k|^p}{(1+\|x\|^2)^{p/2}} dx \le c \int_{\mathbb{R}^N} \|\nabla w_n - \nabla w_k\|^p dx \to 0$$

as $n, k \to \infty$. Thus there exists $w \in \dot{W}^{1,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \frac{|w_n - w|^p}{(1+\|x\|^2)^{p/2}} dx \to 0$$

and $\nabla w_n \to \nabla w$ in $L^p(\mathbb{R}^N; \mathbb{R}^N)$. On the other hand, since $v_n = 0$ outside of $B(0,2)$ we have that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \frac{|v_n - v|^p}{(1+\|x\|^2)^{p/2}} dx \to 0.$$

Since $u_n = v_n + w_n$, it follows that

$$\int_{\mathbb{R}^N} \frac{|u_n - (v+w)|^p}{(1+\|x\|^2)^{p/2}} dx \to 0.$$

But since $\nabla u_n \to 0$ in $L^p(\mathbb{R}^N;\mathbb{R}^N)$, we have that $v+w$ is a constant. Letting $n\to\infty$ in (3) gives

$$\int_{\mathbb{R}^N} \frac{|v+w|^p}{(1+\|x\|^2)^{p/2}}dx = 1,$$

which is a contradiction, since

$$\int_0^\infty \frac{\rho^{N-1}}{(1+\rho^2)^{p/2}}dx = \infty.$$

**Step 2:** Given $u \in \dot{W}^{1,p}(\mathbb{R}^N)$, consider a sequence of functions $u_n \in C_c^\infty(\mathbb{R}^N)$ such that $\nabla u_n \to \nabla u$ in $L^p(\mathbb{R}^N;\mathbb{R}^N)$. Then by the previous step

$$\int_{\mathbb{R}^N} \frac{|u_n-u_k|^p}{(1+\|x\|^2)^{p/2}}dx \le c\int_{\mathbb{R}^N} \|\nabla(u_n-u_k)\|^p dx$$

and so there exists $w \in \dot{W}^{1,p}(\mathbb{R}^N)$ such that $\nabla u = \nabla w$ and

$$\int_{\mathbb{R}^N} \frac{|u_n-w|^p}{(1+\|x\|^2)^{p/2}}dx \to 0.$$

Then $w = u - m_u$ for some constant $m_u$ and applying Step 1 to $u_n$ and letting $n\to\infty$ we get

$$\int_{\mathbb{R}^N} \frac{|u-m_u|^p}{(1+\|x\|^2)^{p/2}}dx \le c\int_{\mathbb{R}^N} \|\nabla u\|^p dx,$$

which concludes the proof. ■

The proof of the following theorem is due to Noah Stevenson.

**Theorem 3** *Let $p > N$ and let $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} \frac{|u(x)-u_B|^p}{(1+\|x\|^2)^{p/2}}dx \le c\int_{\mathbb{R}^N} \|\nabla u(x)\|^p dx$$

*for some constant $c = c(N,p) > 0$.*

**Proof.** Assume that $u \in C^\infty(\mathbb{R}^N)\cap \dot{W}^{1,p}(\mathbb{R}^N)$. By the fundamental theorem of calculus,

$$u(x)-u_B = \frac{1}{\mathcal{L}^N(B)}\int_B (u(x)-u(y))\,dy = \frac{1}{\mathcal{L}^N(B)}\int_B\int_0^1 \nabla u(tx+(1-t)y)\cdot(x-y)\,dtdy.$$

Using Minkowki's inequality for integrals and Hölder's inequality we obtain

$$\left(\int_{\mathbb{R}^N} |u(x)-u_B|^p \frac{dx}{(1+\|x\|^2)^{p/2}}\right)^{1/p}$$

$$\le \frac{1}{\mathcal{L}^N(B)}\left(\int_{\mathbb{R}^N}\left(\int_B\int_0^1 \|\nabla u(tx+(1-t)y)\|\|x-y\|\,dtdy\right)^p \frac{dx}{(1+\|x\|^2)^{p/2}}\right)^{1/p}$$

$$\le \frac{1}{\mathcal{L}^N(B)}\int_0^1\left(\int_{\mathbb{R}^N}\left(\int_B \|\nabla u(tx+(1-t)y)\|\|x-y\|\,dy\right)^p \frac{dx}{(1+\|x\|^2)^{p/2}}\right)^{1/p} dt$$

$$\le \frac{1}{\mathcal{L}^N(B)}\int_0^1\left(\int_{\mathbb{R}^N}\left(\int_B \|\nabla u(tx+(1-t)y)\|^p dy\right)\left(\int_B \|x-y\|^{p'}dy\right)^{p-1} \frac{dx}{(1+\|x\|^2)^{p/2}}\right)^{1/p} dt.$$

Since $B = B(0,1)$, for $y \in B$,

$$\|x-y\|^{p'} \le 2^{p'/2}(1+\|x\|^2)^{p'/2}$$

we have

$$\left(\int_B \|x-y\|^{p'}dy\right)^{p-1}\frac{1}{(1+\|x\|^2)^{p/2}} \le 2^{p/2}(\mathcal{L}^N(B))^{p-1}$$

and so by Tonelli's theorem and the change of variables $tx+(1-t)y=z$,

$$\begin{aligned}
\left(\int_{\mathbb{R}^N}|u(x)-u_B|^p\frac{dx}{(1+\|x\|^2)^{p/2}}\right)^{1/p} &\le \frac{2^{1/2}}{(\mathcal{L}^N(B))^{1/p}}\int_0^1\left(\int_{\mathbb{R}^N}\int_B\|\nabla u(tx+(1-t)y)\|^p dydx\right)^{1/p}dt \\
&= \frac{2^{1/2}}{(\mathcal{L}^N(B))^{1/p}}\int_0^1\left(\int_B\int_{\mathbb{R}^N}\|\nabla u(tx+(1-t)y)\|^p dxdy\right)^{1/p}dt \\
&= \frac{2^{1/2}}{(\mathcal{L}^N(B))^{1/p}}\int_0^1\left(\frac{1}{t^N}\int_B\int_{\mathbb{R}^N}\|\nabla u(z)\|^p dzdy\right)^{1/p}dt \\
&= 2^{1/2}\int_0^1\frac{1}{t^{N/p}}dt\left(\int_{\mathbb{R}^N}\|\nabla u(z)\|^p dz\right)^{1/p} \\
&= \frac{2^{1/2}}{1-N/p}\left(\int_{\mathbb{R}^N}\|\nabla u(z)\|^p dz\right)^{1/p}.
\end{aligned}$$

This concludes the proof. ■

In the case $p = N$ we would expect an inequality of the type

$$\int_{\mathbb{R}^N}\frac{|u(x)-u_B|^N}{(1+\|x\|^2\log^2\|x\|)^{N/2}}dx \le c\int_{\mathbb{R}^N}\|\nabla u(x)\|^N dx \tag{4}$$

for $u \in \dot{W}^{1,N}(\mathbb{R}^N)$. The following example may be found in S. Machihara, T. Ozawa, and H. Wadade, Hardy type inequalities on balls, Tohoku Math. J. (2), 65 (2013), pp. 321–330.

**Example 4** *Let*

$$f_n(r) := \begin{cases} 1 & \text{if } |\log r| \le n \\ 2-|\log r|/n & \text{if } n \le |\log r| < 2n \\ 0 & \text{if } |\log r| > 2n \end{cases}$$

*and define* $u_n(x) := f_n(\|x\|)$, $x \in \mathbb{R}^N$. *Then*

$$\begin{aligned}
\int_{\mathbb{R}^N}\|\nabla u_n\|^N dx &= \beta_N\int_0^\infty r^{N-1}|f_n'(r)|^N dr \\
&= \frac{\beta_N}{n^N}\int_{e^{-2n}}^{e^{-n}}\frac{1}{r}dr + \frac{\beta_N}{n^N}\int_{e^n}^{e^{2n}}\frac{1}{r}dr \\
&= \frac{c}{n^{N-1}} \to 0,
\end{aligned}$$

*while*

$$\int_{\mathbb{R}^N} \frac{|u_n(x)|^N}{(1+\|x\|^2\log^2\|x\|)^{N/2}}dx = \beta_N \int_0^\infty \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}|f_n(r)|^N dr \to \ell > 0.$$

*This shows that the inequality (4) cannot hold without subtracting the average* $(u_n)_B$*. On the other hand,*

$$\begin{aligned}
\frac{1}{\mathcal{L}^N(B)}\int_B u_n\,dx &= \frac{\beta_N}{\alpha_N}\int_0^1 r^{N-1}f_n(r)\,dr \\
&= N\int_{e^{-n}}^1 r^{N-1}dr + N\int_{e^{-2n}}^{e^{-n}} (2+\log r/n)r^{N-1}dr \\
&= \left[r^N\right]_{r=e^{-n}}^{r=1} + 2\left[r^N\right]_{r=e^{-2n}}^{r=e^{-n}} - \frac{1}{n}\int_{e^{-2n}}^{e^{-n}} r^N\frac{1}{r}dr + \frac{1}{n}\left[r^N\log r\right]_{r=e^{-2n}}^{r=e^{-n}} \\
&= 1 - e^{N(-n)} + 2e^{N(-n)} - 2e^{N(-2n)} - e^{N(-n)} + 2e^{N(-2n)} - \frac{1}{Nn}\left(e^{N(-n)} - e^{N(-2n)}\right) \\
&= 1 - \frac{1}{Nn}\left(e^{-nN} - e^{-2nN}\right)
\end{aligned}$$

*and so*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|u_n(x)-(u_n)_B|^N}{(1+\|x\|^2\log^2\|x\|)^{N/2}}dx \\
&= \beta_N\int_0^\infty \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}|f_n(r) - 1 - \frac{1}{Nn}\left(e^{-nN}-e^{-2nN}\right)|^N dr \\
&= \beta_N\int_{\{|\log r|\le n\}} \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}\frac{1}{Nn}\left(e^{-nN}-e^{-2nN}\right)^N dr \\
&\quad + \beta_N\int_{\{n\le|\log r|<2n\}} \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}|1-\frac{1}{n}|\log r| - \frac{1}{Nn}\left(e^{-nN}-e^{-2nN}\right)|^N dr \\
&\quad + \beta_N\int_{\{|\log r|>2n\}} \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}\left|1+\frac{1}{Nn}\left(e^{-nN}-e^{-2nN}\right)\right|^N dr = I + II + III.
\end{aligned}$$

*Note that the function* $g(r) = \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}$ *is integrable since for* $a > 1$*,*

$$\begin{aligned}
\int_a^\infty \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}dr &\le \int_a^\infty \frac{r^{N-1}}{r^N\log^N r}dr = \int_a^\infty \frac{1}{r\log^N r}dr \\
&= \left[\frac{1}{(-N+1)\log^{N-1} r}\right]_{r=a}^{r\to\infty} = \frac{1}{(N-1)\log^{N-1} a} < \infty,
\end{aligned}$$

*while for* $a < 1$*,*

$$\int_0^a \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}dr \le \int_0^a r^{N-1}dr = \frac{a^N}{N}.$$

*Hence,*

$$n^{N-1}|I| \le \frac{n^{N-1}}{Nn}\left(e^{-nN} - e^{-2nN}\right)^N \int_0^\infty g(r)\,dr \to 0.$$

*On the other hand, taking $a = e^n$ gives*

$$n^{N-1}\int_{e^n}^\infty \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}dr \le \frac{1}{(N-1)n^{N-1}},$$

*while*

$$n^{N-1}\int_0^{e^{-n}} \frac{r^{N-1}}{(1+r^2\log^2 r)^{N/2}}dr \le \frac{1}{N}n^{N-1}e^{-nN}.$$

*Hence,*

$$n^{N-1}(|II| + |III|) \le C.$$

*This shows that*

$$n^{N-1}\int_{\mathbb{R}^N} \frac{|u_n(x) - (u_n)_B|^N}{(1+\|x\|^2\log^2\|x\|)^{N/2}}dx \le C,$$

*and since $\int_{\mathbb{R}^N}\|\nabla u_n\|^N dx = \frac{c}{n^{N-1}}$, there is no contradiction to (4).*

In the following section we show that the inequality (4) holds in a (possibly smaller) subspace of $\dot{W}^{1,N}(\mathbb{R}^N)$.

# 2 A Subspace?

Let $N \ge 2$ and $1 \le p < \infty$ and let $\dot{H}^{1,p}(\mathbb{R}^N)$ be the closure of the space of functions $u \in C_c^\infty(\mathbb{R}^N)$ such that $u_B = 0$ with respect to the seminorm $\|\nabla u\|_{L^p(\mathbb{R}^N)}$, that is,

$$\dot{H}^{1,p}(\mathbb{R}^N) := \overline{\{u \in C_c^\infty(\mathbb{R}^N) :\, u_B = 0\}}^{\dot{W}^{1,p}(\mathbb{R}^N)}.$$

**Theorem 5** *Let $N \ge 2$ and $1 \le p < \infty$ and let $u \in \dot{H}^{1,p}(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} \frac{|u(x) - u_B|^p}{(1+\|x\|^2)^{p/2}}dx \le c\int_{\mathbb{R}^N}\|\nabla u(x)\|^p dx \tag{5}$$

*for $p \ne N$ and for some constant $c = c(N,p) > 0$, and*

$$\int_{\mathbb{R}^N} \frac{|u(x) - u_B|^N}{(1+\|x\|^2\log^2\|x\|)^{N/2}}dx \le c\int_{\mathbb{R}^N}\|\nabla u(x)\|^N dx$$

*for $p = N$ and for some constant $c = c(N) > 0$.*

**Proof. Step 1:** Assume $p \neq N$. Let $u \in C_c^\infty(\mathbb{R}^N)$ be such that $\int_{B(0,1)} u\,dx = 0$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be such that $\varphi = 0$ in $B(0,2)$ and $\varphi = 1$ outside $B(0,3)$. Assume $p \neq N$. By applying the previous theorem to $\varphi u$ we get

$$\int_{\mathbb{R}^N \setminus B(0,3)} \frac{|u|^p}{(1+\|x\|^2)^{p/2}} dx = \int_{\mathbb{R}^N \setminus B(0,3)} \frac{|\varphi u|^p}{(1+\|x\|^2)^{p/2}} dx \leq c \int_{\mathbb{R}^N} \|\nabla(\varphi u)\|^p dx$$
$$\leq c\|\varphi\|_\infty^p \int_{\mathbb{R}^N} \|\nabla u\|^p dx + c\|\nabla \varphi\|_\infty^p \int_{B(0,3)} |u|^p dx$$
$$\leq c \int_{\mathbb{R}^N} \|\nabla u\|^p dx,$$

where we used the fact that $\nabla \varphi = 0$ outside $B(0,3)$ and Poincaré's inequality

$$\int_{B(0,3)} |u|^p dx \leq c \int_{B(0,3)} \|\nabla u\|^p dx.$$

On the other hand,

$$\int_{B(0,3)} \frac{|u|^p}{(1+\|x\|^2)^{p/2}} dx \leq c \int_{B(0,3)} |u|^p dx \leq c \int_{B(0,3)} \|\nabla u\|^p dx$$

again by Poincaré's inequality. This proves (5) for all $u \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{B(0,1)} u\,dx = 0$.

**Step 2:** Next let $u \in \dot{H}^{1,p}(\mathbb{R}^N)$ and consider a sequence of functions $u_n \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{B(0,1)} u_n\,dx = 0$ and $\nabla u_n \to \nabla u$ in $L^p(\mathbb{R}^N;\mathbb{R}^N)$. By Poincaré's inequality applied to $u_n - u_m$ in $B(0,k)$, it follows that $\{u_n\}_n$ is a Cauchy sequence in $W^{1,p}(B(0,k))$ for every $k \in \mathbb{N}$. By completeness, there exists $w^{(k)} \in W^{1,p}(B(0,k))$ such that $u_n \to w^{(k)}$ in $W^{1,p}(B(0,k))$. By uniqueness we have that $w^{(k)} = w^{(k+1)}$ in $B(0,k)$ and thus we have a function $w \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ such that $u_n \to w$ in $W^{1,p}(B(0,k))$ for every $k$. Moreover, $\nabla w = \nabla u$. Thus $w$ differs from $u$ by a constant. Note that $\int_{B(0,1)} w\,dx = 0$, which implies that $w = u - u_B$. By selecting a subsequence, not relabelled, we can assume that $u_n \to w$ for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$. Since, by Step 1,

$$\int_{\mathbb{R}^N} \frac{|u_n|^p}{(1+\|x\|^2)^{p/2}} dx \leq c \int_{\mathbb{R}^N} \|\nabla u_n\|^p dx,$$

it follows by Fatou's lemma and the fact that $\nabla u_n \to \nabla u$ in $L^p(\mathbb{R}^N;\mathbb{R}^N)$ that

$$\int_{\mathbb{R}^N} \frac{|w|^p}{(1+\|x\|^2)^{p/2}} dx \leq c \int_{\mathbb{R}^N} \|\nabla u\|^p dx.$$

This concludes the proof in the case $p \neq N$. The case $p = N$ is similar. ■

**Problem 6** *When do the spaces $\dot{H}^{1,p}(\mathbb{R}^N)$ and $\dot{W}^{1,p}(\mathbb{R}^N)$ coincide? For $p \neq N$?*