This note was written in collaboration with Daniel Spector.

1 The Riesz Transform in $B^{1,1}$

Given $j \in \{1, \dots, n\}$ and a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, the *Riesz transform* of f is defined formally as

$$R_j(f)(x) = c_n \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy, \tag{1.1}$$

provided the limit exists. The constant c_n here is given by

$$c_n = \frac{1}{\int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^{(n+1)/2}} dx} = \Gamma((n+1)/2)/\pi^{(n+1)/2}, \tag{1.2}$$

where Γ is the Gamma function. We show that Riesz transforms is bounded from the homogeneous Besov space $\dot{B}^{1,1}(\mathbb{R}^n)$ into itself when $n \geq 2$. Note that this result is not trivial since the Riesz transform does not map $L^1(\mathbb{R}^n)$ into itself, though it is well-known. Its classical proof makes use of the Littlewood–Paley theory (see, e.g., [3] or [4, Section 5.2.2]), though to give a self-contained argument of the results of this paper without recourse to Littlewood-Paley, we give here a different proof that relies on the intrinsic seminorm of $\dot{B}^{1,1}(\mathbb{R}^n)$ and is based on an argument of Devore, Riemenschneider, Sharpley [1].

Theorem 1.1 *For every $f \in B^{1,1}(\mathbb{R}^n)$,*

$$|R_j(f)|_{B^{1,1}(\mathbb{R}^n)} \preceq |f|_{B^{1,1}(\mathbb{R}^n)}.$$

Remark 1.2 *We observe that if $f \in B^{1,1}(\mathbb{R}^n)$, then $f \in W^{1,1}(\mathbb{R}^n)$ (see [2, Theorem 17.66]). If $n = 1$, this implies that $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and in turn, $f \in L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$. On the other hand, if $n \geq 2$, then by the Sobolev–Gagliardo–Nirenberg embedding theorem, we have $f \in L^{n/(n-1)}(\mathbb{R}^n)$. In both cases the Riesz transform of f is well-defined.*

Throughout this note, the expression

$$\mathcal{A} \preceq \mathcal{B} \quad \text{means } \mathcal{A} \leq C\mathcal{B}$$

for some constant $C > 0$ that depends on the parameters quantified in the statement of the result (usually n and p), but not on the functions and their domain of integration.

Definition 1.3 *Given $1 \leq p, q < \infty$ and $0 < s \leq 1$, we say that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the* homogeneous Besov space *$\dot{B}^{s,p}_q(\mathbb{R}^n)$ if*

$$|f|_{B^{s,p}_q(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \|\Delta_h^{\lfloor s \rfloor + 1} f\|^q_{L^p(\mathbb{R}^n)} \frac{dh}{|h|^{n+sq}} \right)^{1/q} < \infty,$$

where $\lfloor s \rfloor$ *is the integer part of* s*. The (non-homogeneous)* Besov space $B_q^{s,p}(\mathbb{R}^n)$ *is the space of all functions* $f \in L^p(\mathbb{R}^n) \cap \dot{B}_q^{s,p}(\mathbb{R}^n)$ *endowed with norm*

$$\|f\|_{B_q^{s,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + |f|_{B_q^{s,p}(\mathbb{R}^n)}.$$

In what follows we will use the equivalent seminorm for $\dot{B}^{1,1}(\mathbb{R}^n)$:

$$|f|^{\infty}_{B^{1,1}(\mathbb{R}^n)} := \int_0^{\infty} \sup_{|h|\leq r} \|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)} \frac{dr}{r^2}$$

(see [2, Proposition 17.17]).

Next, we recall some basic properties of the Riesz transform.

Proposition 1.4 *Let* $1 \leq p < \infty$ *and let* $f \in L^p(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n)$*. Then* $R_j(f)$ *is well-defined with*

$$R_j(f)(x) = \int_{\mathbb{R}^n \setminus B(0,1)} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy + \int_{B(0,1)} [f(x-y) - f(x)] \frac{y_j}{|y|^{n+1}}\, dy. \tag{1.3}$$

Proof. Since $\frac{x_j}{|x|^{n+1}}$ is an odd function,

$$\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy = \int_{\mathbb{R}^n \setminus B(0,1)} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy + \int_{B(0,1)\setminus B(0,\varepsilon)} [f(x-y) - f(x)] \frac{y_j}{|y|^{n+1}}\, dy := I + II.$$

If $p = 1$, then the term I is well-defined since f is integrable, while if $p > 1$, we can use Hölder's inequality to get

$$|I| \leq \|f\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|y|^{np'}}\, dy \right)^{1/p'} < \infty.$$

On the other hand, since $f \in C^{\infty}(\mathbb{R}^n)$,

$$\left| (f(x-y) - f(x)) \frac{y_j}{|y|^{n+1}} \right| \leq \|\nabla f\|_{L^{\infty}(B(x,1))} \frac{1}{|y|^{n-1}},$$

and since the function on the right-hand side is integrable in $B(0,1)$, we can apply the Lebesgue dominated convergence theorem and a change of variables to conclude that (1.3) holds. ■

Fix $t > 0$ and let $\psi \in C_c^{\infty}(\mathbb{R})$ be a nonnegative function such that $\operatorname{supp} \psi = [\frac{1}{2}, 2]$ and

$$\sum_{k=-\infty}^{\infty} \psi_k(x) = 1 \quad \text{for all } x \in \mathbb{R}^n, \tag{1.4}$$

where

$$\psi_k(x) := \psi\left(\frac{|x|}{2^k t}\right). \tag{1.5}$$

Note that

$$\operatorname{supp}\psi_k = \overline{B(0,2^{k+1}t)} \setminus B(0,2^{k-1}t). \tag{1.6}$$

Lemma 1.5 *Let ψ_k be given by (1.5) and define*

$$a_k(x) := \psi_k(x)\frac{x_j}{|x|^{n+1}}, \tag{1.7}$$

and

$$b_k(x) := 2a_k(x) - \frac{1}{2^n}a_k\left(\frac{x}{2}\right) = \left(2\psi_k(x) - \psi_k\left(\frac{x}{2}\right)\right)\frac{x_j}{|x|^{n+1}}. \tag{1.8a}$$

Then

$$\sum_{k=-\infty}^{\infty} b_k(x) = \frac{x_j}{|x|^{n+1}} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.9}$$

Moreover,

$$\int_{\mathbb{R}^n} |a_k(x)|\,dx \preceq 1, \quad \int_{\mathbb{R}^n} |b_k(x)|\,dx \preceq 1, \tag{1.10}$$

and for every $h \in \mathbb{R}^n$ with $|h| \le t$,

$$\int_{\mathbb{R}^n} |\Delta_h^2 a_k(x)|\,dx \preceq \frac{1}{2^{2k}}, \quad \int_{\mathbb{R}^n} |\Delta_h^2 b_k(x)|\,dx \preceq \frac{1}{2^{2k}}. \tag{1.11}$$

Proof. Property (1.9) follows from (1.4) and (1.8a). By (1.5) and (1.6),

$$\begin{aligned}\int_{\mathbb{R}^n} |a_k(x)|\,dx &\le \int_{\overline{B(0,2^{k+1}t)}\setminus B(0,2^{k-1}t)} \psi\left(\frac{|x|}{2^k t}\right)\frac{1}{|x|^n}\,dx \\ &= \int_{\overline{B(0,2)}\setminus B(0,2^{-1})} \psi(|z|)\frac{1}{|z|^n}\,dz \le \beta_n\|\psi\|_\infty \int_{2^{-1}}^{2}\frac{1}{r}\,dr,\end{aligned}$$

where we made the change of variables $z = x/(2^k t)$. This proves (1.10).

On the other hand, by the mean value theorem applied twice, the product rule, and (1.6),

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta_h^2 a_k(x)|\,dx &\preceq |h|^2 \sum_{l=0}^{2}\int_{\mathbb{R}^n}\frac{1}{(2^k t)^l}\left|\nabla^l\psi\left(\frac{|x+\delta h|}{2^k t}\right)\right|\frac{1}{|x+\delta h|^{n+2-l}}\,dx \\ &= \frac{|h|^2}{(2^k t)^2}\sum_{l=0}^{2}\int_{\overline{B(0,2)}\setminus B(0,2^{-1})}\left|\nabla^l\psi(|z|)\right|\frac{1}{|z|^{n+2-l}}\,dz \preceq \frac{1}{2^{2k}}\end{aligned}$$

for some $0 < \delta < 2$ (depending on h and x). Hence, (1.11) holds. ■

Lemma 1.6 *Let b_k be defined as in Lemma 1.5 and let $1 \le p < \infty$. Then for every $f \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$,*

$$R_j(f)(x) = \sum_{k=-\infty}^{\infty} (b_k * f)(x). \tag{1.12}$$

Proof. By (1.4) and (1.8a),

$$\sum_{k=-\infty}^{\infty} |b_k(y)| \le \frac{1}{|y|^n} \sum_{k=-\infty}^{\infty} \left(2\psi_k(y) + \psi_k\left(\frac{y}{2}\right)\right) \le \frac{3}{|y|^n}. \tag{1.13}$$

Hence, if $p = 1$, then

$$\int_{\mathbb{R}^n \setminus B(0,1)} |f(x-y)| \sum_{k=-\infty}^{\infty} |b_k(y)|\, dy \le 3 \int_{\mathbb{R}^n \setminus B(0,1)} |f(x-y)|\, dy < \infty,$$

while if $p > 1$, we can use Hölder's inequality to get

$$\int_{\mathbb{R}^n \setminus B(0,1)} |f(x-y)| \sum_{k=-\infty}^{\infty} |b_k(y)|\, dy \le 3\|f\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|y|^{np'}}\, dy\right)^{1/p'} < \infty,$$

and so, by (1.9) we can write

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n \setminus B(0,1)} f(x-y) b_k(y)\, dy &= \int_{\mathbb{R}^n \setminus B(0,1)} f(x-y) \sum_{k=-\infty}^{\infty} b_k(y)\, dy \qquad (1.14) \\ &= \int_{\mathbb{R}^n \setminus B(0,1)} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy. \end{aligned}$$

Since b_k is odd and in view of (1.6),

$$\int_{B(0,1)} f(x-y) b_k(y)\, dy = \int_{B(0,1)} [f(x-y) - f(x)] b_k(y)\, dy.$$

Using the fact that $f \in C^\infty(\mathbb{R}^n)$,

$$|f(x-y) - f(x)| \sum_{k=-\infty}^{\infty} |b_k(y)| \le 3\|\nabla f\|_{L^\infty(B(x,1))} \frac{1}{|y|^{n-1}}.$$

Since the function on the right-hand side is integrable in $B(0,1)$, by (1.9), we have that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \int_{B(0,1)} f(x-y) b_k(y)\, dy &= \sum_{k=-\infty}^{\infty} \int_{B(0,1)} [f(x-y) - f(x)] b_k(y)\, dy \\ &= \int_{B(0,1)} [f(x-y) - f(x)] \sum_{k=-\infty}^{\infty} b_k(y)\, dy \\ & \qquad\qquad\qquad\qquad (1.15a) \\ &= \int_{B(0,1)} [f(x-y) - f(x)] \frac{y_j}{|y|^{n+1}}\, dy. \end{aligned}$$

Summing the two convergent series in (1.14) and (1.15a) and using (1.3) gives (1.12). ■

We turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Step 1: Given $f \in B^{1,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, define $T_k(f) := b_k * f$. By (1.8a),

$$\begin{aligned} T_k(f)(x) &= \frac{1}{2^n}\int_{\mathbb{R}^n} a_k\left(\frac{y}{2}\right) f(x-y)\,dy - 2\int_{\mathbb{R}^n} a_k(y)\, f(x-y)\,dy \\ &= \int_{\mathbb{R}^n} a_k(z)\,[f(x-2z) - 2f(x-z) + f(x)]\,dz = \int_{\mathbb{R}^n} a_k(z)\,\Delta^2_{-z} f(x)\,dz, \end{aligned} \tag{1.16}$$

where we made the change of variables $z = y/2$ and used the fact that $\int_{\mathbb{R}^n} a_k(z)\,dz = 0$ since ψ_k is even and $\frac{z_j}{|z|^{n+1}}$ odd.

If $k \le 0$, by a change of variables, Tonelli's theorem, (1.6), (1.10), and (1.16), we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\Delta_h^2 T_k(f)(x)|\,dx &\preceq \int_{\mathbb{R}^n} |T_k(f)(x)|\,dx \le \int_{\mathbb{R}^n}\int_{\overline{B(0,2^{k+1}t)}\setminus B(0,2^{k-1}t)} |a_k(z)|\,|\Delta^2_{-z} f(x)|\,dzdx \\ &\preceq \sup_{|z|\le 2^{k+1}t} \int_{\mathbb{R}^n} |\Delta^2_{-z} f(x)|\,dx. \end{aligned} \tag{1.17}$$

If $k > 0$, for $\tau > 0$ and $x \in \mathbb{R}^n$ write

$$\begin{aligned} f(x) &= \frac{1}{\tau^{2n}}\int_{Q(0,\tau)}\int_{Q(0,\tau)} \Delta^2_{y+z} f(x)\,dydz \\ &\quad - \frac{1}{\tau^{2n}}\int_{Q(0,\tau)}\int_{Q(0,\tau)} (f(x+2(y+z)) - 2f(x+y+z))\,dydz =: v_\tau(x) + w_\tau(x). \end{aligned} \tag{1.18}$$

Then, by Tonelli's theorem

$$\int_{\mathbb{R}^n} |v_\tau(x)|\,dx \le \frac{1}{\tau^{2n}}\int_{Q(0,\tau)}\int_{Q(0,\tau)}\int_{\mathbb{R}^n} |\Delta^2_{y+z} f(x)|\,dxdydz \le \sup_{|h|\le 2\sqrt{n}\tau}\int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\,dx. \tag{1.19}$$

Moreover, by [2, Step 3 of the proof of Theorem 17.24],

$$\|\nabla^2 w_\tau\|_{L^1(\mathbb{R}^n)} \preceq \tau^{-2} \sup_{|h|\le c\tau} \|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)}.$$

By the mean value theorem applied twice, a change of variables, and the previous inequality,

$$\|\Delta_h^2 w_\tau\|_{L^1(\mathbb{R})} \preceq |h|^2 \|\nabla^2 w_\tau\|_{L^1(\mathbb{R}^n)} \preceq \frac{|h|^2}{\tau^2} \sup_{|h|\le c\tau} \|\Delta_h^2 f\|_{L^1(\mathbb{R}^n)}. \tag{1.20}$$

Write $\Delta_h^2 T_k(f) = (\Delta_h^2 b_k) * v_\tau + b_k * (\Delta_h^2 w_\tau)$. Then by Tonelli's theorem, a change of variables, and (1.11), and (1.19),

$$\int_{\mathbb{R}^n} |(\Delta_h^2 b_k) * v_\tau)(x)|\,dx \preceq \int_{\mathbb{R}^n} |\Delta_h^2 b_k(z)|dz \int_{\mathbb{R}^n} |v_\tau(y)|\,dy \preceq \frac{1}{2^{2k}} \sup_{|h|\le 2\sqrt{n}\tau} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\,dx. \tag{1.21}$$

Similarly, by Tonelli's theorem, a change of variables, and (1.11), and (1.20),

$$\int_{\mathbb{R}^n} |(b_k*(\Delta_h^2 w_\tau))(x)|\, dx \preceq \int_{\mathbb{R}^n} |b_k(z)| dz \int_{\mathbb{R}^n} |\Delta_h^2 w_\tau(y)|\, dy \preceq \frac{|h|^2}{\tau^2} \sup_{|h|\le c\tau} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx. \tag{1.22}$$

Combining (1.21) and (1.22) and taking $\tau = C2^{k+1}t$, gives

$$\int_{\mathbb{R}^n} |\Delta_h^2 T_k(f)(x)|\, dx \preceq \frac{|h|^2}{(2^k t)^2} \sup_{|h|\le 2^{k+1}t} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx. \tag{1.23}$$

It follows from (1.17) and (1.23) that

$$\sup_{|h|\le t} \int_{\mathbb{R}^n} |\Delta_h^2 T_k(f)(x)|\, dx \preceq \min\{1, 2^{-2k}\} \sup_{|h|\le 2^{k+1}t} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx.$$

Since the function $r \mapsto \sup_{|h|\le r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx$ is increasing, for $k < 0$ we have

$$\sup_{|h|\le 2^{k+1}t} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \preceq \int_{2^k t}^{2^{k+1}t} \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r}$$

while for $k \ge 0$,

$$\frac{1}{(2^k t)^2} \sup_{|h|\le 2^{k+1}t} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \preceq \int_{2^k t}^{2^{k+1}t} \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r^3}.$$

In turn,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \sup_{|h|\le t} \int_{\mathbb{R}^n} |\Delta_h^2 T_k(f)(x)|\, dx &\le \sum_{k=-\infty}^{\infty} \min\{1, 2^{-2k}\} \sup_{|h|\le 2^{k+1}t} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \\ &\preceq \int_0^t \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r} + \int_t^\infty t^2 \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r^3}.\end{aligned}$$

By (1.12), for $|h| \le t$,

$$\int_{\mathbb{R}^n} |\Delta_h^2 R_j(f)(x)|\, dx \le \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} |\Delta_h^2 T_k(f)(x)|\, dx \le \sum_{k=-\infty}^{\infty} \sup_{|h|\le t} \int_{\mathbb{R}^n} |\Delta_h^2 T_k(f)(x)|\, dx.$$

Combining these two inequalities gives

$$\begin{aligned}\sup_{|h|\le t} \int_{\mathbb{R}^n} |\Delta_h^2 R_j(f)(x)|\, dx &\le \sum_{k=-\infty}^{\infty} \sup_{|h|\le t} \int_{\mathbb{R}^n} |\Delta_h^2 T_k(f)(x)|\, dx \\ &\le \int_0^t \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r} + t^2 \int_t^\infty \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r^3}.\end{aligned}$$

Hence,

$$
\begin{aligned}
|R_j(f)|_{B^{1,1}(\mathbb{R}^n)} &= \int_0^\infty \sup_{|h|\le t} \int_{\mathbb{R}^n} |\Delta_h^2 R(f)(x)| \frac{dt}{t^2} \\
&\le \int_0^\infty \int_0^t \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r} \frac{dt}{t^2} + \int_0^\infty \int_t^\infty \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r^3} dt \\
&\le 2 \int_0^\infty \sup_{|h|\le 2r} \int_{\mathbb{R}^n} |\Delta_h^2 f(x)|\, dx \frac{dr}{r^2},
\end{aligned}
$$

where in the last inequality we used Hardy's inequalities ([2, Theorem C.41]).

Step 2: Given $f \in B^{1,1}(\mathbb{R}^n)$, in view of Remark 1.2, we can find $1 < p < \infty$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} \preceq \|f\|_{B^{1,1}(\mathbb{R}^n)}.$$

Let $f_\varepsilon \in B^{1,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ be a mollification of f. By Step 1 and Tonelli's theorem

$$|R_j(f_\varepsilon)|_{B^{1,1}(\mathbb{R}^n)} \preceq |f_\varepsilon|_{B^{1,1}(\mathbb{R}^n)} \le |f|_{B^{1,1}(\mathbb{R}^n)}. \tag{1.24}$$

On the other hand, by the boundedness of the Riesz transform in L^p we have that $R_j(f_\varepsilon) \to R_j(f)$ in $L^p(\mathbb{R}^n)$. By extracting a subsequence, we can assume that $R_j(f_\varepsilon) \to R_j(f)$ pointwise a.e. in $\mathbb{R}^n$. Letting $\varepsilon \to 0^+$ in (1.24) and using Fatou's lemma, we conclude the proof. ■

References

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