1 Slicing in $W^{m,p}(\Omega)$

In what follows, we use the notation (E.2) in Appendix A. Given $x_i' \in \mathbb{R}^{N-1}$ and a set $E \subseteq \mathbb{R}^N$, we write

$$E_{x_i'} := \{x_i \in \mathbb{R} : (x_i', x_i) \in E\}. \tag{1}$$

Moreover, if $v : \Omega \to \mathbb{R}$ is Lebesgue integrable, with a slight abuse of notation, if $\Omega_{x_i'}$ is empty, we set

$$\int_{\Omega_{x_i'}} v(x_i', x_i)\, dx_i := 0,$$

so that by Fubini's theorem

$$\int_\Omega v(x)\, dx = \int_{\mathbb{R}^{N-1}} \Big(\int_{\Omega_{x_i'}} v(x_i', x_i)\, dx_i \Big) dx_i'. \tag{2}$$

Theorem 1 (Absolute continuity on lines) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \le p < \infty$. A function $u \in L^p(\Omega)$ belongs to the space $W^{2,p}(\Omega)$ if and only if there exist functions $\bar{u} : \Omega \to \mathbb{R}$, $\bar{v}_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, N$, such that*

(i) $\overline{u}$ is a precise representative of u,

(ii) for all $i = 1, \ldots, N$ and $\mathcal{L}^{N-1}$-a.e. $x_i' \in \mathbb{R}^{N-1}$ the function $\overline{u}(x_i', \cdot)$ is of class C^1 in $\Omega_{x_i'}$ with $\partial_i \overline{u}(x_i', x_i) = \bar{v}_i(x_i', x_i)$ for all $x_i \in \Omega_{x_i'}$,

(iii) for all $i, j = 1, \ldots, N$ and $\mathcal{L}^{N-1}$-a.e. $x_i' \in \mathbb{R}^{N-1}$ the function $\bar{v}_j(x_i', \cdot)$ is absolutely continuous in $\Omega_{x_i'}$,,

(iv) $\bar{v}_j$ and its first-order (classical) partial derivatives belong to $L^p(\Omega)$.

Moreover the (classical) partial derivatives of $\overline{u}$ agree $\mathcal{L}^N$-a.e. with the weak derivatives of u.

Proof. Step 1: Assume that $u \in W^{2,p}(\Omega)$. Consider a sequence of standard mollifiers $\{\varphi_\varepsilon\}_{\varepsilon>0}$ and for every $\varepsilon > 0$ define $u_\varepsilon := u * \varphi_\varepsilon$ in $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. By Lemma 11.25,

$$\lim_{\varepsilon \to 0^+} \int_{\Omega_\varepsilon} \|\nabla^2 u_\varepsilon(x) - \nabla^2 u(x)\|^p dx = 0.$$

It follows by Fubini's theorem and (2) that for all $i = 1, \ldots, N$,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{N-1}} \Big(\int_{(\Omega_\varepsilon)_{x_i'}} \|\nabla^2 u_\varepsilon(x_i', x_i) - \nabla^2 u(x_i', x_i)\|^p dx_i \Big)\, dx_i' = 0,$$

where $(\Omega_\varepsilon)_{x_i'} := \{x_i \in \mathbb{R} : (x_i', x_i) \in \Omega_\varepsilon\}$, and so we may find a subsequence $\{\varepsilon_n\}_n$ such that for all $i = 1, \ldots, N$ and for $\mathcal{L}^{N-1}$ a.e. $x_i' \in \mathbb{R}^{N-1}$,

$$\lim_{n \to \infty} \int_{(\Omega_{\varepsilon_n})_{x_i'}} \|\nabla^2 u_\varepsilon(x_i', x_i) - \nabla^2 u(x_i', x_i)\|^p dx_i = 0. \tag{3}$$

Set $u_n := u_{\varepsilon_n}$ and $E := \{x \in \Omega : \lim_{n\to\infty} u_n(x)$ exists in $\mathbb{R}$ and $\lim_{n\to\infty} \nabla u_n(x)$ exists in $\mathbb{R}^N\}$. Define

$$\overline{u}(x) := \begin{cases} \lim_{n\to\infty} u_n(x) & \text{if } x \in E, \\ 0 & \text{otherwise,} \end{cases} \qquad \bar{v}_j(x) := \begin{cases} \lim_{n\to\infty} \partial_j u_n(x) & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Since by Theorem C.16, $\{u_n\}_n$ and $\{\nabla u_n\}_n$ converge pointwise to u and ∇u at every Lebesgue point of u and ∇u, the set E contains every point which is a Lebesgue point of both u and ∇u. It follows from Corollary B.119 that $\mathcal{L}^N(\Omega \setminus E) = 0$, and so $\overline{u}$ is a representative of u and $\bar{v}_j$ is a representative of $\partial_j u$. It remains to prove that $\overline{u}$ and $\bar{v}_j$ have the desired properties.

By Fubini's theorem for every $i = 1, \ldots, N$ we have that

$$\int_{\mathbb{R}^{N-1}} \left(\int_{\Omega_{x_i'}} \|\nabla^2 u(x_i', x_i)\|^p dx_i \right) dx_i' < \infty$$

and

$$\int_{\mathbb{R}^{N-1}} \mathcal{L}^1(\{x_i \in \Omega_{x_i'} : (x_i', x_i) \notin E\})\, dx_i' = 0,$$

and so we may find a set $N_i \subset \mathbb{R}^{N-1}$, with $\mathcal{L}^{N-1}(N_i) = 0$, such that for all $x_i' \in \mathbb{R}^{N-1} \setminus N_i$ for which $\Omega_{x_i'}$ is nonempty we have that

$$\int_{\Omega_{x_i'}} \|\nabla^2 u(x_i', x_i)\|^p dx_i < \infty, \tag{4}$$

(3) holds for all $i = 1, \ldots, N$ and $(x_i', x_i) \in E$ for $\mathcal{L}^1$ a.e. $x_i \in \Omega_{x_i'}$.

Fix any such x_i' and let $I \subseteq \Omega_{x_i'}$ be a maximal interval. Fix $t_0 \in I$ such that $(x_i', t_0) \in E$ and let $t \in I$. For all n large, the interval of endpoints t and t_0 is contained in $(\Omega_{\varepsilon_n})_{x_i'}$ and so, since $u_n \in C^\infty(\Omega_{\varepsilon_n})$, by Taylor's formula with integral remainder,

$$u_n(x_i', t) = u_n(x_i', t_0) + \partial_i u_n(x_i', t_0)(t - t_0) + \frac{1}{2} \int_{t_0}^t (t - s) \partial_i^2 u_n(x_i', s)\, ds,$$

$$\partial_j u_n(x_i', t) = \partial_j u_n(x_i', t_0) + \int_{t_0}^t \partial_{ij}^2 u_n(x_i', s)\, ds.$$

Since $(x_i', t_0) \in E$, we have $u_n(x_i', t_0) \to \overline{u}(x_i', t_0) \in \mathbb{R}$ and $\partial_j u_n(x_i', t) \to \overline{v}_j(x_i', t_0) \in \mathbb{R}$. On the other hand, by (3),

$$\lim_{n\to\infty} \int_{t_0}^t |\partial_{ij}^2 u_n(x_i', s) - \partial_{ij}^2 u(x_i', s)|\, ds = 0. \tag{5}$$

Hence as $n \to \infty$,

$$u_n(x_i', t) \to \overline{u}(x_i', t_0) + \overline{v}_i(x_i', t_0)(t - t_0) + \frac{1}{2} \int_{t_0}^t (t - s) \partial_i^2 u(x_i', s)\, ds,$$

$$\partial_j u_n(x_i', t) \to \overline{v}_j(x_i', t_0) + \int_{t_0}^t \partial_{ij}^2 u(x_i', s)\, ds.$$

Note that by the definition of E, $\overline{u}$, and $\overline{v}_j$, this implies, in particular, that $(x_i', t) \in E$ with

$$\overline{u}(x_i', t) = \overline{u}(x_i', t_0) + \overline{v}_i(x_i', t_0)(t - t_0) + \frac{1}{2}\int_{t_0}^{t}(t-s)\partial_i^2 u(x_i', s)\, ds, \qquad (6)$$
$$\overline{v}_j(x_i', t) = \overline{v}_j(x_i', t_0) + \int_{t_0}^{t} \partial_{ij}^2 u(x_i', s)\, ds$$

for *all* $t \in I$. It follows that the identities in (6) hold for all $t, t_0 \in I$. Hence, by Theorem 3.16, the function $\overline{v}_j(x_i', \cdot)$ are absolutely continuous in I with $\partial_i \overline{v}_j(x_i', t) = \partial_{ij}^2 u(x_i', t)$ for $\mathcal{L}^1$ a.e. $t \in I$. In particular, $\overline{v}_i(x_i', \cdot)$ is continuous in I and so $\overline{u}(x_i', \cdot)$ is of class C^1 with $\partial_i \overline{u}(x_i', t) = \overline{v}_i(x_i', t)$ for all $t \in I$.
Step 2: Assume that there exist functions $\overline{u} : \Omega \to \mathbb{R}$, $\overline{v}_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, N$, such that $\overline{u}$ is a precise representative of u, for all $i = 1, \ldots, N$ and $\mathcal{L}^{N-1}$-a.e. $x_i' \in \mathbb{R}^{N-1}$ the function $\overline{u}(x_i', \cdot)$ is of class C^1 in $\Omega_{x_i'}$, $\overline{v}_j(x_i', \cdot)$ is absolutely continuous in $\Omega_{x_i'}$ for all $j = 1, \ldots, N$, the classical derivative of $\overline{u}(x_i', \cdot)$ is $\overline{v}_i(x_i', \cdot)$, and $\overline{v}_j$ and its first-order (classical) partial derivatives belong to $L^p(\Omega)$ Fix $i = 1, \ldots, N$ and let $x_i' \in \mathbb{R}^{N-1}$ be such that $\overline{u}(x_i', \cdot)$ is of class C^1 in $\Omega_{x_i'}$ and $\overline{v}_j(x_i', \cdot)$ absolutely continuous on every connected component of the open set $\Omega_{x_i'}$ for every $j = 1, \ldots, N$. Then for every function $\varphi \in C_c^\infty(\Omega)$, by the integration by parts formula for absolutely continuous functions, we have

$$\int_{\Omega_{x_i'}} \overline{u}(x_i', t)\partial_i \varphi(x_i', t)\, dt = -\int_{\Omega_{x_i'}} \partial_i \overline{u}(x_i', t)\varphi(x_i', t)\, dt,$$
$$\int_{\Omega_{x_i'}} \overline{v}_j(x_i', t)\partial_i \varphi(x_i', t)\, dt = -\int_{\Omega_{x_i'}} \partial_i \overline{v}_j(x_i', t)\varphi(x_i', t)\, dt.$$

Since this holds for $\mathcal{L}^{N-1}$ a.e. $x_i' \in \mathbb{R}^{N-1}$, integrating over $\mathbb{R}^{N-1}$ and using Fubini's theorem yields

$$\int_{\Omega} \overline{u}(x)\partial_i \varphi(x)\, dx = -\int_{\Omega} \partial_i \overline{u}(x)\varphi(x)\, dx,$$
$$\int_{\Omega} \partial_j \overline{u}(x)\partial_i \varphi(x)\, dx = -\int_{\Omega} \partial_i \overline{v}_j(x)\varphi(x)\, dx$$

which implies that $\partial_i \overline{u} \in L^p(\Omega)$ is the weak partial derivative of $\overline{u}$ with respect to x_i and that $\partial_i \overline{v}_j \in L^p(\Omega)$ is the weak partial derivative of $\partial_j \overline{u}$ with respect to x_i. This shows that $u \in W^{2,p}(\Omega)$. ■

A similar result holds in the homogeneous Sobolev space $\dot{W}^{2,p}(\Omega)$.

Theorem 2 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \le p < \infty$. A function $u \in L^p_{\mathrm{loc}}(\Omega)$ belongs to the space $\dot{W}^{2,p}(\Omega)$ if and only if there exist functions $\overline{u} : \Omega \to \mathbb{R}$, $\overline{v}_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, N$, such that $\overline{u}$ is a precise representative of u, for all $i = 1, \ldots, N$ and $\mathcal{L}^{N-1}$-a.e. $x_i' \in \mathbb{R}^{N-1}$ the function $\overline{u}(x_i', \cdot)$ is of class C^1 in $\Omega_{x_i'}$, $\overline{v}_j(x_i', \cdot)$ is locally absolutely continuous in $\Omega_{x_i'}$ for all $j = 1, \ldots, N$, the classical derivative of $\overline{u}(x_i', \cdot)$ is $\overline{v}_i(x_i', \cdot)$, and the first-order (classical) partial* *derivatives of $\bar{v}_j$ belong to $L^p(\Omega)$. Moreover the (classical) partial derivatives of $\overline{u}$ agree $\mathcal{L}^N$-a.e. with the weak derivatives of u.*

Proof. The proof is the same as the previous theorem. ■

Exercise 3 *State and prove the analog of Theorem 1 for $W^{m,p}(\Omega)$.*

When $p > 1$ and $\Omega = \mathbb{R}^N$ we can use the elliptic regularity of solutions of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^N$ to prove a stronger result.

Theorem 4 *Let $1 < p < \infty$. A function $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ belongs to the space $\dot{W}^{2,p}(\mathbb{R}^N)$ if and only if u admits precise representatives $\overline{u}$ and such that for all $i = 1, \ldots, N$ and $\mathcal{L}^{N-1}$-a.e. $x_i' \in \mathbb{R}^{N-1}$ the function $\overline{u}(x_i', \cdot) \in \dot{W}^{2,p}(\mathbb{R})$ with*

$$\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} |\partial_i^2 \bar{u}(x_i', x_i)|^p dx_i dx_i' < \infty. \tag{7}$$

Moreover,

$$\int_{\mathbb{R}^N} \|\nabla^2 \bar{u}(x)\|^p dx \leq C \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i^2 \bar{u}(x)|^p dx$$

for some constant $C = C(N, p) > 0$.

Proof. If $u \in \dot{W}^{2,p}(\mathbb{R}^N)$, the result follows from Theorem 2. Assume that $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ admits precise representatives $\overline{u}$ and such that for all $i = 1, \ldots, N$ and $\mathcal{L}^{N-1}$-a.e. $x_i' \in \mathbb{R}^{N-1}$ the function $\overline{u}(x_i', \cdot) \in \dot{W}^{2,p}(\mathbb{R})$ and (7) holds. Given $R > 0$, since $\overline{u} \in L^p((-R, R)^N)$, by Fubini's theorem we have that $\overline{u}(x_i', \cdot) \in L^p((-R, R))$ for $\mathcal{L}^{N-1}$-a.e. $x_i' \in (-R, R)^{N-1}$. Fix $x_i' \in (-R, R)^{N-1}$ such that $\overline{u}(x_i', \cdot) \in L^p((-R, R))$ and $\overline{u}(x_i', \cdot) \in \dot{W}^{2,p}(\mathbb{R})$. By Gagliardo–Nirenmberg's inequality (see ??),

$$\int_{-R}^{R} |\partial_i \overline{u}(x_i', x_i)|^p dx_i \leq c R^{-p} \int_{-R}^{R} |\overline{u}(x_i', x_i)|^p dx_i + c R^p \int_{-R}^{R} |\partial_i^2 \overline{u}(x_i', x_i)|^p dx_i.$$

Integrating over $(-R, R)^{N-1}$ and using Tonelli's theorem gives

$$\int_{(-R,R)^N} |\partial_i \overline{u}(x)|^p dx \leq c R^{-p} \int_{(-R,R)^N} |\overline{u}(x)|^p dx + c R^p \int_{(-R,R)^N} |\partial_i^2 \overline{u}(x)|^p dx,$$

which shows that $\partial_i \bar{u} \in L^p_{\text{loc}}(\mathbb{R}^N)$.

Then for every function $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}} \overline{u}(x_i', t) \partial_i \varphi(x_i', t)\, dt = -\int_{\mathbb{R}} \partial_i \overline{u}(x_i', t) \varphi(x_i', t)\, dt,$$
$$\int_{\mathbb{R}} \overline{u}(x_i', t) \partial_i^2 \varphi(x_i', t)\, dt = \int_{\mathbb{R}} \partial_i^2 \overline{u}(x_i', t) \varphi(x_i', t)\, dt,$$

Since this holds for $\mathcal{L}^{N-1}$ a.e. $x_i' \in \mathbb{R}^{N-1}$, integrating over $\mathbb{R}^{N-1}$ and using Fubini's theorem yields

$$\int_{\mathbb{R}^N} \overline{u}(x)\partial_i\varphi(x)\,dx = -\int_{\mathbb{R}^N} \partial_i\overline{u}(x)\varphi(x)\,dx,$$
$$\int_{\mathbb{R}^N} \overline{u}(x)\partial_i^2\varphi(x)\,dx = \int_{\mathbb{R}^N} \partial_i^2\overline{u}(x)\varphi(x)\,dx,$$

This shows that $\overline{u} \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ and that it admits weak second order partial derivatives $\partial_i^2\overline{u} \in L^p(\mathbb{R}^N)$. To complete the proof, it remains to show that there exist the weak derivatives $\partial_{ij}^2\overline{u} \in L^p(\mathbb{R}^N)$. Let

$$f := \sum_{i=1}^{N} \partial_i^2\overline{u} \in L^p(\mathbb{R}^N).$$

Then for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, by integrating by parts,

$$\int_{\mathbb{R}^N} \nabla\overline{u}(x)\cdot\nabla\varphi(x)\,dx = -\int_{\mathbb{R}^N} f(x)\cdot\varphi(x)\,dx,$$

which shows that $\bar{u}$ is a weak solution to the Poisson problem $-\Delta\bar{u} = f$ in $\mathbb{R}^N$. It follows by [1, Theorem 2.38] that $\bar{u} \in \dot{W}^{2,p}(\mathbb{R}^N)$ with

$$\|\nabla^2\bar{u}\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}.$$

This completes the proof. ■

Remark 5 *The previous theorem does not hold for $p = 1$. Indeed, Ornstein in [2] constructed a function $u : \mathbb{R}^2 \to \mathbb{R}$ with compact support and such that u, $\partial_1 u$, $\partial_2 u$, $\partial_1^2 u$, $\partial_2^2 u$ belong to $L^1(\mathbb{R}^2)$ but $\partial_{12}^2 u$ does not.*

References

[1] S. Müller, Nonlinear partial differential equations I, Lecture Notes, University of Bonn.

[2] D. Ornstein, A non-equality for differential operators in the L_1 norm. Arch. Ration. Mech. Anal. 11 (1962), 40 49.