

Supplementary Notes to
Differential Geometry, Lie Groups and Symmetric Spaces
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Page 17⁵ means fifth line from top of page 17 and page 81₆ means the sixth line from below on page 81.

Page 28¹⁹. For the applications of Lemma 5.1 in this chapter the local version of the lemma:

$$G(\gamma(t)) = g(t) \quad \text{for } t \text{ near } t_0$$

(the proof of which does not require partition of unity) is sufficient.

Page 101 (middle). The statement $u_p = cX^*(M)$ actually holds in a stronger form:

$$u_p = m_1! \dots m_n! X^*(M)$$

where m_k is the number of entries in the sequence (i_1, \dots, i_p) which equal k and $M = (m_1, \dots, m_n)$. To see this write X_j for X_j^* and note that

$$(t_{i_1} X_{i_1} + \dots + t_{i_p} X_{i_p})^p = \sum_{|M|=p} t^M S_M,$$

where

$$S_M = \sum_{\sigma} X_{i_{\sigma(1)}} \dots X_{i_{\sigma(p)}} = p! u_p.$$

In each term m_k factors equal X_k . One term is $X_1^{m_1} \dots X_n^{m_n}$ and the others are obtained by shuffling. In the sum \sum_{σ} each term will appear $m_1! \dots m_n!$ times. Now

$$(t_1 X_1 + \dots + t_n X_n)^p = p! \sum_{|M|=p} t^M X(M).$$

Here $p!X(M)$ is the sum of the terms obtained from $X_1^{m_1} \dots X_n^{m_n}$ by shuffling, each term appearing *exactly once*. Hence

$$p! u_p = m_1! \dots m_n! p! X(M).$$

Page 188 (middle). G_{21}^- should be $-G_{21}^-$

Page 101 195₄. $X'_{\alpha, \beta}$ should be $X'_{\alpha+\beta}$

Page 325. Proposition 8.10. In response to a question by Adam Korányi, this proposition has the following extension:

Assuming Σ irreducible each automorphism of it extends to an automorphism of Δ .

This follows from Theorem 3.29, Ch. X.

Only the cases (ii) and (iii) have to be considered. The cases \mathfrak{e}_6 and \mathfrak{d}_4 for Σ only occur for the normal form so statement is obvious. This leaves the cases $\mathfrak{a}_\ell (\ell \geq 2)$ and $\mathfrak{d}_\ell (\ell > 4)$ and here we have (by (ii)) to look at one automorphism of Σ which is not induced by $W(\Sigma)$ and show that it is induced by an automorphism of Δ . For the form A1 (Satake diagram, Exercises F, Ch. X) there

is nothing to prove since AI is a normal form. Same for the case \mathfrak{d}_ℓ . For the form AII the extra automorphism is -1 (Exercise B1, Ch. X) so here the statement follows too.

As a corollary we see (considering Theorem 5.4, Ch. IX) that each automorphism of Σ is induced by an automorphism of \mathfrak{u} .

Page 349. Exercise 10. As observed to me by A. Onishchik the connectedness assumption for K is unnecessary here; see G. Fels, “A Note on Homogeneous Locally Symmetric Spaces,” Transformation Groups 2 (1997), 269–277.

Page 396₁₁. The following exercise (response to a question by Morris Hirsch) follows from some results in this chapter.

Proposition. *If G is a connected simple Lie group which does not contain a nontrivial compact subgroup then G is the universal covering group of $\mathbf{SL}(2, \mathbf{R})$.*

Proof: Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Because of Theorem 5.4 the action of \mathfrak{k} on \mathfrak{p} is irreducible. Now \mathfrak{k} is the direct sum of its center \mathfrak{c} and its semisimple part $[\mathfrak{k}, \mathfrak{k}]$. Since any connected Lie group with Lie algebra $[\mathfrak{k}, \mathfrak{k}]$ is compact the assumption implies $\mathfrak{k} = \mathfrak{c}$. Let \tilde{G} denote the universal covering group of G . Let \tilde{K} denote the analytic subgroup corresponding to \mathfrak{k} . Then $\tilde{G}/\tilde{K} = G/K$ and by Prop. 6.2, \mathfrak{k} is 1-dimensional. From Table V on page 518 we see that $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$. Thus $\tilde{G} =$ universal covering group of $SL(2, \mathbf{R})$ and \tilde{K} is isomorphic to \mathbf{R} . Thus $K = \mathbf{R}/N$ where N is a discrete subgroup of \mathbf{R} . Since K is noncompact by assumption, N reduces to the identity. Since the covering map $j : \tilde{G} \rightarrow G$ has kernel contained in \tilde{K} (Theorem 1.1, Ch. VI) and j is injective on \tilde{K} we deduce $G = \tilde{G}$ as claimed.

Page 418₁₀. add “satisfying $J(\mathfrak{h}) = \mathfrak{h}$ ”.

Page 500⁴. “implied” should read “followed from”.

Page 517². Type IV should be Type III.

Page 521₁₂. v_{sl} should be $v_{5\ell}$.

Page 553^{8,9}. This statement follows from the theorem on page 596.

Page 579^{1,2}. $\frac{1}{2}$ should be 2.

Page 580. Solution to Exercise B6. $\text{Aut}(\mathfrak{u})$ acts on \tilde{U} and on \tilde{Z} . If $s \in \text{Aut}(\mathfrak{u})$ acts trivially on \tilde{Z} then by Lemma 6.5, Ch. VII it acts trivially on the lattice $\mathfrak{t}(\mathfrak{u})$ so acts trivially on \mathfrak{t} . Hence by Prop. 5.3, Ch. IX s is in $\text{Int}(\mathfrak{u})$. Thus $\text{Aut}(\mathfrak{u})/\text{Int}(\mathfrak{u})$ acts faithfully on \tilde{Z} so by Theorem 5.4, Ch. IX and Lemma 3.30, Ch. X, $\text{Aut}(R)/W(R)$ acts faithfully on $\tilde{T}(R)/T(R)$.

If $-1 \in W(R)$ then -1 acts trivially on $\tilde{T}(R)/T(R)$ so $-w \equiv w \pmod{T(R)}$, whence $2\tilde{T}(R) \subset T(R)$. Conversely, if $2\tilde{T}(R) \subset T(R)$ then $-w \equiv w \pmod{T(R)}$ so -1 induces the identity map of $\tilde{T}(R)/T(R)$ so $-1 \in W(R)$.

Page 591¹. As the notation implies the top line applies to $j > i$ and then $\epsilon_{ij}(-1)^{i-1} = (-1)^{j-1}$. If $j < i$ then $[X_i, X_j]$ appears at the $(i-1)$ -place so $\epsilon_{ij}(-1)^{i-2}$ is again $(-1)^{j-1}$.

One more proof of Lemma 7.1 is the following: If ω is a bi-invariant p -form then so is the form $J^*\omega$ where J is the map $x \rightarrow x^{-1}$ of G to G . But $(J^*\omega)_e = (-1)^p \omega_e$ so $J^*\omega = (-1)^p(\omega)$. Since d commutes with J^* ((6) I, §3) and $d\omega$ is a bi-invariant $(p+1)$ form,

$$d(-1)^p \omega = dJ^*\omega = J^*d\omega = (-1)^{p+1} d\omega$$

so $d\omega = 0$.

Page 591. Theorem 8.1. Note that in contrast to Chevalley [2], p. 112 this proof does not require the monodromy theorem. However, the following standard result was used in our proof as well as at some other places in the text.

Lemma. *A discrete normal subgroup D of a connected topological group G is contained in the center.*

In fact let $d \in D$ and N a neighborhood of d such that $N \cap D = \{d\}$ and let V be a neighborhood of e in G such that $VdV^{-1} \subset N$. Since D is normal, $\sigma \in V$ implies $\sigma d \sigma^{-1} \in N \cap D = \{d\}$. Since G is connected, V generates G so $gdg^{-1} = d$ for all $g \in G$.

Theorem. *Let G be a connected, locally arcwise connected topological group. Let $H \subset G$ be a closed subgroup and H_0 the identity component of H . Then*

- (i) G/H is connected and locally arcwise connected.
- (ii) The natural map $G/H_0 \rightarrow G/H$ is a covering.
- (iii) If G/H is simply connected then $H = H_0$.
- (iv) If H is discrete, $G \rightarrow G/H$ is a covering.

Proof: (i) The natural map $\pi : G \rightarrow G/H$ is continuous so G/H is connected. If a is in an open subset $V \subset G/H$ take $\tilde{a} \in \pi^{-1}(a)$ and a connected arcwise connected neighborhood W of \tilde{a} contained in $\pi^{-1}(V)$. Then $\pi(W)$ is an arcwise connected neighborhood of a contained in V .

For (ii) let $\pi_0 : G \rightarrow G/H_0$, $\pi : G \rightarrow G/H$ and $\sigma : G/H_0 \rightarrow G/H$ be the natural maps. If $U \subset G/H$ is open then $\sigma^{-1}(U) = \pi_0(\pi^{-1}(U))$ is open so σ is continuous. Also if $V \subset G/H_0$ is open, $\sigma(V) = \pi(\pi_0^{-1}(V))$ is open so σ is a continuous open mapping. H is locally connected so H_0 is open in H . Thus there exists an open subset $V \subset G$ such that $V \cap H = H_0$. Choose a connected neighborhood U of e in G such that $U^{-1}U \subset V$ and $U^{-1}U \cap H \subset H_0$. Then UH is a neighborhood of the origin in G/H .

Consider the inverse image

$$\sigma^{-1}(UH) = \{gH_0 : \sigma(g) \in UH\} = \{UhH_0 \in G/H_0 : h \in H\}$$

the latter equality holding since $gH \in UH$ implies $g = uh$. Each UhH_0 is open in G/H_0 and is connected and their union is $\sigma^{-1}(UH)$. If

$$(1) \quad Uh_1H_0 \cap Uh_2H_0 \neq \emptyset$$

then since H_0 is normal in H ,

$$UH_0h_1 \cap UH_0h_2 \neq \emptyset,$$

and hence $U^{-1}UH_0 \cap H_0h_2h_1^{-1} \neq \emptyset$. Again since H_0 is normal in H ,

$$U^{-1}U \cap H_0h_2^{-1}h_1 \neq \emptyset.$$

Thus $U^{-1}U$ contains an element $h \in H$ so since $U^{-1}U \cap H \subset H_0$, $h \in H_0$ so $h_2^{-1}h_1 \in H_0$. Thus (1) implies

$$(2) \quad Uh_1H_0 = Uh_2H_0.$$

We claim now that for each $h \in H$ the map $\sigma : UhH_0 \rightarrow UH$ is a bijection. In fact if $\sigma(u_1hH_0) = \sigma(u_2hH_0)$ then $u_1h = u_2hh^*$ for some $h^* \in H$ so $u_2^{-1}u_1 \in H$ so by $U^{-1}U \cap H \subset H_0$, $u_1 = u_2h_0$ for

some $h_0 \in H_0$, whence the injectivity of σ . The surjectivity is obvious. The sets UhH_0 being the components of $\sigma^{-1}(UH)$, UH is evenly covered. By translation we see that σ is a covering. Now (iii) and (iv) follow from (ii).

Page 597¹⁰. $d_n = z_n$ should be $d_n = z_n + d^$.*

Page 597¹⁵. $v_i \in V$ should be $v_i \in D$.

Errata

Page and line in $\left\{ \begin{array}{l} \text{above} \\ \text{below} \end{array} \right.$	Instead of:	Read:
249 ₂	automorphism	inner automorphism
250 ₅	g_+	g_{+1}
264 ²	\wedge	Δ
291 ₁₄	lemma	theorem
314 ⁷	linear	affine
373 ¹ , 375 ¹	Hermetion	Hermitian
430 ₇	$u - 1$	$u - I$
432 ⁸	definining	defining
433 _{3,5}	2	$2I$
485 ₈	representatives	representatives ξ', ξ, η'
498 ¹⁷	$\equiv 0$	$= 0$
587 ⁴	$\zeta, \zeta' \in B_r$	$\zeta \in B_r$
592 ₃	=	\in
603 ¹²	remarquablz	remarquable
603 ¹⁶	géométrique	géométrie
603 ₃	Scouten	Schouten
604 ¹¹	halbeinfachen	halbeinfacher
605 ¹¹	les	leurs
606 ¹⁶	enveloppants	enveloppantes
611 ₁₇	geschlossenen	geschlossener
612 ⁹	Gétt	Gött
616 ¹⁴	dei Gruppi e	dei gruppi finiti e
616 ₁₀	Integrabilitetsfaktors	Integrabilitätsfaktors
623 ¹⁶	Lie complexes	Lie semi-simple complexes
625 ²⁴	metrische homogene	metrisch homogenen
633 ₁₃	\mathfrak{h}_{et}	\mathfrak{h}_{p_0}