Hints for solving the exercises in Chapter 13

Hints to Exercise 13.2 With the notation $T^{-1} = (w_{kj}) \in \mathbb{R}^{N\times N}$, one has to verify the regularity of submatrices of the form $(w_{kj})_{1\le k,j\le M} \in \mathbb{R}^{M\times M}$ for $M = 1, 2, \ldots, N$. Here the identities $\mathbf{e}_j = \sum_{k=1}^N w_{kj} v_k$ for $j = 1, 2, \ldots, N$ can be applied, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ denote the unit vectors in $\mathbb{R}^N$. From this an representation of the form $\sum_{j=1}^M \alpha_j \mathbf{e}_j$ can be obtained. This can be used then to appropriately reformulate $\sum_{j=1}^M \alpha_j \mathbf{e}_j \in \text{span}\{ v_{M+1}, v_{M+2}, \ldots, v_N \}$.

Hints to Exercise 13.3 It is sufficient to consider nonsingular matrices $A \in \mathbb{R}^{N\times N}$, and in the second case (tridiagonal structure) for convenience it may be supposed that the matrix is symmetric. One has to show that for a matrix $A \in \mathbb{R}^{N\times N}$ of Hessenberg form and for arbitrary upper triangular matrices $R \in \mathbb{R}^{N\times N}$, also the matrices RA und AR are of Hessenberg form. Show additionally that for arbitrary upper triangular matrices $R \in \mathbb{R}^{N\times N}$, also the inverse matrix R^{-1} is an upper triangular matrix. The conservation of the symmetric tridiagonal form follows from the first part of this exercise and the fact that the QR algorithm conserves symmetry of a matrix.

Hints to Exercise 13.4 First show that

$$\| x_n - y \| = \mathcal{O}(q^n) \quad \text{for} \quad n \to \infty \quad \Longrightarrow \quad \Big\| \frac{x_n}{\| x_n \|} - \frac{y_n}{\| y_n \|} \Big\| = \mathcal{O}(q^n) \quad \text{for} \quad n \to \infty, \qquad (*)$$

with vectors $x_n \in \mathbb{R}^N$ and $0 \ne y \in \mathbb{R}^N$ and some vector norm $\| \cdot \| : \mathbb{R}^N \to \mathbb{R}$.

With the same procedure as in Section 13.7 and a reasoning as in $(*)$, a convergence result is obtained for $\lambda_1^{-k} z^{(k)} / \| z^{(k)} \|$ for $k \to \infty$. The solution of the exercise then follows with Theorem 13.36 on the asymptotical behavior of the Rayleigh quotients.

Hints to Exercise 13.5 The statements (a) and (b) follow with the same technique as in the proof of Theorem 13.34 and with a conclusion of the form $(*)$.

Hints to Exercise 13.6 The Frobenius companion matrix $A \in \mathbb{R}^{n\times n}$ belonging to a given polynomial $p \in \Pi_n$ with leading coefficienst 1 is introduced in the proof of Lemma 5.16. There it is shown that the characteristic polynomial of A is identical with the given polynomial p. One has to find out what the result of the vector iteration $z^{(k+1)} = A^\top z^{(k)}$ for $k = 0, 1, \ldots$ with starting vector $z^{(0)} = (x_{1-n}, x_{2-n}, \ldots, x_0)^\top \in \mathbb{R}^n$ is in that special situation. Then with similar techniques as in Section 13.7, a convergence statement for $\lambda_1^{-k} z^{(k)}$ for $k \to \infty$ can be obtained. (The matrix $A^\top$ may be assumed as diagonalizable.) Then consider for an appropriate index j the corresponding sequence of the j–th entries $z_j^{(k)}$ for $k = 0, 1, \ldots$ and the corresponding sequence of quotients $z_j^{(k+1)} / z_j^{(k)}$ for $k = 0, 1, \ldots$.