Hints for solving the exercises in Chapter 3

Hints to Exercise 3.1 (a) Use that an existence and uniqeness result similar to that for interpolating polynomials also holds for polynomials in $e^{\mathrm{i}x}$: there exists a unique function of the form $(*)\ \sum_{k=-N/2}^{N/2} d_k\, e^{\mathrm{i}k2\pi x/L}$ satisfying the interpolationg conditions. An application of the Euler formula $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$ leads to the representation in the exercise.

(b) We have $y_k = \overline{y_k} = \overline{T(x_k)} \ldots$. The uniqueness of the interpolating function of the form $(*)$ yields the statement of the exercise.

Hints to Exercise 3.2 (a) The first result in the hint to this exercise is obtained by a summation formula for trigonometric functions, and the second result in the hint to the exercise is obtained by using the Euler formula $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$. (Multiplication with complex numbers, then backtransformation to real numbers.)

Hints to Exercise 3.3 (a) For each index $j = 0, 1, \ldots, N-1$ there holds $D_2 w^{(j)} = \lambda_j w^{(j)}$ (verify this for each entry). From $(w^{(j)})^{\mathrm{H}} w^{(\ell)} = \sqrt{N}\delta_{j\ell}$ one then obtains the solution to part (a) of this exerscise .

(b) Show that $\mathcal{F} D_2 c = M \mathcal{F} c$ for $c \in \mathbb{C}^N$, the statement of the exercise then follows easily.

Hints to Exercise 3.4 Here the following definition of the d_j's is needed: $[d_0, d_1, \ldots, d_{N-1}] := 2T[f_0, f_1, \ldots, f_{N-1}]$ with $T = \mathrm{diag}(e^{-\mathrm{i}j0\pi/N}, e^{-\mathrm{i}j1\pi/N}, \ldots, e^{-\mathrm{i}jk2\pi/N})$.

(a) Show that

$$d_j = \frac{2}{N} \sum_{k=0}^{N-1} f_k \cos\Big(\pi j \frac{2k+1}{N}\Big).$$

The statement of the exercise then follows easily.

(b) Apply part (a) with $f_{N-1-k} := f_k$ for $k = 0, \ldots, N/2-1$. For the evaluation of $p(x_k)$ use the complex representation and apply the Euler formula. The discrete inverse Fourier transform finally yields $p(x_k) = f_k$.