Hints for solving the exercises in Chapter 4

Hints to Exercise 4.3 (a) For "$\Longleftarrow$" proceed as for the square root method to compute a Cholesky factorization of symmetric, positiv definite matrices. For "$\Longrightarrow$" show that a LR factorization of A can be used to determine a LR factorization for each of the considered principal submatrices.

Hints to Exercise 4.4 (a) Use the positive definitness of the matrix A with a special well−known vector.

(b) For fixed indices i and j, apply the positive definitness of the matrix A with the vector

$$x = (x_k) \in \mathbb{R}^N \quad with\ x_k = \begin{cases} 1, & k = i, \\ 0, & k \neq i,\ k \neq j, \\ \alpha, & k = j, \end{cases}$$

with $\alpha \in \mathbb{R}$ arbitrary. From that conclude that no real solution α to the corresponding quadratic equation exists. This finally yields the solution to the problem.

(c) A contradictory assumption leads to a contradiction to the statement of part (b) of the exercise.

Hints to Exercise 4.5 Consider the decomposition $(*)$ $A = LL^\top$ with the lower triangular matrix $L = (\ell_{ij}) \in \mathbb{R}^{N\times N}$. For fixed index $i \in \{ m+2, m+3, \ldots, N \}$, prove by mathematical induction w.r.t. $j \in \{ 1, 2, \ldots, i-m-1 \}$ that $\ell_{ij} = 0$ holds. (Use $(*)$ to derive necessary conditions for the numbers ℓ_{ij}).

Hints to Exercise 4.6 Consider the notations

$$U = \left[\begin{array}{c|c|c} u_1 & \ldots & u_N \end{array} \right], \qquad V = \left[\begin{array}{c|c|c} v_1 & \ldots & v_N \end{array} \right], \qquad \langle u, v \rangle_2 = u^\top v$$

and show first that the representation

$$Ax \overset{(*)}{=} \sum_{k=1}^{N} \sigma_k \langle x, u_k \rangle_2 v_k \qquad \text{for } x \in \mathbb{R}^N$$

holds. This representation $(*)$ has to be applied several times in the sequel.

(a) Derive a formula for $\| A \|_2$ in terms of σ_1, and proceed similarly for $\| A^{-1} \|_2$ and σ_N.

(b) Consider $x = \sum_{k=1}^N \alpha_k u_k$ and find out under which conditions on the coefficients $\alpha_1,\ \alpha_2, \ldots, \alpha_N$ the identity $\| b \|_2 = \| A \|_2 \| x \|_2$ is satisfied. Similarly consider $\Delta x = \sum_{k=1}^N \beta_k u_k$ and find out under which conditions on the coefficients $\beta_1,\ \beta_2, \ldots, \beta_N$ the identity $\| \Delta x \|_2 = \| A^{-1} \|_2 \| \Delta b \|_2$ is satisfied. As a consequence from these properties, the solution to the third subproblem in part (b) is obtained.

(c) First reduce the problem and determine vectors $b \in \mathbb{R}^N$ so that $\| x \|_2 = \| A^{-1} \|_2 \| b \|_2$ holds.

Hints to Exercise 4.9 Consider the notations

$$A = \left[\begin{array}{c|c|c} a_1 & \ldots & a_N \end{array} \right], \qquad Q = \left[\begin{array}{c|c|c} q_1 & \ldots & q_N \end{array} \right], \qquad R = (r_{ij}).$$

and compute $|\det A|$ by using the identity $A = QR$. On the other hand there holds $a_j = \sum_{i=1}^j r_{ij} q_i$ with mutually orthonormal vectors $q_1, q_2, \ldots, q_i$. This can be used to determine lower bounds for the number $\| a_j \|_2$.

Hints to Exercise 4.10 In part (a) compute

$$(A + uv^\top)\Big(A^{-1} - \frac{A^{-1}uv^\top A{-}1}{1 + v^\top A^{-1} u} \Big).$$