Hints for solving the exercises in Chapter 5

Hints to Exercise 5.1 In the following, for each of the considered iteration methods in (5.22)–(5.23) the corresponding iteration function is denoted by Φ, respectively. For the methods in (5.22), $\Phi(x^*) = x^*$ and $|\Phi'(x^*)| < 1$ has to be verified. For a method of second order, $\Phi'(x^*) = 0$ is required. For the first method in (5.23) test additionally if the resulting method for the determination of a is realizable. For the second method in (5.23) determine a simple function $g : \mathbb{R} \to \mathbb{R}$ such that for the resulting function $\Phi(x) = \frac{g(x)x \, + \, e^{-x}}{g(x)+1}$ the following holds: $\Phi(x^*) = x^*$ and $\Phi'(x^*) = 0$. The choice $a_j = g(x_j)$ then finally yields the result. Note, however, that the numbers a_j depend also x_j.

Hints to Exercise 5.2 Consider also the noise−free sequence $x_{j+1} = \Phi(x_j)$ for $j = 0, 1, \ldots$. Starting point of the error analysis is the estimate $(*)$ $\| x^\delta_{j+1} - x^* \| \le \| x^\delta_j - x_j \| \, + \, \| x_j - x^* \|$. For the second term on the right−hand side of $(*)$ there exist well−known estimates of the form $\| x_j - x^* \| \le q_j \| x_1 - x_0 \|$ with suitable constants q_j, Then additionally an estimate of the form $\| x_1 - x_0 \| \le L\delta \, + \, \| x^\delta_1 - x^\delta_0 \|$ with an appropriate constant L has to be derived. The first term on the right−hand side of $(*)$ has to estimated as follows, $\| x^\delta_j - x_j \| \le K \| x^\delta_{j-1} - x_{j-1} \| + \delta$. This leads to an estimate of the form $\| x^\delta_j - x_j \| \le c\delta$ with some constant c. Finally these results have to be putted together appropriately.

Hints to Exercise 5.3 (a) Consider $\| \Phi((x,y)^\top) - \Phi((\widehat{x},\widehat{y})^\top) \|_\infty$ with $x = y = z \in (0, \pi/2)$ und $\widehat{x} = \widehat{y} = 0$. Show that $(*)$ $\| \mathcal{D}_{(x,y)^\top}\Phi \|_2 \le K = (5+\sqrt{89})/16 < 1$ holds for all $x, y \in \mathbb{R}$. The matrix is symmetric and therefore for the proof of $(*)$ it is sufficient to estimate the modulus of the eigenvalues of the matrix $\mathcal{D}_{(x,y)^\top}\Phi$. During these computations it is sufficient to estimate the modulus of trigonometric terms by 1.

(b) Apply the a priori and the a posteriori− error estimate of the fixed point theorem of Banach. For the realization of the a posteriori estimate proceed numerically.

Hints to Exercise 5.5 Consider the proof of the behavior of Newton's method for the determination of the largest root of a polynomial. Proceed similiarly to solve this problem.