Hints for solving the exercises in Chapter 7

Hints to Exercise 7.1 Use the auxiliary functions $z_1 := y$ and $z_2 := y'$ to transform the given initial value problem for a system of two differential equations of second order into an initial value problem for a system of four differential equations of first order.

Hints to Exercise 7.2 Follows immediately from Theorem 7.2.

Hints to Exercise 7.4 Integration of the differential equation yields the exact solution. Using the Euler's method leads to a scheme of the form $u_\ell = u_{\ell-1} + g(\ell, h)$ for $\ell = 0, 1, \ldots$, and mathematical induction gives $u_\ell = u_0 + r(\ell, h)$ for $\ell = 0, 1, \ldots$, with suitable functions g and r. For the convergence analysis and for fixed t and $u_0 = 0$ consider the index $\ell = t/h$. This gives a representation of the form $u_\ell = s(t, \ell)$ with an appropriate function s. The limit process $\ell = t/h \to \infty$ yields the stated convergence.

Hints to Exercise 7.6 It may supposed that $N = 1$ holds, i.e., $f : [a, b] \times \mathbb{R} \to \mathbb{R}$.

(a) Derive an explicit representation for the given function $f^{(j)}$ by using mathematical induction. This representation can be used to obtain the iteration function $\varphi(t, y; h)$.

(b) Represent u_n as a function of h and n. The solution of the initial value problem is $y(x) = 1 - e^{-x}$. For the derivation of a representation of the error $u_n - y(1)$ at the point $x = 1$ compute a Taylor expansion $e^{-h} = p_2(h) + e^{\delta h} h^3/6$ with a polynomial $p_2 \in \Pi_2$. Use an identity $e^{-1} = (e^{-1/n})^n$ and an estimate of the form $(*)$ $|(c+b)^n - c^n| \le n|b|c^{n-1}$ for real numbers $c > 0$ and $b < 0$ with $c + b > 0$. Verify also the estimate $(*)$.

Hints to Exercise 7.7 For a verification of

$$\frac{y(t+h) - y(t)}{h} - \frac{1}{6}(k_1 + 4k_2 + k_3) = \mathcal{O}(h^3) \qquad \text{for} \quad h \to 0$$

use Taylor expansions of second order for the functions $k_2 = k_2(h)$ and $k_3 = k_3(h)$ at $h = 0$. For the function y use a Taylor expansion of third order w.r.t. to t.

Hints to Exercise 7.8 For notational convenience assume that p und $n = (b - a)/h$ may be real−valued, in general. The largest possible step size is $h = (\varepsilon/K)^{1/p}$. Show that the resulting computational time $T(p, \varepsilon)$ with $p_{\text{opt}} = p_{\text{opt}}(K, \varepsilon) > 0$ is strictly decreasing (as a function of p) over the interval $[0, p_{\text{opt}}]$. Show moreover that it is strictly increasing over the interval $[p_{\text{opt}}, \infty)$.