Hints for solving the exercises in Chapter 9

Hints to Exercise 9.1 The solution to the first subproblem is easily obtained, and the solvability follows from Exercise 9.3 with the choices $v_0 := v_N := 0$.

Hints to Exercise 9.2 (a) Consider the function $z(x) = \int_0^1 G(x,\xi)\varphi(\xi)\,d\xi$ for $x \in [0,1]$ and prove $z''(x) = \varphi(x)$. Verify also that the boundary conditions are satisfied.

(b) The solution of the boundary value problem $(\Delta u)''(x) = \Delta\varphi(x)$ on $[a,b]$, $(\Delta u)(0) = (\Delta u)(1) = 0$, can be determined by using part (a). Reasonable estimates of $|\Delta u(x)|$ at each point $x \in [a,b]$ then easily follow, if the domain of integration $[a,b]$ is divided into the two subintervals $[0,x]$ and $[x,1]$. The result in fact is $|\Delta u(x)| \le \varepsilon x(1-x)/2$.

(c) This is a discrete version of part (b). The matrix A_0 considered in this exercise concides – up to a normalization – with the matrix A_0 considered in Theorem 9.9. On the other hand, the solution cannot be obtained by simply applying Theorem 9.9, the procedure has to be modified somewhat. One has to use the estimate $|\Delta b| \le \varepsilon\mathbf{e}$, where $|\cdot|$ denotes the modulus function, and $\mathbf{e} := (1, 1, \ldots, 1)^\top$. Moreover the representation of $A_0^{-1}\mathbf{e}$ in the proof of Theorem 9.9 is needed.

Hints to Exercise 9.3 (a) Use the notation $M := \max_{j=0,\ldots,N} v_j$ and prove by a contradictory assumption the following: if for some $1 \le k \le N-1$ the identity $v_k = M$ holds then also $v_{k-1} = v_{k+1} = M$ holds.

(b) The difference $w = u - v$ satisfies $\Delta w \ge 0$, and the rest follows with the statement of part (a).

Hints to Exercise 9.4 Apply Friedrich's inequality to the subintervals.

Hints to Exercise 9.5 (a) One has to show that the bilinear form $[\![\cdot,\cdot]\!]$ is a continuation of the mapping $\langle \mathcal{L}\cdot,\cdot\rangle_2$. In fact, one partial integration gives the required identity $\langle \mathcal{L}u, v\rangle = [\![u,v]\!]$ for $u, v \in \mathcal{D}_{\mathcal{L}}$.

(b) Here Friedrich's inequality as well as the following variant is needed,

$$\|u\|_\infty \quad \le \quad (b-a)^{1/2}\|u'\|_2 \qquad \text{for} \quad u \in C^1_\Delta[a,b] \quad \text{with} \quad u(a) = 0.$$

This estimate also has to be verified which after a consideration of the proof of Friedrich's inequality should be an easy task. Moreover the trivial estimate $\|u\|_2 \le (b-a)^{1/2}\|u\|_\infty$ for $u \in C[a,b]$ is needed.

Hints to Exercise 9.6 The unknown bilinear form is obtained by applying two partial integrations. The boundary condition guarantees that this bilinear form consists only of terms with integrals. The hints of the solution to part (b) of Exercise 9.5 can be used also for this exericse.

Hints to Exercise 9.7 The proof of the classical Cauchy−Schwarz inequality has to be transferred to this more general situation.

Hints to Exercise 9.8 The equivalence of (ii) and (iii) can be easily verified. The implication "(ii) $\Longrightarrow$ (i)" follows similar to proof of the statements in Theorem 9.43 and Corollary 9.44. The implication "(i) $\Longrightarrow$ (ii)" can be obtained by making a contradictory assumption. A consideration of $\|\mathcal{L}(u+tv)-\varphi\|$ for $t \in \mathbb{R}$ then together with an appropriate choice of $t \in \mathbb{R}$ finally yields a contradiction.

Hints to Exercise 9.9 The bilinear form is $[\![u,v]\!] = \int_a^b u'v' + xuv\,dx$ for $u,\ v \in C^1_\Delta[a,b]$.

Hints to Exercise 9.10 (a) The following auxiliary result will be needed:

Theorem *Let $p, r, g \in C[a,b]$ with $r(x) \le 0$ for $x \in [a,b]$. Let the function $y \in C^2[a,b]$ be a solution of the initial value problem*

$$y''(x) + p(x)y'(x) + r(x)y(x) = g(x) \qquad \textit{for} \quad x \in [a,b], \qquad y(a) = \alpha, \quad y'(a) = \beta$$

(for some $\alpha,\ \beta \in \mathbb{R}$). Moreover assume that $z \in C^2[a,b]$ is a solution of the the following initial value problem for a **differential inequality**,

$$z''(x) + p(x)z'(x) + r(x)z(x) \le g(x) \qquad \textit{for} \quad x \in [a,b], \qquad z(a) \le \alpha, \quad z'(a) \le \beta,$$

Show that

$$z(x) \le y(x), \qquad z'(x) \le y(x) \qquad \textit{for} \;\; x \in [a,b].$$

Hints for the proof of this theorem are given after the hints to this exercise.

According to Section 9.4, the function $v = \dfrac{\partial u}{\partial s}(\cdot, s) : [a,b] \to \mathbb{R}$ solves the following initial value problem for a linear differential equation of second order: $v''(x) + p(x)v'(x) + r(x)v(x) = 0$ for $x \in [a,b]$, with certain functions p and q depending on s. For the proof of the inequality $\kappa_1 \le F'(s)$ stated in the exercise apply the above given theorem with the functions $y = v$ and $z(x) = (1 - e^{-L(x-a)})/L$. For the proof of the inequality $F'(s) \le \kappa_2$ given in the exercise apply the above given theorem with the initial value problem

$$y''(x) - Ly'(x) - Ky(x) = 0 \quad \text{for} \;\; x \in [a,b], \qquad y(a) = 0, \quad y'(a) = 1, \tag{$\bullet$}$$

and the function $z = v$. The general solution of $(\bullet)$ (without consideration of the initial conditions) can be determined by consideration of $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$, where $\lambda_{1/2}$ are the roots of the quadratic equation $\lambda^2 - L\lambda - K = 0$.

For the proof of the above given theorem consider the differential operator $\mathcal{L}w := w'' + pw' + rw$. It is sufficient to verify the following:

$$\left\{ \begin{array}{ll} (\mathcal{L}w)(x) \ge 0 & \text{for} \;\; x \in [a,b] \\ w(a) \ge 0, \quad w'(a) \ge 0, & \end{array} \right\} \Longrightarrow \quad w(x) \ge 0, \qquad w'(x) \ge 0 \qquad \text{for} \;\; x \in [a,b]. \tag{$*$}$$

For the proof $(*)$ one has to prove the following stronger version:

$$\left\{ \begin{array}{ll} (\mathcal{L}w)(x) > 0 & \text{for} \;\; x \in [a,b] \\ w(a) \ge 0, \quad w'(a) > 0, & \end{array} \right\} \Longrightarrow \quad w(x) > 0, \qquad w'(x) > 0 \qquad \text{for} \;\; x \in (a,b]. \tag{$**$}$$

For the proof of $(**)$ suppose contradictory that the function w is not monotonically increasing on the interval $[a,b]$. Show then that there exists a number $x^* \in [a,b]$ such that

$$w(x^*) \ge 0, \qquad w'(x^*) = 0, \qquad w''(x^*) \le 0$$

holds. This finally leads to a contradiction.

For the proof of the statement $(*)$ with a function w consider the auxiliary function

$$s(x) = e^{\alpha(x-a)} - 1 \qquad \text{for} \;\; x \in [a,b],$$

where the coefficient $\alpha > 0$ is chosen so that $\alpha^2 - \alpha \max_{x \in [a,b]} |p(x)| + \min_{x \in [a,b]} r(x) > 0$ holds. The statement $(*)$ then follows from $(**)$ applied with the function $w_\delta := w + \delta s$ with $\delta > 0$ and a subsequent limit process $\delta \to 0$.

(b) The solution can be obtained with Banach's fixpoint theorem.

Hints to Exercise 9.11 The general solution of the differential equation $u'' = 100u$ is $u(x) = \alpha e^{10x} + \beta e^{-10x}$. A fitting of the coefficients α and β gives $u(\cdot, s)$. A consideration of the difference $u(3, s_\varepsilon) - u(3, s_*)$ helps to find out if the shooting method is applicable or not.