

Corrections/comments to

Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems

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Note: Negative line numbers indicate lines up from the bottom of the page, so that “line -1” indicates the last line, “line -2” the line above it, etc.

- **p. 6, line 7** insert period before “The flow is given”
- **p. 7, Exercise 1.2.4** Hint: use Stokes’ theorem applied to integrating about the boundary of a rectangle.
- **p. 11, line -8** change $\Delta \subset \Gamma(T\Sigma)$ to $\Delta \subset T\Sigma$
- **p. 12, Exercise 1.3.6.1** Should read: Which of the following subbundles are Frobenius?

$$I_1 = \{dx^1, x^2 dx^3 + dx^4\}$$

$$I_2 = \{dx^1, x^1 dx^3 + dx^4\}$$

- **p. 14, Exercise 1.4.2.2** change $\kappa(s) = \left| \frac{de_1}{ds} \right|$ to $\kappa(s) = \langle e_2, \frac{de_1}{ds} \rangle$
- **p. 17, Def. 1.6.4, line 2** change $\omega|_e$ to ω_e
- **p. 17, line -1** change $\text{Ad}_{a^{-1}}$ to $\text{Ad}(a^{-1})$
- **p. 18, Def. 1.6.9** right-hand side of displayed equation should read $[\omega(X), \theta(Y)] - [\omega(Y), \theta(X)]$ (not +)
- **p. 19, Proof of 1.6.10, last line** the “uniqueness” referred to is in the Frobenius Theorem
- **p. 20, Exercise 1.6.14.3** change “complete the proof of” to “prove”
- **p. 20, Exercise 1.6.14.6** should begin “On a connected Lie group” and displayed equation should read $d\omega_e(X, Y) = -[X, Y]$ (not +)

- p. 22, Exercise 1.7.2.2 change $f = x^n$ to $f = z^n$
- p. 25, line 1 referenced equation should be (1.2.6), not (1.2.7)
- p. 25, 3 lines before Remark 1.8.3 change $d(t)$ to dt and $\omega_2^3 + \tau(t)\omega^1$ to $\omega_2^3 - \tau(t)\omega^1$
- p. 26, Exercise 1.8.4.3 assume $a, b \neq 0$
- p. 27, line -2 add $v(q) = 0$
- p. 28, line 12 between 1 and $\dim M$ (not between 1 and $\dim N$)
- p. 29, line 1 $J^k(M, N)$ (not $J^k(M.N)$)
- p. 29, line -2 $c dx^3$ (not $e dx^3$)
- p. 30, line -5 $1 \leq i, j \leq m - n$ (not $1 \leq i, j \leq n - m$)
- p. 32, Exercise 1.9.16(a) Change question to “At which points is $\frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$ an integral element? What about $\frac{\partial}{\partial y^1} \wedge \cdots \wedge \frac{\partial}{\partial y^n}$?”
- p. 42, Exercise 2.2.2.4(c) $t(v) = \sin(v)$, not $t = \sin(v)$
- p. 42, Exercise 2.3.1.1 change k_1 to $|k_1|$
- p. 42, Exercise 2.3.1.2 Hint: differentiate $\omega_1^3 = k_1\omega^1$ and $\omega_2^3 = k_2\omega^2$
- p. 42, Exercise 2.3.1.3 displayed equation should read

$$[X_1, X_2] = -\frac{k_{1,2}}{k_1 - k_2} X_1 - \frac{k_{2,1}}{k_1 - k_2} X_2$$

- p. 43, Exercise 5 Displayed equation should read

$$-k_1 k_2 = \frac{1}{(k_1 - k_2)^2} ((k_1 - k_2)(k_{2,1,1} - k_{1,2,2}) - k_{1,2} k_{2,2} - k_{2,1} k_{1,1} + 2(k_{1,2}^2 + k_{2,1}^2)).$$

- p. 45, displayed eqn. above (2.11) change $\omega_j^a = h_{ij}^a \omega^j$ to $\omega_i^a = h_{ij}^a \omega^j$
- p. 45, eqn. (2.12) \tilde{h}_{ij}^a instead of $\tilde{h}_i^a j$
- p. 47, Prop. 2.5.3 $\langle II(v, w), e_3 \rangle$ instead of $II(v, w)$
- p. 49, displayed equation last line in displayed equation should be numbered, and reference in lines below should be to this equation, not (2.6).

- **p. 49, line -16** should read “bijective linear maps” (not “linear maps”)
- **p. 51, Exercise 2.6.6** change η^k to $\bar{\eta}^k$ two times
- **p. 51, paragraph below 2.6.7** change \mathbb{E}^{n+s} to \mathbb{E}^{n+r} two times, to avoid confusion with section s .
- **p. 52, proof of 2.6.12, line 1** reference should be to Theorem 1.6.10, not 1.6.11
- **p. 54, Exercise 2(e)** First displayed equation should read

$$\nabla\alpha = (da_i - a_j\eta_i^j)\otimes\eta^i \in \Gamma(T^*M\otimes T^*M).$$

One way to check the minus sign is to note that $Id_T = \eta^i\otimes e_i$ and use the Leibniz rule to verify that $\nabla(Id_T) = 0$.

- **p. 56, second displayed equation** The proper definition of the Laplacian has more complicated signs. One definition is to let $\delta = (-1)^{n(k+1)+1}*d*$ and define $\Delta = \delta d + d\delta$. The introduction of δ is natural because δ is the adjoint of the operator d with respect to the L_2 -inner product on forms on a compact manifold.
- **p. 56, Exercise 2.6.18** The sign error above makes for sign errors here, also, more is needed to do part (c) than discussed in the text. Finally, add the additional exercise to show that with respect to an orthonormal framing e_1, \dots, e_n on a Riemannian manifold, $\Delta f = \sum_i f_{ii} - df(\nabla_{e_i}e_i)$.
- **p. 57, line 8** $u(x^1, \dots, x^n)$ instead of $u(x, y)$
- **p. 57, line before Exercise 2.6.19** ∂x and ∂y should be read as $\partial/\partial x$ and $\partial/\partial y$, respectively
- **p. 57, Exercise 2.6.19(a)** Displayed equation should be $K = -\Delta u$. One obtains the expression in the book if one uses the Laplacian with respect to the flat metric on \mathbb{R}^2 with coordinates x, y . The remark following the exercise is valid for the Laplacian with respect to the flat metric.
- **p. 59, first displayed eqn.** should read

$$d(x, T, N, B) = (x, T, N, B) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{pmatrix} ds.$$

- **p. 59, (2.19)** should read

$$d(x, T, \epsilon, e_3) = (x, T, \epsilon, e_3) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa_g & -\kappa_n \\ 0 & \kappa_g & 0 & -\tau_g \\ 0 & \kappa_n & \tau_g & 0 \end{pmatrix} ds$$

- **p. 59, line -4** Change $\kappa_g(c) = \nabla_{c'} c'$ to $\nabla_{c'} c' = \kappa_g \epsilon$ here, and in Exercise 2.8.1.3 on page 60.
- **p. 60, Exercise 2.8.1.1** change first displayed formula to $\kappa_g = \langle II(T, T), e_3 \rangle$, and change second displayed formula to $\tau_g = \langle II(\epsilon, T), e_3 \rangle$
- **p. 63, picture** Picture is mis-labelled; the curves should be labelled in counter-clockwise order, and δ_1 should be the angle between the tangents to C_1 and C_4 , etc.
- **p. 64, first displayed equation** should read

$$\int_R K dA = 2\pi - \int_{\partial R} \kappa_g ds - \sum_i \delta_i$$

- **p. 65, displayed equation and below** Replace $\text{Pfaff}(\Theta)dV$ by $\text{Pfaff}(\Theta)$ five times.
- **p. 68, last displayed equation** change $t s h_{11}$ to $2t s h_{11}$ and change $t s h_{22}$ to $2t s h_{22}$ in this matrix
- **p. 72, line -3** gP_k instead of $gP_k g^{-1}$
- **p. 75, last displayed eqn.** $|K_x|$ instead of K_x and $\{x\}$ instead of x under lim.
- **p. 78, line 2** Replace line with: “with $\mathcal{F}_{\mathbb{P}V}$, the set of bases of V . We consider $\mathcal{F}_{\mathbb{P}V}$ as a frame bundle over $\mathbb{P}V$, because $(e_1 \bmod e_0) \otimes e^0, \dots, (e_{n+a} \bmod e_0) \otimes e^0$ is a basis of $T_{[e_0]}\mathbb{P}V$. Given $M^n \subset \mathbb{P}V \dots$ ”
- **p. 79, displayed eqn. after Exercise 3.2.8** change \tilde{e}_ν to \tilde{e}_μ in third line
- **p. 79, line -2** change $\tilde{\omega}^\alpha$ to $\tilde{\omega}_0^\alpha$
- **p. 79, last displayed equation** should read

$$\tilde{q}_{\alpha\beta}^\mu = h_0^0 g_\beta^\alpha g_\alpha^\delta h_\nu^\mu q_{\gamma\delta}^\nu$$

- **p. 84, line -16** change “rank one matrices” to “rank one matrices up to scale”

- **p. 84, line 3 above 3.3.6** $\mathbb{P}W$ instead of BPW
- **p. 85, Exercise 3.3.8.3** change “rank one and traceless matrices” to “rank one traceless matrices, up to scale”
- **p. 88, above (3.2)** add: Here and in what follows we identify $\mathbb{F}_{s+1,t+1}(V)$ with the space of \mathbb{P}^s 's contained in \mathbb{P}^t 's contained in $\mathbb{P}V$.
- **p. 89-90** To avoid confusion rename $f = \dim F$ as $\phi = \dim F$ (5 instances).
- **p. 94, line 8** change 3.3.82 to 3.3.8.2.
- **p. 96, Exercise 3.5.3.2** change $\mathcal{F}2$ to \mathcal{F}^2 .
- **p. 96, Def. 3.5.4** change N_{2x} to $N_{2,x}$ in two places
- **p. 96, below Def. 3.5.4** add: Let $T^{(k)}M$ denote the vector bundle defined over general points of M whose fiber at $x \in M$ is $(\hat{T}_x^{(k)}M/\hat{x}) \otimes \hat{x}^*$, so in particular $T_x^{(k)}M \subset T_x\mathbb{P}V$ is a vector subspace.
- **p. 97, k-th fundamental form paragraph** change S^kT_xM to $S^kT_x^*M$; in displayed equation, change X to M , change $\mathbb{F}\mathbb{F}_x^{(k-1)}$ to $\mathbb{F}\mathbb{F}_x^{(k)}$ and change S^{k-1} to S^k
- **p. 97, Notation above Exercise 3.5.10** change $N_{k,x}^*$ to $N_{k-1,x}^*$
- **p. 97, (3.11)** insert right parenthesis before “mod”
- **p. 99, line 3** replace paragraph with: Recall that we may identify $GL(V) \simeq \mathcal{F}_{\mathbb{P}V}$ as sets. If $X = G/P \subset \mathbb{P}V$ is homogeneous for some group $G \subset GL(V)$, we may further identify G with a subbundle of $\mathcal{F}_{\mathbb{P}V}$ and work with this subbundle.
- **p. 99, second display** change $\mathbb{P}S^k$ to S^k
- **p. 99, line -4 and line -2** change $\vec{E}_s^j = e_1 \wedge \dots$ to $\vec{E}_s^j = (-1)^{j-1}e_1 \wedge \dots$; change “and so on” to “and so on with appropriate signs”
- **p. 108, first displayed eqn.** there are missing terms, this is only part of the ambiguity.
- **p. 110, 2nd paragraph** change $\binom{n+a+1}{2} - n$ to $\binom{n+a+2}{2} - (n+1)$ and change $\binom{n+a+1}{2} - n - \binom{n+1}{2}$ to $\binom{n+a+2}{2} - (n+1) - \binom{n+1}{2}$
- **p. 147, Exercise 4.1.8** add new Exercise: Explicitly calculate $A^{(1)}$ when A is the tableau for the Cauchy-Riemann equations.
- **p. 148, line 1** “determine” instead of “determining”

- p. 148, (4.7) note that $2 \leq \sigma \leq n$
- p. 148, last line last equation should read $\dots = C_{\sigma b}^a C_{\rho c}^b p_{11}^c$.
- p. 152, (4.15) change “solution to (4.14)” to “solution to (4.14)”
- p. 156, displayed eqn. on last line, change $s_1 - s$ to $s - s_1$
- p. 159, Theorem 4.6.12 switch the roles of $\dim \Xi_A^{\mathbb{C}} + 1$ and $\widetilde{\deg}(\Xi_A^{\mathbb{C}})$.
- p. 164, Exercise 5.1.2 change “Show that there exist” to “Show that locally there exist”
- p. 173, lines 8-9 change “tableau $A^{(1)}$ associated to (5.7)” to “tableau $A^{(1)}$ associated to (*)” where (*) is the last equation on p. 172.
- p. 174, line 1 change “there exist” to “there locally exist”
- p. 174, line 8 A_{ei}^a (instead of A_{ie}^a)
- p. 174, Remark 5.5.2 Replace with: “Let $B = \{B_a^{ri} v_i \otimes w^a\} \subset V \otimes W^*$ denote the annihilator of A , then if...” now continue with the second line and displayed equation. Then add, “These equations are sometimes called the *symbol relations*.”
- p. 175, second displayed equation $\pi_1 = dr + (\dots)dx + \dots$ instead of $\pi_1 = dr + (\dots) + \dots$
- p. 176, line -6 “out” instead of “our”
- p. 179, Proposition 5.7.1 start with “Let I be a linear Pfaffian system without torsion on a manifold Σ .”
- p. 183, Exercise 5.8.2 add hypothesis $A \neq 0$.
- p. 184, line 13 $= -\omega_j^i \wedge \omega^j$ instead of $= -\omega_j^i \omega^j$
- p. 184, second displayed eqns. $d\theta^2 = -\omega_1^2 \wedge \omega^1 \dots$ instead of $d\theta^2 = -\underline{\omega_1^2} \wedge \omega^1 \dots$
- p. 188, Example 5.8.14 add the assumption that $u \neq 0$
- p. 188, last displayed eqns. change ω_1^2 to $u\omega_1^2$ on both lines
- p. 189, first displayed eqn. change ω_1^2 to $u\omega_1^2$
- p. 192, line 10 change to “ $\mathcal{K}(V) = S_{22}(V)$ is the intersection of $S^2(\Lambda^2 V)$ with the kernel of the skew-symmetrization map”
- p. 192, line 15 precise reference is Exercise A.6.16

- p. 195, line below (5.25) $\{v \lrcorner h^\xi\}^\perp$ instead of $\{v \lrcorner h^\xi\}$
- p. 196, second to last displayed eqns. change ω^k to ω^j on second line
- p. 211, proof of 6.1.19, line 2 change $\pi(U)$ to $\sigma(U)$
- p. 218, Def. 6.3.7 \mathcal{M}_i instead of \mathcal{M}
- p. 218, Prop. 6.3.8, line 3 $dW_i, dX_i \in \Delta_i$ instead of $W_i, X_i \in \Delta_i$
- p. 240, diagram change M_1 to Σ_1
- p. 246, paragraph below 7.1.5 change ϕ to φ two times.
- p. 259, Section 7.5, line 6 suppress f instead of suppress s
- p. 261, line -6 insert a period after “five-manifold”
- p. 270, first displayed eqn. $\frac{1}{\lambda}$ instead of λ in front of last vector.
- p. 275, Exercise 8.1.16 move this exercise to p. 278, just above last paragraph
- p. 276, para. 2, line 3 $C^\infty(TM, TE)$ instead of $C^\infty(TM, E)$
- p. 286, Exercise 8.4.7 displayed equation should read

$$\frac{d^2y}{dx^2} = \Gamma_{22}^1 \left(\frac{dy}{dx}\right)^3 + (2\Gamma_{12}^1 - \Gamma_{22}^2) \left(\frac{dy}{dx}\right)^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2) \left(\frac{dy}{dx}\right) - \Gamma_{11}^2.$$

- p. 286, Prop 8.4.8 displayed equation should read
- $$\frac{d^2y}{dx^2} = A \left(\frac{dy}{dx}\right)^3 + B \left(\frac{dy}{dx}\right)^2 + C \left(\frac{dy}{dx}\right) + D$$
- p. 295, Example 8.6.4 change “Finsler metric” to “reversible Finsler metric”
 - p. 296, Example 8.6.6 This is actually an example of a generalized path geometry, as defined in the correction to p. 298 below, and should be moved to there, just above Example 8.6.14
 - p. 298, paragraph two change *path geometry* to *generalized path geometry*, as the definition here is slightly more general than the one on p. 295
 - p. 300, line 2 below 2nd displayed eqn. replace “is” with “must be”
 - p. 313, last line $v_2 \otimes v_2$ should be $v_1 \otimes v_2$.

- **p. 314, Exercise A.1.8.3** should read: ... we define $\alpha \wedge \beta \in \Lambda^{i+j}V$ by the composition of the inclusion $\Lambda^i V \otimes \Lambda^j V \subset V^{\otimes i+j}$ and then the quotient to $\Lambda^{i+j}V$ by (A.3), divided by $i!j!$.
Remark: given this annoyance of constant, it would have been better to define (A.3) with a $\frac{1}{n!}$.
- **p. 315, line 3** Add the assumption that $\dim V = k$. At the end, add: What happens if $\dim V > k$?
- **p. 322, A.4.8.4** change “then” to “is an $n \times n$ matrix then”
- **p. 323, A.5, line 4** $\frac{n(n+2)(n+1)}{6}$ instead of $\frac{(n+2)(n+1)}{6}$
- **p. 329, line 1** change “ R is an \mathbb{R} -linear” to “ R lies in an \mathbb{R} -linear”
- **p. 335, B.1.2.2** Since local coordinates are not defined and the problem follows almost immediately from their definition, this should probably be a remark instead of a problem.
- **p. 335, B.1.2.3** should read $T^*M = \cup_{x \in M} T_x^*M$, not $T^*M = \cup_{x \in M} T_x M$
- **p. 336, B.1.3.1** should read “only on the function f and the value ...” not “only on the value ...”
- **p. 357, 3.3.15.1, last line** change $II_{X,x}$ to $II_{X',x'}$

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