Corrections/comments to

# *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*

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Note: Negative line numbers indicate lines up from the bottom of the page, so that "line -1" indicates the last line, "line -2" the line above it, etc.

- **p. 6, line 7** insert period before "The flow is given"
- **p. 7, Exercise 1.2.4** Hint: use Stokes' theorem applied to integrating about the boundary of a rectangle.
- **p. 11, line -8** change $\Delta \subset \Gamma(T\Sigma)$ to $\Delta \subset T\Sigma$
- **p. 12, Exercise 1.3.6.1** Should read: Which of the following subbundles are Frobenius?

$$I_1 = \{dx^1, x^2dx^3 + dx^4\}$$
$$I_2 = \{dx^1, x^1dx^3 + dx^4\}$$

- **p. 14, Exercise 1.4.2.2** change $\kappa(s) = \left|\frac{de_1}{ds}\right|$ to $\kappa(s) = \langle e_2, \frac{de_1}{ds}\rangle$
- **p. 17, Def. 1.6.4, line 2** change $\omega|_e$ to $\omega_e$
- **p. 17, line -1** change $\mathrm{Ad}_{a^{-1}}$ to $\mathrm{Ad}(a^{-1})$
- **p. 18, Def. 1.6.9** right-hand side of displayed equation should read $[\omega(X), \theta(Y)] - [\omega(Y), \theta(X)]$ (not +)
- **p. 19, Proof of 1.6.10, last line** the "uniqueness" referred to is in the Frobenius Theorem
- **p. 20, Exercise 1.6.14.3** change "complete the proof of" to "prove"
- **p. 20, Exercise 1.6.14.6** should begin "On a connected Lie group" and displayed equation should read $d\omega_e(X, Y) = -[X, Y]$ (not +)

- **p. 22, Exercise 1.7.2.2** change $f = x^n$ to $f = z^n$
- **p. 25, line 1** referenced equation should be (1.2.6), not (1.2.7)
- **p. 25, 3 lines before Remark 1.8.3** change $d(t)$ to $dt$ and $\omega^3_2 + \tau(t)\omega^1$ to $\omega^3_2 - \tau(t)\omega^1$
- **p. 26, Exercise 1.8.4.3** assume $a, b \neq 0$
- **p. 27, line -2** add $v(q) = 0$
- **p. 28, line 12** between 1 and $\dim M$ (not between 1 and $\dim N$)
- **p. 29, line 1** $J^k(M, N)$ (not $J^k(M.N)$)
- **p. 29, line -2** $c\,dx^3$ (not $e\,dx^3$)
- **p. 30, line -5** $1 \leq i, j \leq m - n$ (not $1 \leq i, j \leq n - m$)
- **p. 32, Exercise 1.9.16(a)** Change question to "At which points is $\frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$ an integral element? What about $\frac{\partial}{\partial y^1} \wedge \ldots \wedge \frac{\partial}{\partial y^n}$? "
- **p. 42, Exercise 2.2.2.4(c)** $t(v) = \sin(v)$, not $t = \sin(v)$
- **p. 42, Exercise 2.3.1.1** change $k_1$ to $|k_1|$
- **p. 42, Exercise 2.3.1.2** Hint: differentiate $\omega^3_1 = k_1\omega^1$ and $\omega^3_2 = k_2\omega^2$
- **p. 42, Exercise 2.3.1.3** displayed equation should read

$$[X_1, X_2] = -\frac{k_{1,2}}{k_1 - k_2} X_1 - \frac{k_{2,1}}{k_1 - k_2} X_2$$

- **p. 43, Exercise 5** Displayed equation should read

$$-k_1 k_2 = \frac{1}{(k_1 - k_2)^2}((k_1 - k_2)(k_{2,1,1} - k_{1,2,2}) - k_{1,2}k_{2,2} - k_{2,1}k_{1,1} + 2(k^2_{1,2} + k^2_{2,1})).$$

- **p. 45, displayed eqn. above (2.11)** change $\omega^a_j = h^a_{ij}\omega^j$ to $\omega^a_i = h^a_{ij}\omega^j$
- **p. 45, eqn. (2.12)** $\tilde{h}^a_{ij}$ instead of $\tilde{h}^a_i j$
- **p. 47, Prop. 2.5.3** $\langle II(v, w), e_3 \rangle$ instead of $II(v, w)$

- **p. 49, displayed equation** last line in displayed equation should be numbered, and reference in lines below should be to this equation, not (2.6).
- **p. 49, line -16** should read "bijective linear maps" (not "linear maps")
- **p. 51, Exercise 2.6.6** change $\eta^k$ to $\overline{\eta}^k$ two times
- **p. 51, paragraph below 2.6.7** change $\mathbb{E}^{n+s}$ to $\mathbb{E}^{n+r}$ two times, to avoid confusion with section $s$.
- **p. 52, proof of 2.6.12, line 1** reference should be to Theorem 1.6.10, not 1.6.11
- **p. 54, Exercise 2(e)** First displayed equation should read

$$\nabla\alpha = (da_i - a_j\eta_i^j)\otimes\eta^i \in \Gamma(T^*M\otimes T^*M).$$

  One way to check the minus sign is to note that $Id_T = \eta^i\otimes e_i$ and use the Leibniz rule to verify that $\nabla(Id_T) = 0$.
- **p. 56, second displayed equation** The proper definition of the Laplacian has more complicated signs. One definition is to let $\delta = (-1)^{n(k+1)+1} * d *$ and define $\Delta = \delta d + d\delta$. The introduction of $\delta$ is natural because $\delta$ is the adjoint of the operator $d$ with respect to the $L_2$-inner product on forms on a compact manifold.
- **p. 56, Exercise 2.6.18** The sign error above makes for sign errors here, also, more is needed to do part (c) than discussed in the text.Finally, add the additional exercise to show that with respect to an orthonormal framing $e_1, \ldots, e_n$ on a Riemannian manifold, $\Delta f = \sum_i f_{ii} - df(\nabla_{e_i}e_i)$.
- **p. 57, line 8** $u(x^1, \ldots, x^n)$ instead of $u(x, y)$
- **p. 57, line before Exercise 2.6.19** $\partial x$ and $\partial y$ should be read as $\partial/\partial x$ and $\partial/\partial y$, respectively
- **p. 57, Exercise 2.6.19(a)** Displayed equation should be $K = -\Delta u$. One obtains the expression in the book if one uses the Laplacian with respect to the flat metric on $\mathbb{R}^2$ with coordinates $x, y$. The remark following the exercise is valid for the Laplacian with respect to the flat metric.
- **p. 59, first displayed eqn.** should read

$$d(x, T, N, B) = (x, T, N, B)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{pmatrix} ds.$$

- **p. 59, (2.19)** should read

$$d(x,T,\epsilon,e_3) = (x,T,\epsilon,e_3)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa_g & -\kappa_n \\ 0 & \kappa_g & 0 & -\tau_g \\ 0 & \kappa_n & \tau_g & 0 \end{pmatrix} ds$$

- **p. 59, line -4** Change $\kappa_g(c) = \nabla_{c'}c'$ to $\nabla_{c'}c' = \kappa_g\epsilon$ here, and in Exercise 2.8.1.3 on page 60.

- **p. 60, Exercise 2.8.1.1** change first displayed formula to $\kappa_g = \langle II(T,T), e_3\rangle$, and change second displayed formula to $\tau_g = \langle II(\epsilon,T), e_3\rangle$

- **p. 63, picture** Picture is mis-labelled; the curves should be labelled in counter-clockwise order, and $\delta_1$ should be the angle between the tangents to $C_1$ and $C_4$, etc.

- **p. 64, first displayed equation** should read

$$\int_R K\,dA = 2\pi - \int_{\partial R} \kappa_g ds - \sum_i \delta_i$$

- **p. 64, Exercise 2.9.4.3** To do this exercise, you need to use the higher-dimensional version of the Poincaré Hopf Theorem; see p.446 in Spivak (reference [182] in the bibliography) for further details.

- **p. 65, displayed equation and below** Replace Pfaff$(\Theta)dV$ by Pfaff$(\Theta)$ five times.

- **p. 68, Exercise 2.10.4** The displayed equation for $Q$ should have the vector $(\omega^1, \omega^2)$ multiplied on the left of the matrix, as well as on the right.

- **p. 68, last displayed equation** change $t\,s\,h_{11}$ to $2t\,s\,h_{11}$ and change $t\,s\,h_{22}$ to $2t\,s\,h_{22}$ in this matrix

- **p. 72, line -3** $gP_k$ instead of $gP_kg^{-1}$

- **p. 75, last displayed eqn.** $|K_x|$ instead of $K_x$ and $\{x\}$ instead of $x$ under lim.

- **p. 78, line 2** Replace line with: "with $\mathcal{F}_{\mathbb{P}V}$, the set of bases of $V$. We consider $\mathcal{F}_{\mathbb{P}V}$ as a frame bundle over $\mathbb{P}V$, because $(e_1 \bmod e_0)\otimes e^0, \dots, (e_{n+a} \bmod e_0)\otimes e^0$ is a basis of $T_{[e_0]}\mathbb{P}V$. Given $M^n \subset \mathbb{P}V$ ..."

- **p. 79, displayed eqn. after Exercise 3.2.8** change $\tilde{e}_\nu$ to $\tilde{e}_\mu$ in third line

- **p. 79, line -2** change $\tilde{\omega}^\alpha$ to $\tilde{\omega}^\alpha_0$

- **p. 79, last displayed equation** should read

$$\tilde{q}^{\mu}_{\alpha\beta} = h^0_0 g^{\alpha}_{\beta} g^{\delta}_{\alpha} h^{\mu}_{\nu} q^{\nu}_{\gamma\delta}$$

- **p. 84, line -16** change “rank one matrices” to “rank one matrices up to scale”
- **p. 84, line 3 above 3.3.6** $\mathbb{P}W$ instead of $BPW$
- **p. 85, Exercise 3.3.8.3** change “rank one and traceless matrices” to “rank one traceless matrices, up to scale”
- **p. 88, above (3.2)** add: Here and in what follows we identify $\mathbb{F}_{s+1,t+1}(V)$ with the space of $\mathbb{P}^s$'s contained in $\mathbb{P}^t$'s contained in $\mathbb{P}V$.
- **p. 89-90** To avoid confusion rename $f = \dim F$ as $\phi = \dim F$ (5 instances).
- **p. 94, line 8** change 3.3.82 to 3.3.8.2.
- **p. 96, Exercise 3.5.3.2** change $\mathcal{F}2$ to $\mathcal{F}^2$.
- **p. 96, Def. 3.5.4** change $N_{2x}$ to $N_{2,x}$ in two places
- **p. 96, below Def. 3.5.4** add: Let $T^{(k)}M$ denote the vector bundle defined over general points of $M$ whose fiber at $x \in M$ is $(\hat{T}^{(k)}_x M/\hat{x})\otimes\hat{x}^*$, so in particular $T^{(k)}_x M \subset T_x\mathbb{P}V$ is a vector subspace.
- **p. 97, k-th fundamental form paragraph** change $S^kT_xM$ to $S^kT^*_xM$; in displayed equation, change $X$ to $M$, change $\mathbb{FF}^{(k-1)}_x$ to $\mathbb{FF}^{(k)}_x$ and change $S^{k-1}$ to $S^k$
- **p. 97, Notation above Exercise 3.5.10** change $N^*_{k,x}$ to $N^*_{k-1,x}$
- **p. 97, (3.11)** insert right parenthesis before “mod”
- **p. 99, line 3** replace paragraph with: Recall that we may identify $GL(V) \simeq \mathcal{F}_{\mathbb{P}V}$ as sets. If $X = G/P \subset \mathbb{P}V$ is homogeneous for some group $G \subset GL(V)$, we may further identify $G$ with a subbundle of $\mathcal{F}_{\mathbb{P}V}$ and work with this subbundle.
- **p. 99, second display** change $\mathbb{P}S^k$ to $S^k$
- **p. 99, line -4 and line -2** change $\vec{E}^j_s = e_1 \wedge ...$ to $\vec{E}^j_s = (-1)^{j-1} e_1 \wedge ...$; change “and so on”to “and so on with appropriate signs”
- **p. 108, first displayed eqn.**There are terms missing from this formula. The following correction was kindly provided by Colleen Robles.

  Recall that we use $\mathfrak{S}$ to denote cyclic summation. If $T_{a_1 \dots a_k}$ and $U_{b_1 \dots b_\ell}$ are symmetric tensors, let $\mathcal{S}_{a_1 \dots a_k, b_1 \dots b_\ell} T_{a_1 \dots a_k} U_{b_1 \dots b_\ell}$ denote the symmetrization of the product.

(This is a summation over the $\binom{k+\ell}{k}$ elements of the symmetric group $S_{k+\ell}$ that symmetrize the product.) For example, $\mathcal{S}_{a_1\dots a_k,b} = \mathfrak{S}_{a_1\dots a_k b}$ and $\mathcal{S}_{ab,cd} = \mathfrak{S}_{abc}\cdot\mathfrak{S}_{bd}$. With this notation in place, let $X^n \subset \mathbb{P}^{n+s}$ and fix index ranges $1 \le a,b \le n$, and $n+1 \le u,v \le n+s$. Assume that $p \ge 2$, and set $p = 2q$ if $p$ is even, and $p = 2q+1$ if $p$ is odd. Then the coefficients of $F_{p+1}$ are given by

$$
\begin{aligned}
r^u_{a_1\dots a_p b}\omega^b_0 \;=\;& - dr^u_{a_1\dots a_p} - (p-1)\, r^u_{a_1\dots a_p}\,\omega^0_0 - r^v_{a_1\dots a_p}\,\omega^u_v \\
&+ \mathfrak{S}_{a_1\dots a_p}\Big((p-2)\, r^u_{a_1\dots a_{p-1}}\omega^0_{a_p} + r^u_{a_1\dots a_{p-1}b}\,\omega^b_{a_p} - q^u_{a_p b}\, r^v_{a_1\dots a_{p-1}}\,\omega^b_v\Big) \\
&- \sum_{k=q+1}^{p-2} \mathcal{S}_{a_1\dots a_k, a_{k+1}\dots a_p}\Big\{ \Big(r^u_{a_1\dots a_k b}\, r^v_{a_{k+1}\dots a_p} + r^u_{a_{k+1}\dots a_p b}\, r^v_{a_1\dots a_k}\Big)\,\omega^b_v \\
&\qquad\qquad + \Big( (k-1)\, r^u_{a_1\dots a_k}\, r^v_{a_{k+1}\dots a_p} \\
&\qquad\qquad\quad + (p-k-1)\, r^u_{a_{k+1}\dots a_p}\, r^v_{a_1\dots a_k}\Big)\,\omega^0_v\Big\} \\
&- (p+1 \bmod 2)\, \mathcal{S}_{a_1\dots a_q, a_{q+1}\dots a_p}\Big\{(q-1)\, r^u_{a_1\dots a_q}\, r^v_{a_{q+1}\dots a_p}\,\omega^0_v \\
&\qquad\qquad + r^u_{a_1\dots a_q b}\, r^v_{a_{q+1}\dots a_p}\,\omega^b_v\Big\},
\end{aligned}
$$

where $q^u_{ab}$ are the components of the projective second fundamental form. Note that the last term appears only when $p$ is even.

- **p. 110, 2nd paragraph** change $\binom{n+a+1}{2} - n$ to $\binom{n+a+2}{2} - (n+1)$ and change $\binom{n+a+1}{2} - n - \binom{n+1}{2}$ to $\binom{n+a+2}{2} - (n+1) - \binom{n+1}{2}$
- **p. 147, Exercise 4.1.8** add new Exercise: Explicitly calculate $A^{(1)}$ when $A$ is the tableau for the Cauchy-Riemann equations.
- **p. 148, line 1** "determine" instead of "determing"
- **p. 148, (4.7)** note that $2 \le \sigma \le n$
- **p. 148, last line** last equation should read ... $= C^a_{\sigma b} C^b_{\rho c} p^c_{11}$.
- **p. 152, (4.15)** change "solution to (4.14)" to "solution to (4.14)"
- **p. 156, displayed eqn.** on last line, change $s_1 - s$ to $s - s_1$
- **p. 159, Theorem 4.6.12** switch the roles of $\dim \Xi^{\mathbb{C}}_A + 1$ and $\widetilde{deg}(\Xi^{\mathbb{C}}_A)$.
- **p. 164, Exercise 5.1.2** change "Show that there exist" to "Show that locally there exist"

- **p. 173, lines 8-9** change "tableau $A^{(1)}$ associated to (5.7)"to "tableau $A^{(1)}$ associated to (*)"where (*) is the last equation on p. 172.

- **p. 174, line 1** change "there exist" to "there locally exist"

- **p. 174, line 8** $A^a_{\epsilon i}$ (instead of $A^a_{i\epsilon}$ )

- **p. 174, Remark 5.5.2** Replace with: "Let $B = \{B^{ri}_a v_i \otimes w^a\} \subset V \otimes W^*$ denote the annihilator of $A$, then if..."now continue with the second line and displayed equation. Then add, "These equations are sometimes called the *symbol relations*."

- **p. 175, second displayed equation** $\pi_1 = dr + (...)dx + ...$ instead of $\pi_1 = dr + (...) + ...$

- **p. 176, line -6** "out" instead of "our"

- **p. 179, Proposition 5.7.1** start with "Let $I$ be a linear Pfaffian system without torsion on a manifold $\Sigma$."

- **p. 183, Exercise 5.8.2**add hypothesis $A \neq 0$.

- **p. 184, line 13** $= -\omega^i_j \wedge \omega^j$ instead of $= -\omega^i_j \omega^j$

- **p. 184, second displayed eqns.** $d\theta^2 = -\omega^2_1 \wedge \omega^1 ...$ instead of $d\theta^2 = -\underline{\omega^2_1} \wedge \omega^1 ...$

- **p. 188, Example 5.8.14** add the assumption that $u \neq 0$

- **p. 188, last displayed eqns.** change $\omega^2_1$ to $u\omega^2_1$ on both lines

- **p. 189, first displayed eqn.** change $\omega^2_1$ to $u\omega^2_1$

- **p. 192, line 10** change to "$\mathcal{K}(V) = S_{22}(V)$ is the intersection of $S^2(\Lambda^2 V)$ with the kernel of the skew-symmetrization map"

- **p. 192, line 15** precise reference is Exercise A.6.16

- **p. 195, line below (5.25)** $\{v \lrcorner h^\xi\}^\perp$ instead of $\{v \lrcorner h^\xi\}$

- **p. 196, second to last displayed eqns.** change $\omega^k$ to $\omega^j$ on second line

- **p. 211, proof of 6.1.19, line 2** change $\pi(U)$ to $\sigma(U)$

- **p. 218, Def. 6.3.7** $\mathcal{M}_i$ instead of $\mathcal{M}$

- **p. 218, Prop. 6.3.8, line 3** $dW_i, dX_i \in \Delta_i$ instead of $W_i, X_i \in \Delta_i$

- **p. 240, diagram** change $M_1$ to $\Sigma_1$

- **p. 239–240, proof of Theorem 6.5.14** Mr. Kazuhiro Shibuya (Hokudai University) has pointed out that the proof given for this theorem is incorrect where it is asserted that "any rank two system inside a three-dimensional Frobenius system is itself Frobenius". This is not true; for example, on $\mathbb{R}^4$ with coordinates $(x, y, p, q)$, the Pfaffian system generated by the one-forms $dy - p\,dx$ and $dp - q\,dx$ lies in the Pfaffian system generated by $dx, dy, dp$, but is not itself Frobenius. This mistake was made in the course of proving the last assertion in the statement of the theorem, i.e., if a Bäcklund tranformation exists between a hyperbolic EDS $\mathcal{I}_1$ of class $k$ and the wave equation system on $\mathbb{R}^5$, then $\mathcal{I}_1$ is Darboux integrable after at most one prolongation.

  We hope to post a simple proof of this assertion here in the near future.

- **p. 246, paragraph below 7.1.5** change $\phi$ to $\varphi$ two times.

- **p. 256, line 6 of Proof of Thm. 7.3.3** $f^a$ should be $F^a$

- **p. 256, (7.7)** To clarify, $\text{codim}_{E_n} \mathcal{V}_n(\mathcal{I})$ means $\text{codim}_{E_n}(\mathcal{V}_n(\mathcal{I}), G_n(T\Sigma))$.

- **p.258–259, Remark 7.4.4** Note that $s_0$ is the same as the rank of the Pfaffian system.

- **p. 259, Section 7.5, line 6** suppress $f$ instead of suppress $s$

- **p. 261, line -6** insert a period after "five-manifold"

- **p. 265, line 10** "In general, any" should be replaced by "Sometimes".

- **p. 270, first displayed eqn.** $\frac{1}{\lambda}$ instead of $\lambda$ in front of last vector.

- **p 274, Defn. 8.1.13** Replace this definition with: Let $G \subset GL(V)$ be a matrix Lie group where $V \simeq \mathbb{R}^n$. A *G-structure* on an $n$-dimensional differentiable manifold $M$ is a reduction of $\mathcal{F}_{GL}$ to a right $G$-bundle $\mathcal{F}_G \subset \mathcal{F}_{GL}$.

- **p. 275, Exercise 8.1.16** move this exercise to p. 278, just above last paragraph

- **p. 276, para. 2, line 3** $C^\infty(TM, TE)$ instead of $C^\infty(TM, E)$

- **p. 286, Exercise 8.4.7** displayed equation should read

$$\frac{d^2y}{dx^2} = \Gamma^1_{22}(\frac{dy}{dx})^3 + (2\Gamma^1_{12} - \Gamma^2_{22})(\frac{dy}{dx})^2 + (\Gamma^1_{11} - 2\Gamma^2_{12})(\frac{dy}{dx}) - \Gamma^2_{11}.$$

- **p. 286, Prop 8.4.8** displayed equation should read

$$\frac{d^2y}{dx^2} = A(\frac{dy}{dx})^3 + B(\frac{dy}{dx})^2 + C(\frac{dy}{dx}) + D$$

- **p. 295, Example 8.6.4** change "Finsler metric" to "reversible Finsler metric"

- **p. 296, Example 8.6.6** This is actually an example of a generalized path geometry, as defined in the correction to p. 298 below, and should be moved to there, just above Example 8.6.14

- **p. 298, paragraph two** change *path geometry* to *generalized path geometry*, as the definition here is slightly more general than the one on p. 295

- **p. 300, line 2 below 2nd displayed eqn.** replace "is" with "must be"

- **p. 313, last line** $v_2 \otimes v_2$ should be $v_1 \otimes v_2$.

- **p. 314, Exercise A.1.8.3** should read: ... we define $\alpha \wedge \beta \in \Lambda^{i+j} V$ by the composition of the inclusion $\Lambda^i V \otimes \Lambda^j V \subset V^{\otimes i+j}$ and then the quotient to $\Lambda^{i+j} V$ by (A.3), divided by $i!\,j!$.

  *Remark*: given this annoyance of constant, it would have been better to define (A.3) with a $\frac{1}{n!}$.

- **p. 315, line 3** Add the assumption that $\dim V = k$. At the end, add: What happens if $\dim V > k$?

- **p. 322, A.4.8.4** change "then" to "is an $n \times n$ matrix then"

- **p. 323, A.5, line 4** $\frac{n(n+2)(n+1)}{6}$ instead of $\frac{(n+2)(n+1)}{6}$

- **p. 329, line 1** change "$R$ is an $\mathbb{R}$-linear" to "$R$ lies in an $\mathbb{R}$-linear"

- **p. 335, B.1.2.2** Since local coordinates are not defined and the problem follows almost immediately from their definition, this should probably be a remark instead of a problem.

- **p. 335, B.1.2.3** should read $T^*M = \cup_{x \in M} T^*_x M$, not $T^*M = \cup_{x \in M} T_x M$

- **p. 336, B.1.3.1** should read "only on the function $f$ and the value ..." not "only on the value ..."

- **p. 338, Fancy definition (iii)** $\eta$ should be $\beta$

- **p. 357, 3.3.15.1, last line** change $II_{X,x}$ to $II_{X',x'}$

We thank Colleen Robles (Univ. Rochester) for providing many corrections.