Corrections to Probability Theory in Finance

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Exercises

(3.5) $f : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$

(3.15) Find a partition $\mathcal{P}$ of Ω such that $\mathcal{F}(\mathcal{P})$ is the σ-field generated by ......

(3.16) $\mathcal{F}_1 \cup \mathcal{F}_2$

(5.7) $\frac{e^{-.2}(.2)^n}{n!}$in place of $\frac{e^{(.2)n}(.2)^n}{n!}$

(5.17) $X = \mathbf{1}_{\{1,2\}} - \mathbf{1}_{\{3,4,5,6\}}$

(5.20) \$1.793 in place of \$1.788

Solutions

Items with (∗) are improvements or alternatives to correct solutions.

(1.6) (a) −\$103.46, (b) +\$166.10.

(5.7) .1815

(5.18) (a) \$.055, (c) −\$.37,

(6.3) 1621/444 in place of 35/222.

(6.14∗) Apply expected values and Definition 6.1 to $\mathbf{1}_A \leq f \leq \mathbf{1}_{B^c}$.

(7.9) $\exp(1/2), \exp(1/2)$, $e^{(-2)} - 2N(-2)$ where N is obtained from the normal tables.

(8.1) $a = 385$.

Text

(page,line), + from the top, − from the bottom.

(5,-8) 5.76% in place of 5.63%

(47,-2) $(B_n)_{n=0}^{\infty}$ in place of $(B_n)_{n=1}^{\infty}$

(87,-2) $ue^{-rT} - de^{-rT}$ in place of $ue^{-rT} - de^{-rT}d$

(111,-1) $\mathbb{E}[X - Y] \geq \mathbb{E}[Y]$ in place of $\mathbb{E}[X - Y] \geq 0$

(94,+1) every $\mathcal{G}$-set is obtained by combining $\mathcal{F}$-sets,

(117,+16) $n + 1$ in place of $2n + 1$

(127,+3) Lemma 6.21 in place of Proposition 6.21

(147,-2) 5.19 in place of 5.18

(189,+10) $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}]$ in place of $\mathbb{E}[\mathbb{E}[X|\mathcal{G}|\mathcal{H}]]$

(214,+3) $(\sqrt{2\pi}\sigma)^{-1}$ in place of $(\sqrt{2\pi}\sigma)^{-1/2}$

(233,+7) $\Delta x = \sqrt{\Delta t}$ in place of $\sqrt{n}(\Delta x) = \sqrt{\Delta t}$

(233,-1) $\int_{\mathbf{R}}$ in place of $\int_{-\infty}^{x}$

(264,-13) $Z_t dW_t$ in place of $Z_t^2 dW_t$

Miscellaneous

1. Proposition 9.15(c) should read: If $\gamma \in \mathbf{R}$, then

$$\lim_{m \longrightarrow \infty} \left(\sup_{t \geq 0} P(\{e^{-\frac{1}{2}\gamma^2 t - \gamma W_t} \geq m\}) \right) = 0$$

In Proposition 9.15(d) X_T should be used in place of X_∞ and one takes $T \geq t \geq s$ in place of $t \geq s$. This has consequences for section 10.3 where one should use Z_T in place of Z_∞.

2. In Chapter 10, Figures 10.1 and 10.2 should be interchanged.

3. Equation (11.19) is not true but it is only needed to prove (11.20) which is correct. The following is a proof of (11.20)-some of this is already in the text but for convenience we give the full proof here.
   Proof. The random variable

$$S_{i,n} := (W_{t^n_{i+1}} - W_{t^n_i})^2 - (t^n_{i+1} - t^n_i)$$

   has a $(t^n_{i+1} - t^n_i) \cdot (X^2 - 1)$ distribution where X is $N(0,1)$. By Exercise 1.3 and Proposition 7.16,

$$\mathbb{E}[(X^2-1)^2] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (x^2-1)^2 e^{-x^2/2} dx =: \alpha < \infty.$$

   Hence $\mathbb{E}[S_{i,n}] = 0$ and $\mathbb{E}[S^2_{i,n}] = (t^n_{i+1} - t^n_i)^2 \alpha$.
   We first suppose $|X_t(\omega)| \leq M$ and $|Y_t(\omega)| \leq M$ for all paths ω and $0 \leq t \leq T$. Let $Z_t = X_t \cdot Y_t$ for $t \geq 0$. Since the process $(Z_t)_{t \geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ it follows that $Z_{t^n_i}$ is independent of $S_{j,n}$ for $i \leq j$. Hence

$$\begin{aligned}
\mathbb{E}[|\sum_{i=0}^{k_n-1} Z_{t^n_i} S_{i,n}|^2] &= \sum_{i=0}^{k_n-1} \mathbb{E}[Z^2_{t^n_i} S^2_{i,n}] + \sum_{0 \leq i < j \leq k_n - 1} \mathbb{E}[Z_{t^n_i} Z_{t^n_j} S_{i,n} S_{j,n}] \\
&= \sum_{i=0}^{k_n-1} \mathbb{E}[Z^2_{t^n_i} S^2_{i,n}] + \sum_{0 \leq i < j \leq k_n - 1} \mathbb{E}[Z_{t^n_i} Z_{t^n_j} S_{i,n}] \mathbb{E}[S_{j,n}] \\
&\leq M^2 \sum_{i=0}^{k_n-1} \mathbb{E}[S^2_{i,n}] \\
&= M^2 \alpha \sum_{i=0}^{k_n-1} (t^n_{i+1} - t^n_i)^2.
\end{aligned}$$

   Since

$$\sum_{i=0}^{k_n-1} (t^n_{i+1} - t^n_i)^2 \leq \sup_{0 \leq i < k_n} |t^n_{i+1} - t^n_i| \cdot \sum_{i=0}^{k_n-1} (t^n_{i+1} - t^n_i) = T \sup_{0 \leq i < k_n} |t^n_{i+1} - t^n_i|$$

tends to 0 as $n \longrightarrow 0$ we have $\mathbb{E}[|\sum_{i=0}^{k_n-1} Z_{t_i^n} S_{i,n}|^2] \longrightarrow 0$ as $n \longrightarrow \infty$. By Proposition 11.6 this implies

$$\sum_{i=0}^{k_n-1} Z_{t_i^n} S_{i,n} = \sum_{i=0}^{k_n-1} X_{t_i^n} Y_{t_i^n} \big((W_{t_{i+1}^n} - W_{t_i^n})^2 - (t_{i+1}^n - t_i^n)\big) \longrightarrow 0$$

in measure as $n \longrightarrow \infty$. By using a *diagonal process*, as in the proof of Proposition 11.14, we can remove the boundedness hypothesis on $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ and, from Section 11.3, we see that

$$\sum_{i=0}^{k_n-1} X_{t_i^n} Y_{t_i^n} (t_{i+1}^n - t_i^n) \longrightarrow \int_0^T X_t Y_t dt$$

almost surely as $n \longrightarrow \infty$. Hence

$$\sum_{i=0}^{k_n-1} X_{t_i^n} Y_{t_i^n} (W_{t_{i+1}^n} - W_{t_i^n})^2 \longrightarrow \int_0^T X_t Y_t dt \tag{1}$$

in measure as $n \longrightarrow \infty$. If we let $X_t(\omega) = Y_t(\omega) = 1$ for all $t \geq 0$ and all paths ω, this implies $\sum_{i=0}^{k_n-1}(W_{t_{i+1}^n} - W_{t_i^n})^2 \longrightarrow T$ in measure as $n \longrightarrow \infty$.

It remains to show that we can replace $X_{t_i^n} Y_{t_i^n}$ by $X_{t_{i*}^n} Y_{t_{i**}^n}$ in (1). By considering subsequences of subsequences it suffices to show that this is possible whenever we have almost sure convergence of the sequence $\big(\sum_{i=0}^{k_n-1}(W_{t_{i+1}^n} - W_{t_i^n})^2\big)_{n=1}^{\infty}$. As $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are both continuous processes we may suppose that the almost sure convergence is pointwise convergence over a set A consisting of X and Y continuous paths. If $\omega \in A$ and $\epsilon > 0$ are arbitrary we can choose, using Proposition 7.5, n_0 sufficiently large so that

$$\Big|\sum_{i=0}^{k_n-1}(W_{t_{i+1}^n}(\omega) - W_{t_i^n}(\omega))^2\Big| < 2T \text{ and } |X_{t_{i*}^n}(\omega) Y_{t_{i**}^n}(\omega) - X_{t_i^n}(\omega) Y_{t_i^n}(\omega)| < \epsilon$$

for all $n \geq n_0$ and all choices of t_{i*}^n and t_{i**}^n. Hence

$$\Big|\sum_{i=0}^{k_n-1}(X_{t_i^n}(\omega) Y_{t_i^n}(\omega) - X_{t_{i*}^n} Y_{t_{i**}^n})(W_{t_{i+1}^n}(\omega) - W_{t_i^n}(\omega))^2\Big| \leq 2T\epsilon$$

and, by (1), $\sum_{i=0}^{k_n-1} X_{t_{i*}^n} Y_{t_{i**}^n}(W_{t_{i+1}^n}(\omega) - W_{t_i^n}(\omega))^2 \longrightarrow \int_0^T X_t Y_t dt$ in measure as $n \longrightarrow \infty$. This completes the proof.