NOTES AND ERRATA FOR RECURRENCE AND TOPOLOGY

JOHN M. ALONGI

p.11 Replace

$$\rho(t,z) = \int_0^t \sin^2(\tau + \arg z) + 1 - \left[\frac{|z|e^\tau}{1+|z|(e^\tau - 1)}\right]^2 d\tau.$$

with

$$\rho(t,z) = \int_0^t \sin^2(\tau + \arg z) + 1 - |z|^2\left[\frac{|z|e^\tau}{1+|z|(e^\tau - 1)}\right]^2 d\tau.$$

p.11 Replace

$$\frac{\partial\rho}{\partial t}(t,z) = \sin^2(t + \arg z) + 1 - \left[\frac{|z|e^t}{1+|z|(e^t - 1)}\right]^2 > 0$$

with

$$\frac{\partial\rho}{\partial t}(t,z) = \sin^2(t + \arg z) + 1 - |z|^2\left[\frac{|z|e^t}{1+|z|(e^t - 1)}\right]^2 > 0$$

p.11 Replace

$$1 - |z|^2 \leq \sin^2(\tau + \arg z) + 1 - \left[\frac{|z|e^\tau}{1+|z|^2(e^\tau - 1)}\right]^2$$

with

$$1 - |z|^2 \leq \sin^2(\tau + \arg z) + 1 - |z|^2\left[\frac{|z|e^\tau}{1+|z|^2(e^\tau - 1)}\right]^2$$

p.12 Replace Example 1.3.6 by the following. (The previous Example 1.3.6 was correct, but its purpose is to provide a reparametrization in Example 1.3.10 which contains an error.)

Example 1.3.6. Let ϕ^t be a smooth flow on a manifold M. Let $\beta : M \to \mathbb{R}$ be a smooth bounded positive function. Define $\rho : \mathbb{R} \times M \to \mathbb{R}$ so that for each $x \in M$ the function $\rho(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is the solution of the initial value problem

$$\begin{aligned} \dot{u} &= \beta(\phi^u(x)) \\ u(0) &= 0. \end{aligned}$$

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Let $x \in M$. Because ϕ^t and β are smooth, the function $\rho(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is smooth. In particular,

$$\frac{\partial \rho}{\partial t}(t,x) = \beta(\phi^{\rho(t,x)}(x)) > 0.$$

Thus, $\rho(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is strictly increasing.

Since the function $\rho(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is increasing, either the limit

$$\lim_{t\to\infty} \rho(t,x)$$

exists, or $\rho(t,x) \to \infty$ as $t \to \infty$. We claim that $\rho(t,x) \to \infty$ as $t \to \infty$. By means of contradiction, assume that there exists a real number S so that

$$\lim_{t\to\infty} \rho(t,x) = S.$$

The continuity of ρ, ϕ and β implies that

$$\lim_{t\to\infty} \frac{\partial \rho}{\partial t}(t,x) = \lim_{t\to\infty} \beta(\phi^{\rho(t,x)}(x)) = \beta(\phi^S(x)) > 0.$$

However, because β is bounded and $\rho(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is smooth and increasing,

$$\lim_{t\to\infty} \frac{\partial \rho}{\partial t}(t,x) = 0.$$

This is a contradiction. Therefore, $\rho(t,x) \to \infty$ as $t \to \infty$. Similarly, $\rho(t,x) \to -\infty$ as $t \to -\infty$.

Since the function $\rho(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing, and $\rho(t,x) \to \pm\infty$ as $t \to \pm\infty$, the function $\rho(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is surjective. Therefore, ρ is a reparametrization.

p.13 Replace

$$\begin{aligned} \dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2) \\ \dot{\theta} &= \sin^2\theta + 1 - r^2 \end{aligned}$$

with

$$\begin{aligned} \dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4) \\ \dot{\theta} &= \sin^2\theta + 1 - r^4 \end{aligned}$$

p.14 Replace

$$\rho(t,z) = \int_0^t \sin^2(\tau + \arg z) + 1 - \left[\frac{|z|e^\tau}{1 + |z|(e^\tau - 1)}\right]^2 d\tau$$

with

$$\rho(t,z) = \int_0^t \sin^2(\tau + \arg z) + 1 - |z|^2\left[\frac{|z|e^\tau}{1 + |z|(e^\tau - 1)}\right]^2 d\tau$$

p.14 Replace

$$\rho(t,(r,\theta)) = \int_0^t \sin^2(\tau+\theta) + 1 - \left[\frac{re^\tau}{1+r(e^\tau-1)}\right]^2 d\tau$$

with

$$\rho(t,(r,\theta)) = \int_0^t \sin^2(\tau+\theta) + 1 - r^2\left[\frac{re^\tau}{1+r(e^\tau-1)}\right]^2 d\tau$$

p.14 Replace

$$\begin{pmatrix} r(1-r) \\ 1 \end{pmatrix} (\sin^2\theta + 1 - r^2)$$

with

$$\begin{pmatrix} r(1-r) \\ 1 \end{pmatrix} (\sin^2\theta + 1 - r^4)$$

p.14 Replace

$$\begin{pmatrix} r(1-r)(\sin^2\theta + 1 - r^2 \\ \sin^2\theta + 1 - r^4 \end{pmatrix}$$

with

$$\begin{pmatrix} r(1-r)(\sin^2\theta + 1 - r^4) \\ \sin^2\theta + 1 - r^4 \end{pmatrix}$$

pp.14–15 Replace Example 1.3.10 by the following.

Example 1.3.10. Let ϕ^t be the flow of a smooth vector field f on a manifold M. Let $\beta : M \to \mathbb{R}$ be a smooth bounded positive function. We will show that scaling the vector field f by the real-valued function β generates a flow which is topologically equivalent to ϕ^t.

Define $\rho : \mathbb{R} \times M \to \mathbb{R}$ so that $\rho(t,x)$ is the solution of the initial value problem

$$\begin{aligned} \dot{u} &= \beta(\phi^u(x)) \\ u(0) &= 0 \end{aligned}$$

for each $x \in M$. By Example 1.3.6 the function ρ is a reparametrization.

By the Chain Rule,

$$\begin{aligned} \frac{d}{dt}\phi^{\rho(t,x)}(x) &= f(\phi^{\rho(t,x)}(x))\,\frac{\partial\rho}{\partial t}(t,x) \\ &= f(\phi^{\rho(t,x)}(x))\beta(\phi^{\rho(t,x)}(x)) \\ &= (\beta f)(\phi^{\rho(t,x)}(x)) \end{aligned}$$

for all $x \in M$. If ψ^t is the flow of the vector field βf on M, then

$$\frac{d}{dt}\psi^t(x) = (\beta f)(\psi^t(x))$$

for all $x \in M$. Thus, for each $x \in M$, the functions $\psi^t(x)$ and $\phi^{\rho(t,x)}(x)$ are both solutions of the differential equation

$$\dot{y} = (\beta f)(y).$$

Since $\psi^0(x) = x = \phi^0(x) = \phi^{\rho(0,x)}(x)$ for all $x \in M$, the existence and uniqueness theorem for ordinary differential equations implies that

$$\psi^t(x) = \phi^{\rho(t,x)}(x)$$

for all $x \in M$ and all $t \in \mathbb{R}$. Therefore, ϕ^t is topologically equivalent to ψ^t.

pp.28–29 In Proposition 2.2.4, the hypothesis that $\{p\}$ is closed is unnecessary.

Proposition 2.2.4. *If p is a periodic point of a flow ϕ^t with period T, then $\phi^T(p) = p$.*

Proof. Let $H = \{\tau \in \mathbb{R} \mid \phi^\tau(p) = p\}$. If H is discrete, then H is (topologically) closed, so that $T = \inf\{\tau > 0 \mid \phi^\tau(p) = p\} \in H$. Hence, $\phi^T(p) = p$. If H is not discrete, then there is a real number s which is an accumulation point of H. For each $\epsilon > 0$ there exists $\tau \in H$ so that $|\tau - s| < \epsilon$. By the group property of flows and Proposition 1.1.2 the set H is a subgroup of $(\mathbb{R}, +)$. Thus, $\phi^{|\tau-s|}(p) = p$. Consequently, $T = \inf\{\tau > 0 \mid \phi^\tau(p) = p\} = 0$, in which case $\phi^T(p) = \phi^0(p) = p$. □

p.31 The last sentence of Example 2.2.8 should read "If $\beta \neq 0$, then since $\sin \beta t$ and $\cos \beta t$ are periodic functions, and both have period $2\pi/|\beta|$, every point in $\mathbb{R}^2$ is a periodic point of ϕ^t, and the period of each point other than the fixed point at $(0, 0)$ is $2\pi/|\beta|$."

p.36 Proposition 2.2.13 should read:

Proposition 2.2.13. *Every periodic orbit of a flow is compact, and every periodic orbit of a flow with a Hausdorff phase space is compact.*

The last sentence of the proof should read, "Every compact subset of a Hausdorff space is closed."

p.36 Replace $\dot{\theta} = \sin^2\theta + 1 - r^2$ by $\dot{\theta} = \sin^2\theta + 1 - r^4$.

p.36 Replace $\dot{\theta} = \sin^2\theta + 1 - r_0^2$ by $\dot{\theta} = \sin^2\theta + 1 - r_0^4$.

p.39 In Example 2.3.2 replace the first two sentences of the second paragraph by:

"On the other hand, if $x \in \mathcal{O}(p)$, then there exists a positive real number τ such that $x = \phi^\tau(p)$. So,

$$\lim_{n\to\infty} \phi^{\pm n\tau}(p) = x$$

by Proposition 2.2.5."

p.40 In the proof of Proposition 2.3.3 (ii) define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = |\phi^t(x) - x|,$$

and replace $\phi^\tau(x) - x$ in l.-3 by $|\phi^\tau(x) - x|$.

pp.43–45 Proposition 2.3.6 (v) should read:

The sets $\omega(x)$ and $\alpha(x)$ are invariant.

The first-countability of X is not necessary.

p.48 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2)\\ \dot{\theta} &= \sin^2\theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4)\\ \dot{\theta} &= \sin^2\theta + 1 - r^4\end{aligned}$$

p.62 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2)\\ \dot{\theta} &= \sin^2\theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4)\\ \dot{\theta} &= \sin^2\theta + 1 - r^4\end{aligned}$$

p.74 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2)\\ \dot{\theta} &= \sin^2\theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4)\\ \dot{\theta} &= \sin^2\theta + 1 - r^4\end{aligned}$$

p.75 Replace

$$\dot{r} = r(1-r)(\sin^2\theta + 1 - r^2) > 0$$

with

$$\dot{r} = r(1-r)(\sin^2\theta + 1 - r^4) > 0$$

p.99 The last sentence of Example 3.1.2 should read:

"Therefore,

$$\mathrm{Fix}(\phi^t) \subseteq \mathrm{Per}(\phi^t) \subseteq M(\phi^t)$$

for any flow ϕ^t on a Hausdorff phase space."

pp.105–106 There is a stronger version of Proposition 3.2.7:

Proposition 3.2.7. *Let ϕ^t be flow on a topological space X. If $A \subseteq X$ is topologically transitive with respect to ϕ^t, then for each pair of nonempty open sets U and V in A, there exists a nonnegative real number T such that $\phi^T(U) \cap V$ is nonempty.*

Proof. Let U and V be nonempty open sets in A. There exist open sets U_0 and V_0 in X so that $U = U_0 \cap A$ and $V = V_0 \cap A$. Since A is topologically transitive, there exists $x \in A$ such that $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the topology on X. So, there exist nonnegative real numbers r and s such that $\phi^{-r}(x) \in U_0$ and $\phi^s(x) \in V_0$. By Proposition 3.2.6 the set A is invariant with respect to ϕ^t. Hence, $\phi^{-r}(x) \in U$ and $\phi^s(x) \in V$. Let $T = r + s$. The number T is nonnegative, and $\phi^s(x) \in \phi^s(\phi^r(U)) \cap V = \phi^T(U) \cap V$. Therefore, $\phi^T(U) \cap V$ is nonempty. □

p.106 The word "nonempty" is missing from each part of the Birkhoff Transitivity Theorem:

Theorem 3.2.10.The Birkhoff Transitivity Theorem. Let ϕ^t be a flow on a nonempty second-countable Baire space X.

(i) *If $\bigcup_{t\geq 0} \phi^t(U)$ is dense in X for every nonempty open subset U of X, then there exists $\mathcal{D}^- \subseteq X$ such that $\mathcal{D}^-$ is residual in X, and $\mathcal{O}^-(x)$ is dense in X for all $x \in \mathcal{D}^-$.*

(ii) *If $\bigcup_{t\leq 0} \phi^t(U)$ is dense in X for every nonempty open subset U of X, then there exists $\mathcal{D}^+ \subseteq X$ such that $\mathcal{D}^+$ is residual in X, and $\mathcal{O}^+(x)$ is dense in X for all $x \in \mathcal{D}^+$.*

(iii) *If $\bigcup_{t\geq 0} \phi^t(U)$ and $\bigcup_{t\leq 0} \phi^t(U)$ are dense in X for every nonempty open subset U of X, then there exists $\mathcal{D} \subseteq X$ such that $\mathcal{D}$ is residual in X, and $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in X for all $x \in \mathcal{D}$.*

p.107 Proposition 3.2.11 is false. Here is a correct proposition:

Proposition 3.2.11. *Let ϕ^t be flow on a topological space X. Let $A \subseteq X$ be a nonempty second-countable Baire space in the subspace topology and invariant with respect to ϕ^t. If for each pair of nonempty open sets U and V in A, there exists a nonnegative real number T such that $\phi^T(U) \cap V$ is nonempty, then A is topologically transitive with respect to ϕ^t.*

Proof. For each pair of nonempty open subsets U and V in A there exist nonnegative real numbers T and S such that $\phi^T(U) \cap V$ and $\phi^S(V) \cap U$ are nonempty. Consequently, $\phi^{-S}(U) \cap V$ is nonempty. Since A is invariant with respect to ϕ^t, the sets $\bigcup_{t\geq 0} \phi^t(U)$ and $\bigcup_{t\leq 0} \phi^t(U)$ are dense in A with respect to the subspace topology for each nonempty open subset U in A.

By the Birkhoff Transitivity Theorem, there exists $\mathcal{D} \subseteq A$ such that $\mathcal{D}$ is residual in A, and for each $x \in \mathcal{D}$ the sets $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the subspace topology. Since A is a nonempty Baire space, the set $\mathcal{D}$ is nonempty. Hence, there exists $x \in A$ such that $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the subspace topology.

Let W be an open subset of X such that $W \cap A$ is nonempty. The set $W \cap A$ is an open set in A. Since $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the subspace topology, there exist nonnegative real numbers r and s such that $\phi^{-r}(x) \in W \cap A$ and $\phi^s(x) \in W \cap A$. Thus, $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the topology on X. Therefore, A is topologically transitive with respect to ϕ^t.

□

pp.132–133 There is an easier proof of the forward implication of Proposition 3.4.30.

Let $x, y \in X$ be chain equivalent with respect to ϕ^t, and let (A, A^*) be an attracting-repelling pair for ϕ^t. The collection

$$\{A \cap \mathcal{R}(\phi^t), A^* \cap \mathcal{R}(\phi^t)\}$$

is a partition of $\mathcal{R}(\phi^t)$ into closed, disjoint sets. If A is empty, then $x, y \in A^*$. If A^* is empty, then $x, y \in A$. Since x and y are chain equivalent, the points x and y are elements of the same connected component of $\mathcal{R}(\phi^t)$. Therefore, $x, y \in A$ or $x, y \in A^*$.

p.166 Replace

$$\begin{aligned} \dot{\theta} &= \sin^2\theta + 1 - \left(\frac{\phi}{\pi}\right)^2 \\ \dot{\phi} &= \phi\left(1 - \frac{\phi}{\pi}\right)\left[\sin^2\theta + 1 - \left(\frac{\phi}{\pi}\right)^2\right] \end{aligned}$$

with

$$\begin{aligned} \dot{\theta} &= \sin^2\theta + 1 - \left(\frac{\phi}{\pi}\right)^4 \\ \dot{\phi} &= \phi\left(1 - \frac{\phi}{\pi}\right)\left[\sin^2\theta + 1 - \left(\frac{\phi}{\pi}\right)^4\right] \end{aligned}$$

p.167 Replace

$$-\frac{\phi(t)}{\pi}\left[1 - \frac{\phi(t)}{\pi}\right]\left(\sin^2[\theta(t)] + 1 - \left[\frac{\phi(t)}{\pi}\right]^2\right)$$

with

$$-\frac{\phi(t)}{\pi}\left[1 - \frac{\phi(t)}{\pi}\right]\left(\sin^2[\theta(t)] + 1 - \left[\frac{\phi(t)}{\pi}\right]^4\right)$$

p.195 In Exercise 20, replace

$$\mathcal{R}(\phi^t) = \mathrm{Fix}(\phi^t) = p(\mathcal{R}(\phi^t))$$

with

$$\mathcal{R}(\Phi^t) = \mathrm{Fix}(\Phi^t) = p(\mathcal{R}(\phi^t)).$$

p.198 Replace

$$\begin{aligned} \dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2) \\ \dot{\theta} &= \sin^2\theta + 1 - r^2 \end{aligned}$$

with

$$\begin{aligned} \dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4) \\ \dot{\theta} &= \sin^2\theta + 1 - r^4 \end{aligned}$$

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