

Notes and Errata for Recurrence and Topology

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October 19, 2009

p.11 Replace

$$\rho(t, z) = \int_0^t \sin^2(\tau + \arg z) + 1 - \left[\frac{|z|e^\tau}{1 + |z|(e^\tau - 1)} \right]^2 d\tau.$$

with

$$\rho(t, z) = \int_0^t \sin^2(\tau + \arg z) + 1 - |z|^2 \left[\frac{|z|e^\tau}{1 + |z|(e^\tau - 1)} \right]^2 d\tau.$$

p.11 Replace

$$\frac{\partial \rho}{\partial t}(t, z) = \sin^2(t + \arg z) + 1 - \left[\frac{|z|e^t}{1 + |z|(e^t - 1)} \right]^2 > 0$$

with

$$\frac{\partial \rho}{\partial t}(t, z) = \sin^2(t + \arg z) + 1 - |z|^2 \left[\frac{|z|e^t}{1 + |z|(e^t - 1)} \right]^2 > 0$$

p.11 Replace

$$1 - |z|^2 \leq \sin^2(\tau + \arg z) + 1 - \left[\frac{|z|e^\tau}{1 + |z|^2(e^\tau - 1)} \right]^2$$

with

$$1 - |z|^2 \leq \sin^2(\tau + \arg z) + 1 - |z|^2 \left[\frac{|z|e^\tau}{1 + |z|^2(e^\tau - 1)} \right]^2$$

p.12 Replace Example 1.3.6 by the following. (The previous Example 1.3.6 was correct, but its purpose is to provide a reparametrization in Example 1.3.10 which contains an error.)

Example 1.3.6. Let ϕ^t be a smooth flow on a manifold M . Let $\beta : M \rightarrow \mathbb{R}$ be a smooth bounded positive function. Define $\rho : \mathbb{R} \times M \rightarrow \mathbb{R}$ so that for each $x \in M$ the function $\rho(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is the solution of the initial value problem

$$\begin{aligned} \dot{u} &= \beta(\phi^u(x)) \\ u(0) &= 0. \end{aligned}$$

Let $x \in M$. Because ϕ^t and β are smooth, the function $\rho(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is smooth. In particular,

$$\frac{\partial \rho}{\partial t}(t, x) = \beta(\phi^{\rho(t, x)}(x)) > 0.$$

Thus, $\rho(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing.

Since the function $\rho(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, either the limit

$$\lim_{t \rightarrow \infty} \rho(t, x)$$

exists, or $\rho(t, x) \rightarrow \infty$ as $t \rightarrow \infty$. We claim that $\rho(t, x) \rightarrow \infty$ as $t \rightarrow \infty$. By means of contradiction, assume that there exists a real number S so that

$$\lim_{t \rightarrow \infty} \rho(t, x) = S.$$

The continuity of ρ , ϕ and β implies that

$$\lim_{t \rightarrow \infty} \frac{\partial \rho}{\partial t}(t, x) = \lim_{t \rightarrow \infty} \beta(\phi^{\rho(t, x)}(x)) = \beta(\phi^S(x)) > 0.$$

However, because β is bounded and $\rho(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is smooth and increasing,

$$\lim_{t \rightarrow \infty} \frac{\partial \rho}{\partial t}(t, x) = 0.$$

This is a contradiction. Therefore, $\rho(t, x) \rightarrow \infty$ as $t \rightarrow \infty$. Similarly, $\rho(t, x) \rightarrow -\infty$ as $t \rightarrow -\infty$.

Since the function $\rho(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, strictly increasing, and $\rho(t, x) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$, the function $\rho(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is surjective. Therefore, ρ is a reparametrization.

p.13 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2 \theta + 1 - r^2) \\ \dot{\theta} &= \sin^2 \theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2 \theta + 1 - r^4) \\ \dot{\theta} &= \sin^2 \theta + 1 - r^4\end{aligned}$$

p.14 Replace

$$\rho(t, z) = \int_0^t \sin^2(\tau + \arg z) + 1 - \left[\frac{|z|e^\tau}{1 + |z|(e^\tau - 1)} \right]^2 d\tau$$

with

$$\rho(t, z) = \int_0^t \sin^2(\tau + \arg z) + 1 - |z|^2 \left[\frac{|z|e^\tau}{1 + |z|(e^\tau - 1)} \right]^2 d\tau$$

p.14 Replace

$$\rho(t, (r, \theta)) = \int_0^t \sin^2(\tau + \theta) + 1 - \left[\frac{re^\tau}{1 + r(e^\tau - 1)} \right]^2 d\tau$$

with

$$\rho(t, (r, \theta)) = \int_0^t \sin^2(\tau + \theta) + 1 - r^2 \left[\frac{re^\tau}{1 + r(e^\tau - 1)} \right]^2 d\tau$$

p.14 Replace

$$\left(\begin{array}{c} r(1-r) \\ 1 \end{array} \right) (\sin^2 \theta + 1 - r^2)$$

with

$$\left(\begin{array}{c} r(1-r) \\ 1 \end{array} \right) (\sin^2 \theta + 1 - r^4)$$

p.14 Replace

$$\left(\begin{array}{c} r(1-r)(\sin^2 \theta + 1 - r^2) \\ \sin^2 \theta + 1 - r^4 \end{array} \right)$$

with

$$\left(\begin{array}{c} r(1-r)(\sin^2 \theta + 1 - r^4) \\ \sin^2 \theta + 1 - r^4 \end{array} \right)$$

pp.14–15 Replace Example 1.3.10 by the following.

Example 1.3.10. Let ϕ^t be the flow of a smooth vector field f on a manifold M . Let $\beta : M \rightarrow \mathbb{R}$ be a smooth bounded positive function. We will show that scaling the vector field f by the real-valued function β generates a flow which is topologically equivalent to ϕ^t .

Define $\rho : \mathbb{R} \times M \rightarrow \mathbb{R}$ so that $\rho(t, x)$ is the solution of the initial value problem

$$\begin{aligned} \dot{u} &= \beta(\phi^u(x)) \\ u(0) &= 0 \end{aligned}$$

for each $x \in M$. By Example 1.3.6 the function ρ is a reparametrization.

By the Chain Rule,

$$\begin{aligned} \frac{d}{dt}\phi^{\rho(t,x)}(x) &= f(\phi^{\rho(t,x)}(x)) \frac{\partial \rho}{\partial t}(t, x) \\ &= f(\phi^{\rho(t,x)}(x))\beta(\phi^{\rho(t,x)}(x)) \\ &= (\beta f)(\phi^{\rho(t,x)}(x)) \end{aligned}$$

for all $x \in M$. If ψ^t is the flow of the vector field βf on M , then

$$\frac{d}{dt}\psi^t(x) = (\beta f)(\psi^t(x))$$

for all $x \in M$. Thus, for each $x \in M$, the functions $\psi^t(x)$ and $\phi^{\rho(t,x)}(x)$ are both solutions of the differential equation

$$\dot{y} = (\beta f)(y).$$

Since $\psi^0(x) = x = \phi^0(x) = \phi^{\rho(0,x)}(x)$ for all $x \in M$, the existence and uniqueness theorem for ordinary differential equations implies that

$$\psi^t(x) = \phi^{\rho(t,x)}(x)$$

for all $x \in M$ and all $t \in \mathbb{R}$. Therefore, ϕ^t is topologically equivalent to ψ^t .

pp.28–29 In Proposition 2.2.4, the hypothesis that $\{p\}$ is closed is unnecessary.

Proposition 2.2.4. *If p is a periodic point of a flow ϕ^t with period T , then $\phi^T(p) = p$.*

Proof. Let $H = \{\tau \in \mathbb{R} \mid \phi^\tau(p) = p\}$. If H is discrete, then H is (topologically) closed, so that $T = \inf\{\tau > 0 \mid \phi^\tau(p) = p\} \in H$. Hence, $\phi^T(p) = p$. If H is not discrete, then there is a real number s which is an accumulation point of H . For each $\epsilon > 0$ there exists $\tau \in H$ so that $|\tau - s| < \epsilon$. By the group property of flows and Proposition 1.1.2 the set H is a subgroup of $(\mathbb{R}, +)$. Thus, $\phi^{|\tau-s|}(p) = p$. Consequently, $T = \inf\{\tau > 0 \mid \phi^\tau(p) = p\} = 0$, in which case $\phi^T(p) = \phi^0(p) = p$. \square

p.36 Proposition 2.2.13 should read:

Proposition 2.2.13. *Every periodic orbit of a flow is compact, and every periodic orbit of a flow with a Hausdorff phase space is compact.*

The last sentence of the proof should read, “Every compact subset of a Hausdorff space is closed.”

p.36 Replace $\dot{\theta} = \sin^2 \theta + 1 - r^2$ by $\dot{\theta} = \sin^2 \theta + 1 - r^4$.

p.36 Replace $\dot{\theta} = \sin^2 \theta + 1 - r_0^2$ by $\dot{\theta} = \sin^2 \theta + 1 - r_0^4$.

p.39 In Example 2.3.2 replace the first two sentences of the second paragraph by:

“On the other hand, if $x \in \mathcal{O}(p)$, then there exists a positive real number τ such that $x = \phi^\tau(p)$. So,

$$\lim_{n \rightarrow \infty} \phi^{\pm n\tau}(p) = x$$

by Proposition 2.2.5.”

p.40 In the proof of Proposition 2.3.3 (ii) define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = |\phi^t(x) - x|,$$

and replace $\phi^\tau(x) - x$ in 1.-3 by $|\phi^\tau(x) - x|$.

pp.43–45 Proposition 2.3.6 (v) should read:

The sets $\omega(x)$ and $\alpha(x)$ are invariant.

The first-countability of X is not necessary.

p.48 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2) \\ \dot{\theta} &= \sin^2\theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4) \\ \dot{\theta} &= \sin^2\theta + 1 - r^4\end{aligned}$$

p.62 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2) \\ \dot{\theta} &= \sin^2\theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4) \\ \dot{\theta} &= \sin^2\theta + 1 - r^4\end{aligned}$$

p.74 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^2) \\ \dot{\theta} &= \sin^2\theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2\theta + 1 - r^4) \\ \dot{\theta} &= \sin^2\theta + 1 - r^4\end{aligned}$$

p.75 Replace

$$\dot{r} = r(1-r)(\sin^2\theta + 1 - r^2) > 0$$

with

$$\dot{r} = r(1-r)(\sin^2\theta + 1 - r^4) > 0$$

p.99 The last sentence of Example 3.1.2 should read:

“Therefore,

$$\text{Fix}(\phi^t) \subseteq \text{Per}(\phi^t) \subseteq M(\phi^t)$$

for any flow ϕ^t on a Hausdorff phase space.”

pp.105–106 There is a stronger version of Proposition 3.2.7:

Proposition 3.2.7. *Let ϕ^t be flow on a topological space X . If $A \subseteq X$ is topologically transitive with respect to ϕ^t , then for each pair of nonempty open sets U and V in A , there exists a nonnegative real number T such that $\phi^T(U) \cap V$ is nonempty.*

Proof. Let U and V be nonempty open sets in A . There exist open sets U_0 and V_0 in X so that $U = U_0 \cap A$ and $V = V_0 \cap A$. Since A is topologically transitive, there exists $x \in A$ such that $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the topology on X . So, there exist nonnegative real numbers r and s such that $\phi^{-r}(x) \in U_0$ and $\phi^s(x) \in V_0$. By Proposition 3.2.6 the set A is invariant with respect to ϕ^t . Hence, $\phi^{-r}(x) \in U$ and $\phi^s(x) \in V$. Let $T = r + s$. The number T is nonnegative, and $\phi^s(x) \in \phi^s(\phi^r(U)) \cap V = \phi^T(U) \cap V$. Therefore, $\phi^T(U) \cap V$ is nonempty. \square

p.106 The word “nonempty” is missing from each part of the Birkhoff Transitivity Theorem:

Theorem 3.2.10. The Birkhoff Transitivity Theorem. Let ϕ^t be a flow on a nonempty second-countable Baire space X .

- (i) *If $\bigcup_{t \geq 0} \phi^t(U)$ is dense in X for every nonempty open subset U of X , then there exists $\mathcal{D}^- \subseteq X$ such that \mathcal{D}^- is residual in X , and $\mathcal{O}^-(x)$ is dense in X for all $x \in \mathcal{D}^-$.*
- (ii) *If $\bigcup_{t \leq 0} \phi^t(U)$ is dense in X for every nonempty open subset U of X , then there exists $\mathcal{D}^+ \subseteq X$ such that \mathcal{D}^+ is residual in X , and $\mathcal{O}^+(x)$ is dense in X for all $x \in \mathcal{D}^+$.*
- (iii) *If $\bigcup_{t \geq 0} \phi^t(U)$ and $\bigcup_{t \leq 0} \phi^t(U)$ are dense in X for every nonempty open subset U of X , then there exists $\mathcal{D} \subseteq X$ such that \mathcal{D} is residual in X , and $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in X for all $x \in \mathcal{D}$.*

p.107 Proposition 3.2.11 is false. Here is a correct proposition:

Proposition 3.2.11. *Let ϕ^t be flow on a topological space X . Let $A \subseteq X$ be a nonempty second-countable Baire space in the subspace topology and invariant with respect to ϕ^t . If for each pair of nonempty open sets U and V in A , there exists a nonnegative real number T such*

that $\phi^T(U) \cap V$ is nonempty, then A is topologically transitive with respect to ϕ^t .

Proof. For each pair of nonempty open subsets U and V in A there exist nonnegative real numbers T and S such that $\phi^T(U) \cap V$ and $\phi^S(V) \cap U$ are nonempty. Consequently, $\phi^{-S}(U) \cap V$ is nonempty. Since A is invariant with respect to ϕ^t , the sets $\bigcup_{t \geq 0} \phi^t(U)$ and $\bigcup_{t \leq 0} \phi^t(U)$ are dense in A with respect to the subspace topology for each nonempty open subset U in A .

By the Birkhoff Transitivity Theorem, there exists $\mathcal{D} \subseteq A$ such that \mathcal{D} is residual in A , and for each $x \in \mathcal{D}$ the sets $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the subspace topology. Since A is a nonempty Baire space, the set \mathcal{D} is nonempty. Hence, there exists $x \in A$ such that $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the subspace topology.

Let W be an open subset of X such that $W \cap A$ is nonempty. The set $W \cap A$ is an open set in A . Since $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the subspace topology, there exist nonnegative real numbers r and s such that $\phi^{-r}(x) \in W \cap A$ and $\phi^s(x) \in W \cap A$. Thus, $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in A with respect to the topology on X . Therefore, A is topologically transitive with respect to ϕ^t .

□

pp.132–133 There is an easier proof of the forward implication of Proposition 3.4.30.

Let $x, y \in X$ be chain equivalent with respect to ϕ^t , and let (A, A^*) be an attracting-repelling pair for ϕ^t . The collection

$$\{A \cap \mathcal{R}(\phi^t), A^* \cap \mathcal{R}(\phi^t)\}$$

is a partition of $\mathcal{R}(\phi^t)$ into closed, disjoint sets. If A is empty, then $x, y \in A^*$. If A^* is empty, then $x, y \in A$. Since x and y are chain equivalent, the points x and y are elements of the same connected component of $\mathcal{R}(\phi^t)$. Therefore, $x, y \in A$ or $x, y \in A^*$.

p.166 Replace

$$\dot{\theta} = \sin^2 \theta + 1 - \left(\frac{\phi}{\pi}\right)^2$$

$$\dot{\phi} = \phi \left(1 - \frac{\phi}{\pi}\right) \left[\sin^2 \theta + 1 - \left(\frac{\phi}{\pi}\right)^2\right]$$

with

$$\begin{aligned}\dot{\theta} &= \sin^2 \theta + 1 - \left(\frac{\phi}{\pi}\right)^4 \\ \dot{\phi} &= \phi \left(1 - \frac{\phi}{\pi}\right) \left[\sin^2 \theta + 1 - \left(\frac{\phi}{\pi}\right)^4\right]\end{aligned}$$

p.167 Replace

$$-\frac{\phi(t)}{\pi} \left[1 - \frac{\phi(t)}{\pi}\right] \left(\sin^2[\theta(t)] + 1 - \left[\frac{\phi(t)}{\pi}\right]^2\right)$$

with

$$-\frac{\phi(t)}{\pi} \left[1 - \frac{\phi(t)}{\pi}\right] \left(\sin^2[\theta(t)] + 1 - \left[\frac{\phi(t)}{\pi}\right]^4\right)$$

p.198 Replace

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2 \theta + 1 - r^2) \\ \dot{\theta} &= \sin^2 \theta + 1 - r^2\end{aligned}$$

with

$$\begin{aligned}\dot{r} &= r(1-r)(\sin^2 \theta + 1 - r^4) \\ \dot{\theta} &= \sin^2 \theta + 1 - r^4\end{aligned}$$

Thanks to T.S. Huang, Michael Hurley, Thanate Dhirasakdanon, Keith Burns and Katrin Gelfert.