TEACHING PLAN FOR GRADUATE ALGEBRA: NONCOMMUTATIVE VIEW

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As indicated in the introduction, our goal in *Graduate Algebra: Noncommutative View* is to provide a common framework for algebraic structure theory in terms of representations into matrix algebras. For use as a textbook, one can lay the foundations by building on the pioneering work of Wedderburn, Artin, Noether, and Jacobson. Here are suggestions for various courses, pinned on suitable theorems.

0.1. **The basic one-semester graduate course in noncommutative algebra.**

The main theme is that much of noncommutative algebra revolves around matrices and their properties (especially the trace). A basic course in noncommutative algebra could start with the structure of Artinian rings, focusing on semisimple rings and their characterization in the Wedderburn-Artin Theorem, continue with tensor products, and then turn to group representations and Maschke's theorem, and culminate with a major application – Burnside's theorem on the solvability of finite groups of order $p^i q^j$ for p and q prime. One could trim a bit, ending with Frobenius' Theorem, or continue on to Artin's Theorem on induced characters (Theorem 20.44). This requires the following basic material:

- **Basic facts about matrices, matrix units, and idempotents** (Chapter 13 thru Remark 13.35).

  Here one obtains a key tool (albeit technical) in Proposition 13.9, the internal description of matrix rings in terms of matrix units.
- **Hom and Representations of algebras into matrices** (Definition 13.35′ – Remark 13.50).

  The key tool is:

  Proposition 13.39 (Schur's Lemma). Every nonzero map $f : M \to N$ between simple modules is an isomorphism,

  which leads to the following major theorem:

  Theorem 13.48. A ring R is a finite direct sum of minimal left ideals if and only if

  $$R \cong \prod_{i=1}^{t} M_{n_i}(D_i)$$

  for suitable t, $n_i \in \mathbb{N}$, and division rings D_i.

- **Semisimple modules and complements, and the Wedderburn-Artin Theorem** (Chapter 14 thru Theorem 14.24).

  A description of semisimple modules in terms of other useful module-theoretic properties is given in

  Theorem 14.13. The following conditions are equivalent for a module M:
  - M is semisimple.
  - M is complemented.
  - M has no proper large submodules.

  Viewing R as $\operatorname{End}_R R$; yields a very elegant proof of perhaps our most basic structure theorem:

  Theorem 14.24 (Wedderburn-Artin). The following properties are equivalent for a ring R:
  - $R \cong M_n(D)$.
  - R is simple and left Artinian.
  - R is simple with a minimal left ideal.
  - R is simple and semisimple.

- **Primitive, prime, and semiprime rings and ideals** (Chapter 15 thru Remark 15.15).
  Since representation theory hinges largely on the set of simple modules of a ring, it is natural to study primitive rings, i.e., those with a faithful simple module. These could have many nonisomorphic simple modules, but one does have:

  Proposition 15.7. Suppose a primitive ring R has a minimal nonzero left ideal L. Then every faithful simple R-module is isomorphic to L.

  This provides the uniqueness of n and D ini the description $R = M_n(D)$ (Corollary 15.9) as well as determining all of the isomorphism classes of a semisimple ring (Corollary 15.10). Prime rings provide a natural, more intrinsically ring-theoretic, generalization.
- **The Jacobson radical** (Definition 15.16).
  Jac(R) is the intersection of the primitive ideals of R, thereby indicating the obstruction to studying R via its simple modules. The Jacobson radical unifies many different concepts in ring theory, one of which is given in Proposition 15.17.
- **Left Artinian rings** (Theorems 15.18 thru 15.21).
  The Jacobson radical is used to prove many important properties about a left Artinian ring R.

  Theorem 15.18.
  - Jac(R) is the intersection of finitely many maximal left ideals of R.
  - $R/\operatorname{Jac}(R)$ is a semisimple ring;
  - Each primitive ideal of R is maximal, and there are only a finite number of these.
  - Jac(R) is a nilpotent ideal.

  Theorem 15.19. Any prime left Artinian ring is simple.

  Theorem 15.21 (Hopkins-Levitzki. R is also left Noetherian.

- **Tensor products of modules and algebras** (Chapter 18 thru Remark 18.37).
  This critical construction in algebra is defined abstractly in Definition 18.3, in order to obtain quickly the underlying universal property in Proposition 18.4. The transition to homomorphisms is given in:

  Proposition 18.5. Suppose $f : M \to M'$ is a map of right R-modules and $g : N \to N'$ is a map of R-modules. Then there is a group homomorphism denoted

  $$f \otimes g : M \otimes N \to M' \otimes N'$$

  given by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.

  This enables one to define the tensor product as a module (Proposition 18.7) and obtain its properties of associativity and commutativity up to isomorphism (Propositions 18.10 – 18.16). Then one moves to tensor products of algebras (Theorem 18.21).

  Theorem 18.25. The following algebra isomorphisms hold for any algebras A_1, A_2, and A_3 over a commutative ring C:

  $$A_1 \otimes_C C \cong C \otimes_C A_1 \cong A_1, \qquad A_1 \otimes A_2, \cong A_2 \otimes A_1, \qquad A_1 \otimes (A_2 \otimes A_3) \cong (A_1 \otimes A_2) \otimes A_3.$$

  A nice application, although not critical to the course, is Proposition 18.29′, which describes separable field extensions in terms of tensor products.

  Corollary 18.34 shows that the tensor product of simple algebras is simple, and Proposition 18.35 shows that tensors are compatible with Hom. This enables one to describe the tensor product of matrices in Remark 18.37. The rest of Chapter 18 (application to algebraic geometry and the adjoint isomorphism) also is important, but could be skipped if there is time pressure.
- **Group representations** (Chapter 19 thru Proposition 19.12).
  The degree 1 group representations are given in Proposition 19.5 in terms of G/G'. The correspondence to G-modules is given in Proposition 19.12.
- **Group algebras and Maschke's Theorem** (Definition 19.13 thru Theorem 19.26).
  The n ext observation enables one to pass from one theory to another:

  Proposition 19.18. Given a vector space V over a field F, we have a 1:1 correspondence between:
  - group representations $\rho : G \to \mathrm{GL}(V)$;

- algebra representations $F[G] \to \operatorname{End}_F V$;
- G-space structures on the vector space V;
- $F[G]$-module structures on V.

This leads to the crucial tool in studying representations of finite groups:

Theorem 19.26 (Maschke's Theorem). $F[G]$ is a semisimple ring, for any finite group G whose order is not divisible by char(F).

The proof given here is via the averaging process of Lemma 19.25.

- **Splitting fields** (Definition 19.30 thru Example 19.40).

  The splitting field F enables us to view a group algebra as a finite direct product of matrix algebras over F. $\mathbb{C}$ is a splitting field of every finite group, which is the reason we focus on complex representations.
- **The center of the group algebra** (Remark 19.41 thru Corollary 19.43).

  Next one characterizes the center of the group algebra in terms of the sum of the elements of a conjugacy class.

Theorem 19.42. For any commutative ring C, $\operatorname{Cent}(C[G])$ is free as a C-module having base $\{z_{\mathcal{C}} : \mathcal{C} \text{ is a conjugacy class of } G\}$.

Corollary 19.43. Suppose F is a splitting field for a finite group G. The following numbers are equal:
- The number of conjugacy classes of G.
- The number of inequivalent irreducible representations of G.
- The number of simple components of $F[G]$.
- $\dim_F \operatorname{Cent}(F[G])$.

- **Group characters** (Chapter 20 thru Theorem 20.23).

  Now one gets to the main point, using group representations to study the group elements in terms of matrix properties, especially the character as given in Definition 20.1. Some of their main arithmetic properties are given in Proposition 20.4, but the main idea here is Schur's Theorem (20.5) that the irreducible characters of a finite group are an orthonormal base of the algebra of class functions, with respect to a natural inner product described in terms of the group elements.

  This leads one to the celebrated **character table**, the matrix of the irreducible characters acting on representatives of the conjugate classes, and a magical reformulation (Theorem 20.15) of Schur's Theorem. Using the theory of integral extensions enables one to prove

Theorem 20.18 (Frobenius). The degree of each irreducible representation divides $|G|$.

It does not take much more work to arrive at

Theorem 20.23 (Burnside's Theorem). There does not exist a nonabelian simple group G of order $p^u q^v$ (for p, q prime)

- **Tensor products of representations**.

  Returning to representations of groups that need not be finite, one can use tensor products to describe products of characters (Proposition 20.29 and Theorem 20.32) and induced representations (Definition 20.35ff.), leading to the Frobenius Reciprocity Theorem (20.42) and

Theorem 20.44 (Artin). Every complex character (for a group G) is a combination (over $\mathbb{Q}$) of complex characters induced from cyclic subgroups of G.

The proofs of these results have been selected for the purposes of introducing fundamental concepts and getting most directly to the goal. If time permits, one could continue in various directions.

- One can study the Young-Frobenius theory of the group algebra of the symmetric group (Theorems 19.60, 19.61), classifying its irreducible representations,
- Schur's theorem (19A.9) says that any finite generated periodic subgroup of matrices is finite.

- Rather than continue with group representations, one could turn to the basic theory of projective (Definition 25.8) and their categorical dual, injective modules (Definition 25.28). These lie at the foundation of homology theory.

  Proposition 25.10 The following assertions are equivalent for an R-module P:
  - P is projective.
  - P is a direct summand of a free module.
  - There is a split epic $f\ F \to P$ with F free. (Moreover, if P is generated by n elements, then we can take $F = R^{(n)}$.)
  - Every epic $f\ M \to P$ splits.
  - If the sequence of modules $M \xrightarrow{f} N \to Q$ is exact and $h\ P \to N$ with $gh = 0$, then h lifts to a map $\hat{h}\ P \to M$ with $h = f\hat{h}$;

  Another equivalent property is the Dual Basis Lemma (Proposition 25.13). This all leads to projective resolutions (Definition 25.18) and projective dimension (and, dually, injective dimension).
- Along a slightly different vein, to emphasize the structure theory, one could introduce non-Artinian rings such as polynomial rings (13.51). The Weyl algebra (13.A.1) is simple left Noetherian but non-Artinian. One then continues with the theory of Noetherian rings in Chapter 16, focusing on Goldie's Theorems (16.23 and 16.29).

0.2. **A second semester in noncommutative algebra.**
There are several reasonable ways to continue in noncommutative algebra from the first semester.

- **Growth of groups and algebras** (Chapter 17).
  Chapter 17 goes in a different direction, describing algebraic structures in terms of generators and relations. One goes on to the growth of algebraic structures, and the Milnor-Wolf Theorem (17.61), that every finitely generated, virtually solvable group of subexponential growth is virtually nilpotent.
- **Algebraic groups** (Appendices 19A and B).
  One could continue towards linear groups and algebraic groups. Theorem 19B.19, that every affine algebraic group has a faithful finite dimensional representation, is reasonably accessible Much more ambitious is the celebrated Tits' alternative:

  Theorem 19B.21) Every f.g. linear group either is virtually solvable or contains a free subgroup.

- **Polynomial identities** (Chapter 23).
  A structural ring-theoretic approach to representations is developed by means of the PI-theory in Chapter 23. One main theme is the existence of central polynomials for matrices discovered by Formanek and Razmyslov (Theorem 23.26), which imply that ideals of semiprime PI-rings intersect the center, leading to a stronger version of Goldie's Theorem. One can go on to the powerful Theorem 23.39 which describes the structure of an algebra in terms of a central polynomial. The important application to affine algebras is given in Appendix 23A.
- **Nonassociative algebras –Lie algebras**(Chapters 21 and 22).
  A nonassociative algebra over a commutative ring C can be viewed as a C-module A together with a multiplication map $\mu : A \otimes_C A \to A$; multiplication is bilinear over C if and only if μ is a C-module isomorphism. The most important special case, Lie algebras, is covered in Chapter 21, with the major structure theorems of Lie and Engel and the Killing form, and Cartan's classification of semieimple Lie algebras in terms of root systems in Theorem 21.108; one can continue with the classification of simple Lie algebras in terms of Dynkin diagrams (Theorem 22.11), generalized to Coxeter graphs in Theorem 22.28, and perhaps Lie cohomology in Chapter 25.
- **The Brauer group** (Chapter 24).
  To explore division algebras, one could introduce the quaternion algebra (Example 14.29), continue with tensor products of simple algebras (Corollary 18.34), and go on to Chapter 24. Here the main goal would be the Brauer group, and its properties (Theorems 24.62 and 24.66). The Merkurjev-Suslin Theorem, tieing the Brauer group to K-theory, is discussed, but its proof is beyond the scope of this volume. The Brauer group is tied in with the second cohomology

group in Example 25.66, and the lovely generalization of separable algebras is given in Appendix 25B.

- **Hopf algebras (Chapter 26)**.

  This topic unifies group algebras and Lie algebras, as well as more general structures, and could be brought in later. The difficulty with Hopf algebras in a course is that one has to set up an elaborate framework before proving major theorems, but there are some nice theorems (such as Nichols-Zoeller, Corollary 26.28, and the generalization of Maschke's Theorem in Theorem 26.30) which are reasonably accessible.

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