The Mathematics of Voting and Elections: A Hands-On Approach

Instructor's Manual

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Contents

Preface

It's been about five years now since we wrote *The Mathematics of Voting and Elections: A Hands-On Approach*. Since then, the course for which the book was written has been taught four times at Grand Valley State University and once at Appalachian State University. Each time we have taught the course, we have gained new insights about this fascinating area of mathematics and how to help students engage its central ideas.

The goal of this instructor's manual is to share with you, the instructor, the things we have learned from our experiences and the strategies we have employed in our own classrooms. Each chapter contains (at least) the following sections:

- A **chapter summary**, which gives a quick overview of the main ideas in the chapter.
- **Learning objectives**, which identify the primary goals of the investigations and discussions contained in the chapter.
- **Teaching notes**, which provide thoughts, suggestions, and observations that are particularly useful to instructors teaching the topic for the first time.
- **Reading quiz questions**, which we have used at the beginning of each class as a lighthearted warmup and as a way to ensure that students have completed the assigned readings.
- **Questions for class discussion**, many of which have been suggested by our students as part of their reading assignments.
- A **discussion of selected questions**, which includes solutions to many of the non-starred questions in the text.
- **Supplementary questions**, which can be used for either exams or additional assignments.

As we continue to teach the mathematics of voting and elections, we expect that this instructor's manual will grow to incorporate new ideas and insights. To this end, we invite you to contribute your own comments and suggestions, which we will consider for inclusion in a future update. If you have any thoughts to share, or any questions for us, please feel free to contact us at the e-mail addresses listed below.

We wish you all the best, and we thank you for choosing our book.

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Chapter 1

What's So Good about Majority Rule?

Chapter Summary

Chapter 1 begins with the simplest of all elections: those that involve only two candidates. **Majority rule** seems to be the obvious choice for deciding such elections, but is it the best choice? By considering three alternatives to majority rule, readers learn that majority rule satisfies a number of desirable properties that other systems do not. For instance, a **dictatorship** does not treat all of the voters equally, thereby violating the property of **anonymity**. **Imposed rule** fails to treat all of the candidates equally, thus violating **neutrality**. **Minority rule** violates **monotonicity**, meaning that it can be detrimental to a candidate to receive additional votes.

In contrast, majority rule is shown to satisfy all three of the desirable properties of anonymity, neutrality, and monotonicity. Readers then investigate **May's theorem**, which states that majority rule with an odd number of voters is the only voting system for elections with two candidates that satisfies all three of these properties and avoids the possibility of ties. The proof of May's theorem involves first showing that in a two-candidate election, every voting system that is anonymous, neutral, and monotone must be a **quota system**. Readers then argue that the only quota system that avoids the possibility of ties is majority rule with an odd number of voters.

Learning Objectives

After completing Chapter 1, the reader should be able to . . .

- . . . define, compare, and contrast majority rule, dictatorship, imposed rule, and minority rule.
- . . . define the properties of anonymity, neutrality, and monotonicity.
- . . . explain why a given voting system does or does not satisfy each of the properties of anonymity, neutrality, and monotonicity. Give examples of voting systems that do and do not satisfy each of these properties.
- . . . state May's theorem, and explain its implications for voting in elections with only two candidates.
- . . . define quota system, and understand the relationship between majority rule and other quota systems.
- . . . understand and explain the main ideas behind the proof of May's theorem.

Teaching Notes

In many ways, Chapter 1 sets the tone for the first half of the book by introducing the analytical framework that is common to much of social choice theory. The basic process of introducing new systems and analyzing these systems according to precisely defined fairness criteria is one that will be employed throughout subsequent chapters, in particular Chapters 2 through 5.

At first, students may not see the point of studying systems like dictatorship, imposed rule, and minority rule. They may see these systems as being inherently bad choices, and this initial reaction is valid in many ways. The question to ask is why are these systems not desirable, and can we pinpoint ways in which they fail to satisfy our internal notion of fairness? Attempting to answer these questions will highlight the importance of being precise and specific about what the word "fair" means to us. It's worth pointing out that there are several potentially competing notions of fairness, and that in order to make meaningful comparisons between various voting systems, we are going to need nail down exactly which notions of fairness are most important.

At some point during the discussion of this chapter, students may begin to wonder where the mathematics is. This response is particularly common in general education courses, in which many students' prior experiences with mathematics will have been focused primarily on algebraic manipulation. It's important to note that what we are doing (by defining terms precisely, reasoning from these definitions, making and proving conjectures, etc.) is also highly mathematical, and perhaps even closer to the heart of what mathematics is than the more familiar computational exercises of the past.

Reading Quiz Questions

1. What was the name of the city referenced in the opening example from Chapter 1? **Stickeyville**

2. Which of the following people or places were not mentioned in the opening example from Chapter 1?

    (a) Stickeyville

    (b) **George W. Bush**

    (c) Mike Dowell

    (d) Laura Stutzman

3. Which of the following properties applies to voting systems that treat all candidates equally?

    (a) Anonymity

    (b) **Neutrality**

    (c) Monotonicity

    (d) Equitability

4. Which of the following properties applies to voting systems that treat all voters equally?

    (a) **Anonymity**

    (b) Neutrality

    (c) Monotonicity

    (d) Equitability

5. Which of the following terms would best describe a voting system in which all but one of the voters' ballots are discarded?

(a) Majority rule

(b) Minority rule

(c) Imposed rule

(d) **Dictatorship**

6. Which of the following terms would best describe a voting system in which the outcome is decided before any of the ballots are cast?

   (a) Majority rule

   (b) Minority rule

   (c) **Imposed rule**

   (d) Dictatorship

7. True or **false**: A dictatorship treats all voters equally.

8. **True** or false: Imposed rule treats all voters equally.

9. In which of the voting systems from Chapter 1 can it be detrimental to a voter to receive additional votes? **Minority rule**

10. Which of the following is not a property of minority rule?

    (a) It treats all of the candidates equally.

    (b) It treats all of the voters equally.

    (c) **It is beneficial for a candidate to receive additional votes.**

11. Which of the following properties of voting systems were not discussed in today's reading?

    (a) Anonymity

    (b) Monotonicity

    (c) **Solubility**

    (d) Neutrality

12. Which one of the following voting systems is not anonymous?

    (a) **Dictatorship**

(b) Imposed Rule

(c) Minority Rule

(d) Majority Rule

13. Which one of the following properties is not satisfied by imposed rule?

    (a) Anonymity

    (b) **Neutrality**

    (c) Monotonicity

14. **True** or false: Majority rule is anonymous, neutral, and monotone.

15. True or **false**: Majority rule always avoids the possibility of ties.

16. Fill in the blank: The important theorem in Chapter 1 pertaining to majority rule is named after Kenneth <u>**May**</u>.

17. Who won the 1876 U.S. presidential election? **Rutherford B. Hayes** (*Note: This question requires students to have completed Question 1.37, a Question for Further Study.*)

18. **True** or false: If a voting system for an election with two candidates is anonymous, neutral, and monotone, then it must be a quota system.

19. **True** or false: Majority rule is one of many voting systems that are anonymous, neutral, and monotone.

20. True or **false**: Majority rule is the only quota system.

21. **True** or false: A quota system may produce two winners or two losers.

22. **True** or false: The quota in a quota system can depend on the number of voters in the election.

23. **True** or false: The definition of a quota system includes the phrase "if and only if."

24. Which of the following is an example of a quota system?

    (a) Imposed rule

    (b) Dictatorship

    (c) **Majority rule**

(d) Minority rule

25. Suppose majority rule is used in an election with 121 voters. What would the quota be in this case? **61**

26. **True** or false: For a two-candidate election with an odd number of voters, majority rule is the only quota system that avoids ties.

Questions for Class Discussion

1. What are some ways that a tie could be resolved in an election with an even number of voters? How likely do you think a tie (two candidates receiving the exact same number of votes) is in an actual election, and how would this depend on the number of voters in the election?

2. In what types of elections would it make sense to use a quota system other than majority rule? Give specific examples.

3. Would a quota system other than majority rule prevent "tyranny of the majority?" Explain.

4. What are the potential advantages and disadvantages of using a quota system in an election with more than two candidates?

5. Why would anyone vote if imposed rule or a dictatorship were in place?

6. Are there situations in which minority rule would be considered a useful or acceptable voting method?

7. Are there situations in which the properties of anonymity, neutrality, and/or monotonicity would not be desirable?

8. If majority rule were not an option, which voting system would you prefer: dictatorship, imposed rule, or minority rule?

9. Which property do you think is least important: anonymity, neutrality, or monotonicity?

10. Are there any situations where the mathematical advantages of majority rule would be outweighed by other social or cultural factors? Explain.

11. Would it ever make sense to use a quota system with a quota of zero? Would it ever make sense to use a quota system with a quota equal to the number of voters?

12. Is majority rule the best way to pick a leader since there is the potential that almost half of the voters could be dissatisfied with the results?

13. How many different voting systems are there?

14. Is there a point at which the number of voters in an election makes a tie unlikely or even impossible?

Discussion of Selected Questions

Questions 1.5 – 1.7. Minority rule does treat all voters equally and all candidates equally. However, under minority rule, it can be detrimental to a candidate to receive additional votes.

Question 1.11. Note that in a dictatorship, if the dictator votes for candidate A, then candidate A will win regardless of how any of the remaining voters vote. However, if the dictator trades ballots with a voter who votes for candidate B, then the outcome of the election will change to B, thus violating anonymity.

Question 1.12. In a dictatorship, if every voter changes their vote, then the dictator will necessarily change his or her vote as well, thus changing the outcome of the election. Thus, every dictatorship is neutral. Similarly, in a dictatorship, candidate A is declared a winner if and only if the dictator votes for A. Even if A gains additional votes from other non-dictator voters, these votes do not affect the dictator's vote and thus cannot cause A to become a losing candidate. Thus, every dictatorship is monotone.

Question 1.13. Imposed rule is anonymous, since the outcome of the election is unaffected by any ballot changes, and is specifically unaffected by two voters trading ballots. Imposed rule is not neutral for the same reason – if every voter changes their vote from one candidate to the other, the outcome of the election will remain unchanged. Likewise, imposed rule is monotone, since receiving more (or less) votes can never affect whether a candidate is declared a winner or not. Thus, a winning candidate can *never* become a losing candidate, regardless of changes in votes.

Question 1.14. Minority rule is anonymous, since the winner is determined exclusively by the number of votes received. If two voters trade ballots, the total number of votes received by each candidate will remain unchanged, and thus the winner of the election will remain unchanged. Minority rule is also neutral. To see this, note that if every voter changes their vote from one candidate to the other, then the candidate who received the most votes will now receive the fewest, and the candidate who received the fewest votes will now receive the most. Thus, the winner will become the loser and the loser will become the winner. In the case of a tie, neither voter's total number of votes will change, and thus the outcome will remain unchanged. Minority rule is clearly not monotone. Any candidate who receives less than half of the total number of votes will be declared a winner, but if that candidate were to receive all of the votes, then he or she would no longer be a winning candidate.

Question 1.16. Majority rule is anonymous and neutral by the same argument used with minority rule in Question 1.14. For monotonicity, suppose that candidate A wins an election (or is tied for the win) by receiving at least as many votes as candidate B. If some voters change their votes from B to A and no voters change their votes from A to B, then candidate A will still receive more votes than candidate B, and will thus either remain the unique winner or go from being tied for the win to being the unique winner.

Question 1.17. Ties can actually be useful in some election situations; for instance, narrowing a field of ten candidates down to two potential winners is certainly progress. But if a two candidate election results in a tie, there is no similar benefit. No candidates can be eliminated for a potential second round of voting, and the election has accomplished nothing from a decision-making standpoint.

Question 1.21. The only voting system in Chapter 1 that is a quota system is majority rule. A dictatorship is not a quota system – if it were, the quota would have to be 1, since it is possible for a candidate to be declared a winner while only receiving one vote (from the dictator). However, if that candidate does not receive the vote of the dictator, but receives a vote from one other voter, he or she will not be declared a winner. Thus, the quota cannot be 1. In imposed rule, it is possible for a candidate to win with no votes, and so the only possible quota for imposed rule is 0. However, the quota for imposed rule cannot be 0 since this would imply that both candidates would always be declared winners, which we know is not the case. For minority rule, the argument is similar – the only possible quota is 0, but this cannot be the quota since any candidate who receives all of the votes (and, in doing so, exceeds the quota) will lose.

Question 1.23. We can conclude that V is a quota system with quota 2. Since Jen can win the election by receiving the votes of Joel and Grace, by anonymity, she would also win the election by receiving the votes of *any* two voters. Similarly, since Jen will lose the election if she receives only Joel's vote, it follows that she will lose the election whenever she receives exactly one vote, regardless of who that vote comes from. Monotonicity then implies that Jen will win the election if she receives two *or more* votes, and she will lose the election if she receives fewer than two votes. Finally, by neutrality, the same conclusions apply to Brian. Thus, in this election, a candidate will be declared a winner if and only if he or she receives at least two votes.

Questions 1.24 – 1.26. These three questions help the reader discover and understand one way to prove Theorem 1.22. The suggested proof relies on determining the quota by "asking" a voting system V (which is assumed to be anonymous, neutral, and monotone) a series of questions. It's important to note here that V can answer the questions asked because that is exactly what voting systems do – they tell who the winner or winners of an election should be given any possible combination of votes. The desired quota is established by the first question to which the answer is yes. This yields a set of q particular voters who can force a win for candidate A. By anonymity, any set of q voters could also force a win for candidate A. By monotonicity, any set of more than q voters could also force a win for A. Since some particular set of $q-1$ voters is unable to force a win for A, it follows by anonymity that every set of $q-1$ voters will be unable to force a win for A. By monotonicity again, the same conclusion applies to any set of less than $q-1$ voters. Thus, a set of voters can force a win for A if any only if that set contains q or more voters. By neutrality, the same argument holds for candidate B, thus establishing that the system being considered, V, is in fact a quota system with quota q.

Questions 1.27 – 1.31. This sequence of questions makes the final connection between Theorem 1.22 and May's theorem. The basic idea is that in a two-candidate election with an odd number, n, of voters, the only quota system that avoids the possibility of ties is the one with a quota of $(n+1)/2$ (or $n/2$ rounded up), exactly the quota for majority rule. Thus, if a voting system for a two candidate election is anonymous, neutral, and monotone, then it must be a quota system. If such a system is to also avoid the possibility of ties, then the quota must be exactly that of majority rule. The assumption that there are an odd number of voters is important because any quota system with an even number of voters is susceptible to ties. Of course, it is also worth noting that majority rule with an even number of voters is still anonymous, neutral, and monotone, and less prone to ties than any other quota system for an even number of voters.

Question 1.33. This question illustrates a nuance in the idea of two voters "cancelling out" each others' votes. Suppose there were 50 voters in the congregation (including Greg and Gail) and that the outcome of the election was 33 votes in favor of the recall and 17 against. In this situation, the pastor would just barely keep his job, since 34 votes in favor would be required for a recall. However, if Greg and Gail had decided not to vote, the outcome would have been 32 votes in favor versus 16 against, which would have then resulted in the pastor being recalled.

Questions 1.35 and 1.37. Because of the electoral college, majority rule does not dictate the outcome of United States presidential elections. There are several instances throughout history in which no candidate, including the winning candidate, received a majority of the votes cast. In some of these cases (1824, 1888, and 2000), even the candidate who received the *most* votes did not win. The most grievous example of this phenomenon occurred in 1876, when Samuel Tilden did receive a majority (approximately 51%) of the popular vote but subsequently lost to Rutherford B. Hayes in the electoral college. The election returns in several states were disputed, and Colorado's electors were ultimately appointed by Congress and not elected in a popular vote.

Question 1.36. With only two candidates running, the winner of each state's electoral votes (all of them!) is determined by majority rule. There are, however, two exceptions to this rule – in both Maine and Nebraska, electoral votes can theoretically be split among the candidates instead of awarded to just one of them. Such a split occurred for the first time in history during the 2008 election, when Nebraska awarded 4 electoral votes to John McCain and 1 to Barack Obama. Although this type of distribution had always been possible, it had never actually occurred until Obama made the decision to campaign aggressively in key areas of the state – specifically, the 2^{nd} congressional district, which consists mainly of Omaha. Obama's strategy paid off; he won this district, and consequently its single electoral vote.

Question 1.38. To override a presidential veto, a 2/3 majority vote is required. Thus, the system used is a quota system, but is not equivalent to majority rule.

Supplementary Questions

Question 1.42. Consider an election with two candidates, Mark and Lara, and three voters, Al, Bethany, and Candice. Suppose that if Al and Bethany vote for Mark, and Candice votes for Lara, then Mark will win. Suppose also that the voting system being used is anonymous, neutral, and monotone. Using only this information, determine what the outcome of the election would be for each of the other 7 combinations of votes. Clearly explain your reasoning, including where you used each of the properties of anonymity, neutrality, and monotonicity.

Question 1.43. Consider a voting system for an election with two candidates in which each voter casts a vote for one of the two candidates, and a candidate is declared a winner if and only if he or she receives more than half of the votes of the female voters in the election **and** more than half of the votes of the male voters in the election.

(a) Is the system described above anonymous? Give a convincing argument or example to justify your answer.

(b) Is the system described above neutral? Give a convincing argument or example to justify your answer.

(c) Is the system described above monotone? Give a convincing argument or example to justify your answer.

Question 1.44. Repeat Question 1.43, but this time assume that a candidate is declared a winner if and only if he or she receives more than half of the votes of the female voters in the election **and** *less than half* of the votes of the male voters in the election.

Question 1.45. Suppose that in an election with two candidates, a candidate is declared a winner if and only if he or she receives an even number of votes. Decide whether such a system is anonymous, neutral, and/or monotone. Give a convincing argument or example to justify your answer for each property.

Question 1.46. Research the tie-breaking methods used in various states for general elections. In which state is it possible for the winner to be decided by a game of poker?

Question 1.47. A devious politician has hired you to find or invent a voting system that violates all three of the properties of anonymity, neutrality, and monotonicity. Does such a voting system exist? If so, describe one such system. If not, explain why no such system can exist.

Chapter 2

Perot, Nader, and Other Inconveniences

Chapter Summary

Chapter 2 steps away from the simple context of two-candidate elections and begins to consider the subtleties and complications that arise from the inclusion for additional candidates. The chapter begins by considering the **spoiler effect** in the 1992 and 2000 **U.S. presidential elections**. These examples formally introduce the **plurality method**, and further questions explore the distinction between **plurality** and **majority rule**. The **2003 California recall election** is then used to illustrate how it is possible, in an election with a large number of candidates, for the plurality winner to receive a very small percentage of the actual votes cast.

After investigating plurality, the reader is introduced to the **Borda count** via an example from the sports world. This example demonstrates that the Borda count violates the **majority criterion**, which states that if any candidate receives a majority of the first-place votes cast in an election, then that candidate must be declared the winner. Further investigations introduce **preference orders** and **preference schedules**.

To close out the chapter, the reader considers how the definitions of **anonymity**, **neutrality**, and **monotonicity** must be modified in order to apply to elections with more than two candidates. Finally, a brief journey back to **May's theorem** reveals that both plurality and the Borda count satisfy all of the properties of anonymity, neutrality, and monotonicity. Any apparent contradictions to May's theorem are quickly resolved by noting that the theorem applies only to elections with exactly two candidates.

Learning Objectives

After completing Chapter 2, the reader should be able to ...

- ... describe the plurality method, and explain how it differs from majority rule.
- ... define and use the Borda count to decide the winner of an election with more than two candidates.
- ... describe several real-life examples of elections involving the Borda count, and explain any surprising features of these elections or their outcomes.
- ... define the majority criterion, and explain why it is or is not satisfied by each of majority rule, plurality, and the Borda count.
- ... explain how the definitions of anonymity, neutrality, and monotonicity must be modified in order to apply to elections with more than two candidates.
- ... explain why the Borda count satisfies each of the properties of anonymity, neutrality, and monotonicity.
- ... discuss how the Borda count is related to both the hypotheses and the conclusion of May's theorem.

Teaching Notes

In spite of its widespread use, most students will have never seriously studied the plurality system prior to this chapter. In particular, they are likely to be surprised by the unexpected and undesirable behavior that can occur when plurality is used in elections with numerous candidates. For some students, examples like the 2000 U.S. presidential election or the 2003 California gubernatorial recall election will be persuasive. Such students are likely to embrace potential alternatives to plurality such as the Borda count. Others may feel that, in spite of its flaws, plurality is still the best system. It's important to give voice to both sides in this debate, and to let students talk through the pros and cons of each system. It's also important to highlight both the practical and the theoretical arguments for or against each system. For instance, plurality is easy to use and widely accepted, whereas the Borda count requires more information from voters, is harder to understand, and would likely encounter resistance if proposed for use in political elections. These practical considerations provide both balance and context to the more theoretical results explored within the text. Thus, they should be incorporated into the discussion whenever possible.

Chapter 2 is the first time that students will be exposed to preference schedules and societal preference orders. Both are used consistently throughout Chapters 3 through 5, and so it is essential that students master the relevant notation and terminology before moving on. Also important are the modifications to the definitions of anonymity, neutrality, and monotonicity that appear at the end of the chapter. In addition to their use in subsequent chapters, these definitions also demonstrate how the relationships between individual and societal preference orders can be used to define fairness criteria.

Reading Quiz Questions

1. Which of the following politicians were not mentioned in the opening example from Chapter 2?

   (a) George W. Bush

   (b) Al Gore

   (c) **Pat Buchanan**

   (d) Ralph Nader

2. **True** or false: For an election with exactly two candidates, the words majority and plurality mean the same thing.

3. True or **false**: For an election with more than two candidates, the words majority and plurality mean the same thing.

4. Which famous actor turned politician was discussed in Question 2.5? **Arnold Schwarzenegger**

5. Fill in the blank: One of the systems studied in Chapter 2 is called the __**Borda**__ count.

6. Fill in the blank: The desirable property of voting systems introduced in Chapter 2 is called the __**majority**__ criterion.

7. True or **false**: The Borda count satisfies the majority criterion.

8. The ranking of the candidates produced by a voting system is called a:
   (a) Normalized candidate ranking
   (b) **Societal preference order**
   (c) Voter-candidate analysis
   (d) Collective preference vector
9. In a five-candidate election with the winner chosen by the Borda count, how many points would a first-place vote be worth? **4**
10. In a five-candidate election with 10 voters, which candidate would be ranked higher by the Borda count?
    (a) One who was ranked first by 5 voters and last by 5 voters
    (b) **One who was ranked third by 9 voters and second by 1 voter**
11. **True** or false: The definitions of anonymity, neutrality, and/or monotonicity require modifications in order to apply to elections with more than two candidates.
12. For an election with more than two candidates, which of the following properties are violated by plurality?
    (a) Anonymity
    (b) Neutrality
    (c) Monotonicity
    (d) The majority criterion
    (e) All of the above
    (f) **None of the above**
13. For an election with more than two candidates, which of the following properties are violated by the Borda count?
    (a) Anonymity
    (b) Neutrality
    (c) Monotonicity
    (d) **The majority criterion**

(e) All of the above

(f) None of the above

14. True or **false**: The Democratic National Committee ran advertisements supporting Ralph Nader in the weeks prior to the 2000 U.S. presidential election. (*Note: It was the Republican National Committee that ran ads supporting Nader.*)

Questions for Class Discussion

1. Is it unwise to let candidates on the ballot who stand virtually no chance of winning the election? If yes, why? If not, what would a candidate have to do to prove that they should be on the ballot?

2. Would it ever make sense to use a different point scheme with a Borda-type system? If so, give an example, and specify how many points each ranking should be worth in your example.

3. Do you think there are any practical difficulties that might arise if the Borda count was implemented in a large election for public office?

4. How easy do you think it would be to explain the Borda count to a member of the general public, and how do you think average citizens would respond to a proposal to use the Borda count in a public election?

5. What is the best criticism or defense of plurality that you have heard so far?

6. Can you think of any other modifications to plurality that would prevent spoiler candidates from having a significant effect on election outcomes?

7. Do you think a Borda count system would be effective for U.S. elections? If so, in what context? If not, explain why.

8. Is the majority criterion a desirable criterion? Does it ever make sense to not elect a candidate who receives a majority of first-place votes?

9. What are some criteria, other than the ones we have discussed so far, that may be desirable for voting systems to satisfy?

10. Would it be advantageous to combine either plurality or the Borda count with some kind of runoff system? What would be the advantages and disadvantages of doing so?

11. If the Borda count was used in the U.S. for national elections, how would this affect the way candidates campaigned? In your opinion, would the change be positive or negative?

12. In elections with large numbers of candidates, is it practical to have voters rank every candidate? Would reducing the number of potential rankings (for instance, asking voters to rank the top 10 instead of all 135 in the case of the California gubernatorial recall election) be unfair to candidates who may secure more points toward the middle of the scale?

13. How could ties and/or incomplete ballots be incorporated into the Borda count? What different approaches could be used, and what are the advantages and disadvantages of each of these approaches?

14. Under the Borda count, would a voter ever have an incentive to vote insincerely by misrepresenting his or her preferences? If so, what kind of misrepresentation do you think would be most common or most beneficial to the voter?

15. How do you think voter ignorance and/or apathy could affect the results of the Borda count?

Discussion of Selected Questions

Question 2.5. Schwarzenegger did not receive a majority of the first place votes, although he did receive a plurality. If all of the 8,657,915 votes had been distributed as evenly as possible among the 135 candidates, then 95 of the candidates would have received 64,133 votes and the remaining 40 would have received 64,132 votes. So, Schwarzenegger could have theoretically tied for first place with 64,133 votes, but would have needed 64,134 (0.74%) to have had a chance of winning the election outright. In either of these situations, all of the remaining voters (99.26% of the electorate) could have ranked Schwarzenegger last among all of the candidates.

Question 2.10. In an election with n candidates and m voters, a candidate cannot possibly win outright with fewer than $\lceil \frac{m}{n} \rceil$ first-place votes, and always has at least a chance of winning with $\lceil \frac{m}{n} \rceil + 1$ first-place votes. These numbers obviously decrease (and in fact approach zero) as n increases. Moreover, a candidate winning with this minimal number of first-place votes could theoretically be ranked last by all of the remaining voters. Question 2.5 gives an example of one such worst-case scenario. While an example that extreme would be highly unlikely to occur in an actual election, there are many examples throughout history where the plurality winner has received a relatively small percentage of the popular vote and has been disliked by a large percentage of the remaining voters.

Question 2.19. Suppose that a change unfavorable to a candidate caused that candidate to experience an increase in rank on the resulting societal preference order. Then reversing the change would constitute a change favorable to the candidate and would result in a decrease in rank on the resulting societal preference order, a violation of the definition of monotonicity.

Question 2.21. Both plurality and the Borda count satisfy anonymity, neutrality, and monotonicity. This is not a contradiction to May's theorem since both systems are equivalent to majority rule for elections with only two candidates.

Question 2.22. For elections with more than two candidates, plurality is not a quota system. By Question 2.10, if plurality were a quota system, then the quota would have to be no more than $\lceil \frac{m}{n} \rceil + 1$, where m and n represent the respective numbers of voters and candidates in the election. But there are clearly situations in which a candidate could receive much more than this number of votes and still lose. In order to win under plurality, a candidate must not only exceed a certain minimum number of votes, but must also receive more votes than any of the other candidates that also exceed this minimum value. This fact does not contradict May's theorem, since plurality is equivalent to majority rule (a quota system) in elections with only two candidates.

Question 2.23. In an election with exactly three candidates, a candidate can tie for first place without receiving any first-place votes, but must receive at least one first place vote in order to win outright. For elections with four or more candidates, there are many ways for a candidate to win without receiving any first-place votes. To be *guaranteed* a win, a candidate must receive more than $\frac{c-1}{c} \cdot v$ votes, where c is the number of candidates and v is the number of voters. To prove this, suppose candidate A receives x first-place votes. For A to be guaranteed a win over every other candidate, A must be able to beat a candidate who receives x second-place votes and $v - x$ first-place votes (the best that any other candidate could do with A receiving x first-place votes), even if A receives last-place votes from all $v - x$ of the voters who did not rank A first. For this to happen, it must be the case that

$$x \cdot (c-1) + (v-x) \cdot 0 > x \cdot (c-2) + (v-x) \cdot (c-1).$$

Solving this inequality for x yields

$$x > \frac{c-1}{c} \cdot v.$$

Question 2.26. If the 3 voters in the far right column of the table switch the order of Filiz and Gerald on their preference orders, then the new societal preference order will be $H \succ G \succ F \succ I$. In this example, no individual voters changed their preferences between Helen and Gerald, and yet the ranking of these two candidates was reversed on the resulting societal preference order. A similar phenomenon occurs if the voters in the second and third columns switch their rankings of Filiz and Ivan; now Filiz beats both Gerald and Helen, even though none of the voters changed their individual rankings of either Filiz and Gerald or Filiz and Helen.

Question 2.27. A good strategy in this situation would be to introduce a candidate that emulated Filiz's views, potentially splitting the votes of those who support Filiz between Filiz and the new candidate.

Question 2.28. There have been numerous U.S. presidential elections in which the winner received a plurality, but not a majority, of the popular vote. These include: 1996, 1992, 1968, 1960, 1948, 1916, 1912, 1892, 1884, 1880, 1860, 1856, 1848, 1844. There have been four U.S. presidential elections in which the winner did not receive a plurality of the popular vote: those in 2000, 1888, 1876, and 1824.

Question 2.29. Most political scientists agree that if the Borda count had been used, Gore would have won Florida and thus the entire election. It is possible, however, to construct a scenario in which, by voting insincerely (for instance, $B \succ N \succ G$ instead of $B \succ G \succ N$), voters could have chosen Nader as the winner in Florida.

Question 2.30. Had McCain run as an independent, it is almost certain that Gore would have won under plurality. Under the Borda count, however, is is likely that McCain would have secured first or second place votes from much of the electorate, leading to a probable win for McCain in the general election.

Question 2.32. The poll in this question does not illustrate a violation of the majority criterion, since no single team received a majority of the first place votes.

Question 2.33. Each of the 62 voters awarded a total of $25 + 24 + 23 = 72$ points for first, second, and third place votes. Thus, a total of $62 \times 72 = 4{,}464$ points were awarded for these top three rankings. If any of Florida State, Notre Dame, or Nebraska had been ranked below third place on any ballot, then the total number of points awarded between these three teams would have been less than 4,464. But $1{,}523 + 1{,}494 + 1{,}447 = 4{,}464$, and so each of these three teams was ranked either first, second, or third by each of the 62 voters.

Question 2.34. In the 1990 UPI poll, Georgia Tech and Colorado finished first and second, respectively. Georgia Tech and Colorado were clearly the two best teams that year, so it is possible that the coach of either team could have attempted to manipulate the vote by leaving the opposing team off the ballot altogether.

Question 2.35. Since there are only 28 voters in the poll, and Ken Griffey, Jr. won all 28 first-place votes in 1997, it must be that each first-place vote is worth $392 \div 28 = 14$ points. Since Juan Gonzalez received 21 first-place votes in 1998, these first-place votes must have accounted for $21 \times 14 = 294$ of his 357 points. The remaining 63 points must have come from his 7 second-place votes. Thus, each second place vote must be worth 9 points.

Question 2.36. In the 2001 poll, the Mariners' Ichiri Suzuki won with 289 points, defeating runner-up Jason Giambi from the A's, who had 281 points. However, if the normal Borda count had been used, Giambi would have defeated Suzuki, 249 points to 245 points.

Question 2.37. The Heisman Trophy is college football's most sought-after and prestigious award. (See `http://www.heisman.com` for more information.) The winners of the Heisman trophy are selected by a group consisting of 870 members of the media, all of the past Heisman Trophy winners, and 1 fan vote (for a total of 925 voters in the 2007 contest). The media electors are appointed by six sectional representatives, and each state is allocated a number of votes proportional to its size and number of media outlets. In 1956, Paul Hornung, Notre Dame's "Golden Boy", became the only Heisman trophy winner ever chosen from a team with a losing record (2–8). Hornung was also the first winner to not receive a plurality of the first-place votes. In fact, Tom McDonald from Oklahoma received 205 first-place votes, 8 more than Hornung's 197.

Supplementary Questions

Question 2.39. Decide whether each of the following statements are true or false. Give a convincing argument or example to justify your each of your answers.

(a) In a three-candidate election that does not result in a tie, the Borda count winner must receive at least one first-place vote.

(b) In a four-candidate election that does not result in a tie, the Borda count winner must receive at least one first-place vote.

Question 2.40. Find a copy of the article "Would the Borda Count Have Avoided the Civil War?" by Alexander Tabarrok and Lee Spector in the *Journal of Theoret-*

ical Politics. Write a summary of the article, including the authors' answer to the question posed in the article's title.

Question 2.41. Consider an election with four candidates and the preferences shown below:

	Number of Voters		
Rank	51	25	24
1	A	C	D
2	B	B	B
3	C	D	C
4	D	A	A

(a) Who would win this election under any system that satisfies the majority criterion?

(b) Who would win this election under the Borda count?

(c) Which of the outcomes from parts (a) and (b) do you think is most fair? In your opinion, which best represents the will of the voters?

(d) Do your answers to parts (a)–(c) affect your opinion of the majority criterion in any way? Explain.

Question 2.42. Investigate the results of voting for the 2008 Heisman trophy, and write a detailed summary of your findings. Which player received the most first-place votes, and in what place did this player finish in the overall rankings?

Chapter 3

Back into the Ring

Chapter Summary

Chapter 3 continues previous investigations of elections with more than two candidates, beginning with the **1998 Minnesota gubernatorial election**, where former pro wrestler Jesse "The Body" Ventura, running on a Reform Party ticket, defeated both his Republican and Democratic opponents to become the 38th governor of the state of Minnesota. Upon further analysis, the reader learns that Ventura was a **Condorcet loser**, meaning that he would have lost to either the Republican or the Democratic candidate in a head-to-head contest.[1] Moreover, the Republican candidate, St. Paul mayor Norm Coleman, was a **Condorcet winner**, meaning that he would have defeated either of his opponents in a head-to-head race. Thus, plurality is shown to violate both the **Condorcet winner criterion** (because it can fail to elect a Condorcet winner when one exists) and the **Condorcet loser criterion** (because it can elect a Condorcet loser).

Further properties of Condorcet winners and losers are explored, and **sequential pairwise voting** is introduced as an alternative to plurality voting that does satisfy the Condorcet winner criterion. However, sequential pairwise voting is also shown to violate **neutrality** due to the manipulability of its **agenda**, which is determined apart from the voting process.

Finally, the **instant runoff** voting system is introduced and studied in detail. Readers discover that although instant runoff has a long history of vocal support

[1] Although several other authors also make this claim, it is not undisputed. In fact, a paper by Lacy and Monson (*Anatomy of a third-party victory: Electoral support for Jesse Ventura in the 1998 Minnesota gubernatorial election*, available at `http://polmeth.wustl.edu/media/Paper/lacy00b.pdf`) suggests the opposite.

from various philosophers, politicians, and election reform groups, the system actually violates both monotonicity and the Condorcet winner criterion. The chapter concludes by asking readers to summarize which criteria are satisfied and violated by each of the voting systems studied in the first three chapters.

Learning Objectives

After completing Chapter 3, the reader should be able to ...

- ... discuss the 1998 Minnesota gubernatorial election and its significance.
- ... define Condorcet winner and Condorcet loser, and give examples to illustrate both terms.
- ... define the Condorcet winner and Condorcet loser criteria, and give examples of voting systems that satisfy these criteria and others that do not.
- ... describe in detail the relationship between the Condorcet winner criterion and the majority criterion.
- ... describe how sequential pairwise voting works, and in particular how societal preference orders are constructed from individual preferences.
- ... explain why sequential pairwise voting satisfies the Condorcet winner criterion but violates neutrality.
- ... describe how instant runoff works, and explain why instant runoff is capable of violating both the Condorcet winner criterion and monotonicity.

Teaching Notes

There are plenty of surprises in this chapter, from Jesse Ventura's unexpected win in the 1998 Minnesota gubernatorial election to instant runoff's paradoxical ability to violate monotonicity. With these surprises comes ample opportunity for discussion and debate. For instance, even if we admit that Jesse Ventura was a Condorcet loser, did he win the election fair and square? Should he have won the election? Does the fact that he won the election put a nail in the coffin of our familiar friend, plurality? These are all legitimate questions, and students are likely to offer a variety of perspectives on them.

It's important to take plenty of time to discuss the significance of pairwise comparisons, not just within the formal system of sequential pairwise voting, but as a framework from which to analyze election outcomes as well. Consider, for example, the preferences from Question 3.4. This classic example, known as Condorcet's paradox, illustrates a disturbing possibility. In particular, when the societal preferences over pairs of possible outcomes are cyclic, it is possible that *no matter what voting system is used and no matter what outcome is selected, there will be some other outcome that is preferred by a significant portion of the electorate.* This bothersome observation ought to add a dose of realism to the class discussion of various voting systems. Specifically, while some voting systems may be better than others according to our fairness criteria, it is possible that some level of voter dissatisfaction will occur no matter what voting system we use.

Some students may have trouble understanding the importance of the discussion on pages 44–45 regarding societal preference orders for sequential pairwise voting (SPV). This discussion is included mainly for the purposes of illustrating the potential non-transitivity of pairwise comparisons, and so that SPV can be modeled using the same framework that is used to model every other voting system throughout the text. The most important point of this discussion is that, by requiring ties in the societal preference order, we are eliminating potentially inaccurate pairwise comparisons between non-winning candidates.

Finally, of all of the potential alternatives to plurality discussed in the text, instant runoff is arguably the one that has gained the most traction for political elections. This fact alone merits further discussion, and it provides context for some of the more theoretical results pertaining to instant runoff. Question 3.32 in the Questions for Further Study asks students to investigate the arguments for instant runoff advanced by the Center for Voting and Democracy. This question is worth assigning if only for the fact that the Center's admittedly biased web site provides a wealth of information on recent implementations of instant runoff voting. One of the more notable items is a bill that was recently passed by Vermont's state legislature but vetoed by the state's governor, which would have adopted instant runoff for all of the state's federal elections.

Reading Quiz Questions

1. Which recent election did the opening example from Chapter 3 refer to? **1998 Minnesota gubernatorial election**

2. The winner of the election in the opening example from Chapter 3 was a:

   (a) Condorcet winner
   (b) **Condorcet loser**
   (c) Neither
   (d) Both

3. True or **false**: Every election has at least one Condorcet winner.

4. True or **false**: An election can have more than one Condorcet winner.

5. **True** or false: Sequential pairwise voting satisfies the Condorcet winner criterion.

6. True or **false**: If a voting system satisfies the majority criterion, then it must satisfy the Condorcet winner criterion.

7. **True** or false: If a voting system satisfies the Condorcet winner criterion, then it must satisfy the majority criterion.

8. Fill in the blank: In sequential pairwise voting, the order in which head-to-head elections are conducted is called an **agenda**.

9. Which of the following voting systems do not satisfy anonymity?

   (a) Plurality
   (b) The Borda count
   (c) Sequential pairwise voting
   (d) Instant runoff
   (e) All of the above
   (f) **None of the above**

10. Which of the following voting systems do not satisfy neutrality?

   (a) Plurality
   (b) The Borda count
   (c) **Sequential pairwise voting**
   (d) Instant runoff
   (e) All of the above

(f) None of the above

11. Which of the following voting systems do not satisfy monotonicity?

    (a) Plurality

    (b) The Borda count

    (c) Sequential pairwise voting

    (d) **Instant runoff**

    (e) All of the above

    (f) None of the above

12. Which philosopher was quoted in Chapter 3 with regard to instant runoff?

    (a) **John Stuart Mill**

    (b) Bertrand Russell

    (c) Plato

    (d) Howard Stern

13. True or **false**: Instant runoff satisfies the Condorcet winner criterion.

Questions for Class Discussion

1. What would be some of the practical obstacles to implementing sequential pairwise voting in a political election? How could these obstacles be overcome?

2. Can you think of a situation in which it would make sense not to elect a Condorcet winner? Can you think of a situation in which it would make sense to elect a Condorcet loser?

3. Should Jesse Ventura have won the Minnesota gubernatorial election? Why or why not?

4. Is it always possible to determine if the winner of an election is a Condorcet winner, a Condorcet loser, or neither? What kind of information is required to make this determination?

5. Should the fact that sequential pairwise voting violates neutrality automatically eliminate it from consideration as a reasonable alternative to plurality? Why or why not?

6. Was there a Condorcet winner in the 2000 U.S. presidential election, either in Florida or nationwide? Was there a Condorcet loser?

7. How could plurality be modified to satisfy the Condorcet loser criterion?

8. Which do you think is the most important criterion for a voting system to satisfy: the majority criterion, the Condorcet winner criterion, or the Condorcet loser criterion?

9. For which types of elections, if any, would sequential pairwise voting be most appropriate, and why?

10. Can you think of a situation apart from political elections in which sequential pairwise voting, or a system similar to it, might be used to make an important decision?

11. Would sequential pairwise voting be more fair if the agenda was chosen randomly? Would the resulting system be a good alternative to plurality?

12. Should the fact that instant runoff violates monotonicity automatically eliminate it from consideration as a reasonable alternative to plurality? Why or why not?

13. How likely do you think it is that voters would exploit the fact that instant runoff violates monotonicity in order to influence the outcome of an election? What voting strategies would they have to employ in order to do so?

14. If instant runoff were used in U.S. presidential elections, would it make sense to simply skip the party nominations and have all of the candidates run against each other until one winner remained?

15. Some have argued that instant runoff should not be considered as a viable alternative to plurality because it has the potential to eliminate a consensus or compromise candidate who may be acceptable to a large proportion of the electorate and yet not receive many first-place votes. Is this a valid criticism if instant runoff? Why or why not?

16. How do you think instant runoff compares to other voting systems in terms of manipulability?

17. Of the four systems we have discussed so far (for elections with more than two candidates), which do you think is most fair or best representative of the will of the electorate? Does your answer depend on the type of election that the system is used for?

18. Of all the fairness criteria we have discussed, which do you think are the most important? Which are the least important?

19. How do you think the use of ranked systems such as the Borda count and instant runoff would affect voter turnout compared to plurality?

Discussion of Selected Questions

Warmup 3.1 and Question 3.3. These first two questions demonstrate that plurality is capable of electing a Condorcet loser, even in the presence of a Condorcet winner.

Question 3.4. In this example, combining the results of each of the possible head-to-head contests leads to a *cyclic* societal preference, namely

$$A \succ B \succ C \succ A \succ B \succ C \succ \cdots$$

Notice that no matter which candidate is selected as the winner, some other candidate will be preferred by two thirds of the voters.

Question 3.11. Sequential pairwise voting does satisfy the Condorcet loser criterion, since the sequential pairwise voting winner must win at least one head-to-head contest, which a Condorcet loser can never do.

Question 3.13. One possible agenda for electing Filiz is I, H, F, G. To elect Gerald, we could use the agenda F, I, H, G.

Question 3.14. When every voter switches the positions of Ivan and Helen in their preference orders, the outcome of the election remains unchanged. Because Helen does not become the new winner, this example demonstrates that sequential pairwise voting violates neutrality.

Question 3.15. In sequential pairwise voting, each head-to-head contest is determined entirely by the number of voters who prefer each candidate. A swap of two voters' ballots would leave these numbers unchanged, and thus would not affect the outcome of any individual head-to-head contest. Thus, the overall outcome would remain unaffected by such swaps, which proves that sequential pairwise voting is anonymous. To see why sequential pairwise voting is monotone, note that candidate A wins under sequential pairwise voting if and only if A wins every head-to-head contest in which he or she competes. Since the outcome of head-to-head contests not involving A does not depend on A's position, it follows that changes involving A will not affect the outcome of any of the contests prior to when A first appears on the agenda. Thus, changes favorable to A can have no effect on who A's first opponent is. Since A would defeat this first opponent and all subsequent opponents even before the favorable changes, A will still defeat all of these opponents after the changes are made (and by a larger margin). Thus, sequential pairwise voting is monotone. A more detailed argument can be found in Appendix A.

Question 3.18. In instant runoff, eliminations are based entirely on the number of first-place votes received by each candidate at any given stage of the election. Since two voters trading ballots would not affect these numbers in any way, such trades have no impact on the order in which candidates are eliminated. Since the winner of the election and the resulting societal preference order are determined uniquely by the order in which candidates are eliminated, it follows that instant runoff is anonymous. For neutrality, suppose that the positions of candidates A and B and swapped on every voter's ballot. For the purposes of instant runoff, the effect of this swap is to exchange the number of first-place votes held by A and B at each stage of the election. Now suppose also that A was eliminated prior to B before the swap was made. Then at some stage of the election, A had fewer first-place votes than any of the remaining candidates in the election, including B. After the swap, however, the tables would be turned: at that same stage in the election, B would have fewer first-place votes than any of the remaining candidates, including A. Thus, B would be eliminated prior to A. A similar argument can be made assuming that B is eliminated prior to A (pre-swap) or that A and B are eliminated at the same time (again, pre-swap). Thus, exchanging the positions of A and B on every voter's ballot is guaranteed to change the societal preference order accordingly, which implies that instant runoff is neutral.

Question 3.19. Once a candidate obtains a majority of the first-place votes, subsequent eliminations will maintain and possibly add votes to that majority. Thus, a majority winner at any stage of the election will ultimately emerge as the overall winner.

Question 3.21. In this election, even though A is the Condorcet winner, A also receives the least number of first-place votes and is thus eliminated in the first round of the election. Because this is possible, it follows that instant runoff does not satisfy the Condorcet winner criterion.

Question 3.25. Although most familiar systems that satisfy CWC also satisfy CLC, this relationship is not guaranteed to hold. For instance, imagine a voting system that elects a Condorcet winner whenever one exists and then reverts to plurality if there is no Condorcet winner. Such a system would satisfy CWC but not CLC.

There are also plenty of systems that satisfy CLC but not CWC, including the Borda count and instant runoff.

Question 3.26. Under the Borda count, Melissa would win with 31 points. Gary would come in second with 30 points, Dallas third with 27 points, and Sydnie last with 14 points. Melissa would also win under sequential pairwise voting with *any* agenda, since she is the Condorcet winner. Note also that Sydnie is the Condorcet loser.

Question 3.27. This question is actually not that challenging, since any example in which the Borda count fails to elect a majority winner will suffice. An interesting example for discussion is the following:

	Number of Voters		
Rank	51	25	24
1	A	C	D
2	B	B	B
3	C	D	C
4	D	A	A

Here the Borda count elects B handily, even though A is the majority winner, and thus the Condorcet winner. There are plenty of good questions raised by this example, such as:

- How well does B reflect the will of the voters in the election? Is B a better choice than A? Why or why not?
- Would any of the other methods we have discussed also elect B?
- Could A have received even more first place votes and still lost to B? If so, how many? Would B be a good choice in this scenario?

Note that a similar situation could have occurred in the 2003 California recall election. Suppose that 51% of the voters liked Arnold Schwarzenneger best, Leo Gallagher second best, and Gary Coleman last (that is, 135^{th}). Suppose also that the remaining 49% of the voters liked Gallagher first, Coleman second, and Schwarzenegger last. Then Schwarzenegger would have been elected under plurality, even though he was the last choice of 49% of the voters. Moreover, the Borda count winner in this scenario would have been Leo Gallagher, who was the first or second choice (out of 135 candidates!) of *all* of the voters.

Question 3.28. The Borda count does satisfy the Condorcet loser criterion, although this is not easy to show. The basic idea behind the argument is that a Condorcet loser always will always be awarded a below average number of points when compared to the other candidates in the election. Because of this, there must be other candidates in the election with more points, which guarantees the Condorcet loser's defeat.

To flesh out the details of this argument, we will begin by considering the special case of an election with three candidates, A, B, and C. We will let n denote the number of voters, and we will assume that C is a Condorcet loser. We will then argue that C cannot win under the Borda count.

Suppose that the preferences of the n voters in the election are represented by the preference schedule shown below. (Note that we have used variables to denote the number of voters with each possible preference order.)

	Number of Voters					
Rank	u	v	w	x	y	z
1	A	A	B	B	C	C
2	B	C	A	C	A	B
3	C	B	C	A	B	A

We know that C must lose to A. Thus,

$$u + v + w > x + y + z.$$

Furthermore, since C must also lose to B, we know that

$$u + w + x > v + y + z.$$

Combining the previous two inequalities, we find that

$$2(u + w) + x + v > 2(y + z) + x + v.$$

This implies that $u + w > y + z$, which means that the number of voters who rank C last must exceed the number of voters who rank C first. Let $k = y + z$, so that k denotes the number of voters who rank C first. By our previous argument, $k < \frac{n}{2}$, and at least $k + 1$ voters rank C last. This means that at most $n - k - (k + 1) = n - 2k - 1$ rank C in second place. Thus, the Borda count awards C at most $2k + (n - 2k - 1) = n - 1$ points. But each voter distributes a total of 3 points among the three candidates, and so the average number of points received by each candidate is n. Since $n-1 < n$, there must be some candidate, say A, who receives more than n points. Thus, A will be ranked higher than C by the Borda count, and C, the Condorcet loser, will not be elected.

To extend this argument to elections with more than 3 candidates, we will use a technique known as *mathematical induction*. Essentially, we will show that if our result is true for all elections with m candidates, then it must also be true for all elections with $m + 1$ candidates. Since we already know our result to be true for elections with 3 candidates, this *inductive step* will imply that it is true for elections with 4 candidates, which will in turn imply that it is true for elections with 5 candidates, and so on. So let's assume that in any election with m candidates, the Borda count will award a Condorcet losing candidate a below average number of points. Consider any election with $m + 1$ candidates, and let C be a Condorcet loser in this larger election. If we remove any candidate other than C, say D, then the resulting election will have m candidates, and C will still be a Condorcet loser. Thus, in the smaller election, our previous assumption (technically called the *induction hypothesis*) allows us to conclude that C will be awarded a below average number of points. Since each voter's ballot is worth

$$0 + 1 + 2 + \cdots + (m - 1) = \frac{m \cdot (m - 1)}{2}$$

points, and there are n ballots, the total number of points awarded is equal to

$$\frac{n \cdot m \cdot (m - 1)}{2}.$$

This implies that the average number of points received per candidate is

$$\frac{1}{m} \cdot \frac{n \cdot m \cdot (m - 1)}{2} = \frac{n \cdot (m - 1)}{2}.$$

Thus, we can conclude that C receives fewer than $\frac{n \cdot (m-1)}{2}$ points in the smaller election.

Now let's place D back on each voter's ballot, at the same location that D was originally located in the larger election. Doing so will cause C's score to increase by 1 for every ballot in which D was originally ranked lower than C. But since C is a Condorcet loser, D must defeat C in a head-to-head contest, which means that D must be ranked lower than C on fewer than half of the ballots. This implies that C gains at most $\frac{n-1}{2}$ points when D is added. So, in the original election, C receives fewer than

$$\begin{aligned} \frac{n \cdot (m - 1)}{2} + \frac{n - 1}{2} &= \frac{n \cdot (m - 1) + n - 1}{2} \\ &= \frac{n \cdot m - 1}{2} \end{aligned}$$

points. This, however, is less than the average number of points for the larger election, which is

$$\frac{1}{m+1} \cdot n \cdot (0+1+2+\cdots+m) = \frac{1}{m+1} \cdot \frac{n \cdot (m+1) \cdot m}{2} = \frac{n \cdot m}{2}.$$

All of this shows that in any election, the Borda count will award a Condorcet loser fewer points than the average number of points received per candidate. This implies that some other candidate will receive more points and thus defeat the Condorcet loser.

Question 3.29. Instant runoff does not satisfy the Condorcet loser criterion, as a Condorcet loser could win in the first round by receiving more first-place votes than two other candidates who had the same number of first-place votes. For instance, in an election with 10 voters and 3 candidates, suppose candidate A receives 4 first-place votes, while candidates B and C each receive 3 first-place votes. Then candidates B and C would be eliminated in the first round of instant runoff voting, even if A was ranked last by all 6 voters who did not rank A first. (In this case, A would be a Condorcet loser.)

Question 3.33. The system used in the 1991 Lousisiana gubernatorial election involved what is known as a *jungle primary*, where all candidates from all parties are placed on the same ballot, and the top two candidates (under plurality) advance to a runoff election, even if they are from the same party. The 1991 election received significant media attention since the runner-up in the primary was Republican David Duke, a former leader of the Ku Klux Klan. He received 32% of the popular vote, compared to 34% for Edwin Edwards, the top Democratic candidate, and 27% for Buddy Roemer, a Republican who placed third in the primary. Another Republican, Clyde Holloway, placed fourth in the primary with 5% of the vote. In the runoff election, Edwards defeated Duke by a significant margin: 61% to 39%. In spite of Edwards' win, it seems likely that Roemer was the Condorcet winner. Since 64% of the electorate voted for a Republican in the primary, Roemer would have likely defeated Edwards, a Democrat, in a head-to-head contest. In order for Roemer to defeat Duke in a head-to-head contest, he would have needed to gain the support of a little over half of the voters who voted for Edwards or Holloway in the primary. Given Duke's controversial background, this level of support for Roemer seems likely. Duke's likely loss to Roemer, combined with his poor showing against Edwards in the general election, makes Duke a fairly clear Condorcet loser.

Question 3.34. French presidential elections are conducted using plurality with a runoff, where the top two candidates in the first round compete head-to-head in a second round to determine the overall winner. In 2002, incumbent Jacques Chirac won the first round with approximately 20% of the vote. Second place went to Jean-Marie Le Pen, an alleged anti-semite who had been tried and convicted by both French and German courts for denying the Holocaust. Chirac was controversial in his own right, in part due to allegations of corruption when he was the mayor of France. Nevertheless, he won the second round of the election in one of the largest landslides in French political history: 82% to 18%. Many of Chirac's votes came somewhat grudgingly; one infamous poster exhorted voters to "vote for a crook, not a Fascist."

Question 3.35. Voting for Olympic host cities is conducted via a sequence of runoff elections, with the candidate receiving the fewest votes eliminated until one candidate receives a majority. This method is very similar to instant runoff except that each round involves a separate ballot, and thus a separate election. Provided that voter preferences remain fixed between rounds of voting, the system behaves identically to and satisfies the same properties as instant runoff.

Question 3.36. The procedure used to select the nominees for the Academy Awards is similar to instant runoff. Each of the approximately 6000 members of the Academy belongs to exactly one category or branch (for instance, the acting branch, the directing branch, etc.). Members of each branch vote for awards corresponding to that branch – that is, actors vote for acting awards, directors for directing awards, and so on. For each award, members provide a preference ranking containing up to five names. A quota is then established for each award by dividing the total number of ballots by the number of desired nominees plus one. (So, for example, if 1000 ballots were submitted for an award with 4 nominees, the quota would be $1000/(4+1) = 200$.) If any candidate reaches this quota based solely on first-place votes, then they are automatically a nominee, and the ballots containing these first-place votes are set aside and not used throughout the rest of the process. Then, the candidate with the fewest first-place votes is eliminated, and the second-place candidate on each of the corresponding ballots is promoted to first-place (as in instant runoff). The process repeats like this, with any candidate reaching quota automatically receiving a nomination, and candidates with the fewest first-place votes being eliminated until all 4 slots have been filled, or there are only 4 candidates left in the running. One additional caveat is that if all of the candidates on a particular ballot are eliminated, then that ballot is declared void for the rest of the process, and the quota is adjusted accordingly. So, if after several rounds of voting, 100 of our 1000 ballots had been declared void, the new quota would be $900/(4 + 1) = 180$.

Question 3.37. The Coombs voting system is like instant runoff, but in each stage of the election, the candidate with the most last-place votes is eliminated. The Coombs system is anonymous and neutral, and it satisfies the Condorcet loser criterion. Like instant runoff, the Coombs system violates both monotonicity and the Condorcet winner criterion. Unlike instant runoff however, the Coombs system also violates the majority criterion, since a very controversial or polarizing candidate could simultaneously receive a majority of first-place votes and a plurality of last-place votes.

Question 3.38. The voting system used on *Survivor* is similar to the Coombs system in that it involves casting a plurality vote for the *least preferred* contestant, with the candidate receiving the most votes being eliminated. One difference between the *Survivor* system and the Coombs system is that each round of voting in the *Survivor* system requires a separate ballot, and thus a separate election. Consequently, voter preferences could theoretically change between rounds of voting. Another key difference is that the voters and the candidates are the same collection of people. Thus, every elimination of a candidate also eliminates a voter.

Supplementary Questions

Question 3.40. Come up with a preference schedule for an election with four candidates in which plurality, the Borda count, sequential pairwise voting (with some agenda you specify), and instant runoff would all yield different outcomes, **and** for which the outcome of one of these four methods would demonstrate a violation of the Condorcet loser criterion.

Question 3.41. For each of the following statements, give a brief argument either for or against the statement. Support your argument with a specific example (for instance, a preference schedule that supports the behavior indicated in the statement).

(a) The Borda count is easier to manipulate than instant runoff voting.

(b) The Borda count is less likely to elect a consensus or compromise candidate than other methods.

(c) In certain situations, it is possible that no matter what voting method is used, and no matter what candidate is elected, some other candidate will be preferred by a substantial majority of the voters.

(d) There are situations in which the Condorcet winner may not be the most socially desirable outcome.

(e) Plurality is more likely than other voting systems to elect a Condorcet loser.

Question 3.42. Go to the web site for the Center for Voting and Democracy (`http://www.fairvote.org`), and search for "instant runoff and monotonicity." You should find a page that argues that the potential for instant runoff to violate monotonicity is not a fatal flaw. Summarize and critique this argument.

Question 3.43. Consider an election with the following preference schedule:

	Number of Voters			
Rank	7	6	5	3
1	A	B	C	D
2	B	A	B	C
3	C	C	A	B
4	D	D	D	A

What would the effect be on the outcome of instant runoff voting if the 3 voters in the rightmost column moved A from last place to first place?

Appendix A: Why Sequential Pairwise Voting Is Monotone, and Instant Runoff Is Not

Suppose that candidate A wins an n-candidate election under sequential pairwise voting with agenda

$$C_1, C_2, \ldots, C_{k-1}, A, C_{k+1}, C_{k+2}, \ldots, C_n,$$

where each C_i denotes one of the other candidates in the election (so that $A = C_k$). Suppose also that, in a new election, some voters make changes to their ballots, but these changes consist only of increasing A's position on certain voters' preference ballots.

Such changes do not affect the relative rankings of candidates other than A on any individual voter's preference ballot. Thus, in the new election, the outcome of the pairwise contests between any two candidates not including A will remain the same. Consequently, A's first opponent in the new election will be the same as in the original election. For convenience, call this first opponent C_a.

Since A defeated C_a in the original election, more than half of the voters originally ranked A higher than C_a on their preference ballots. In the new election, A

will be ranked at least as high on each ballot as in the original election. Thus, A will continue to be ranked higher than C_a on more than half of the ballots, and A will defeat C_a in the new election.

Since A won the original election, A must have originally defeated each of C_{k+1}, C_{k+2}, $\ldots$, C_n. By an argument similar to that in the previous paragraph, we can conclude that A will defeat each of these candidates in the new election as well. Thus, A will remain a winner in the new election, which proves that sequential pairwise voting is monotone.

Note that a similar argument would break down with instant runoff (as we would expect it to, since instant runoff violates monotonicity). With sequential pairwise voting, changes favorable to a particular candidate on individual preference ballots can never affect whether or when other candidates are eliminated. This, however is not true with instant runoff, as we have seen in previous examples. In fact, promoting a candidate, say A, under instant runoff could cause A to receive more first-place votes during a particular stage of the runoff, thereby causing an opponent, say B, to receive fewer first-place votes and thus be eliminated. If B was a weaker candidate in comparison to A than the candidate that would have been eliminated prior to A's promotion, then B's elimination could very well cause A to lose the election in the end. In other words, candidate A can lose an election by having extra first-place votes at a stage of the runoff in which A is not in danger of being eliminated. These extra votes come at the expense of another potentially beatable candidate, who may be prematurely eliminated due to A's premature advancement.

Chapter 4

Trouble in Democracy

Chapter Summary

In Chapter 4, the reader finally arrives at **Arrow's theorem**, the result to which the first three chapters have been building. The chapter begins by considering **Black's system**, a modification of the Borda count that also satisfies the Condorcet winner criterion. Black's system, however, is shown to have a major flaw: it violates the **independence of irrelevant alternatives (IIA)** criterion, which states that the societal preference between any two candidates in an election can depend only on the individual voters' preferences between those two candidates, and not on the voters' preferences between other, potentially irrelevant candidates. Upon further investigation, the reader learns that nearly all of the systems considered in previous chapters also violate IIA, and that the only systems that do seem to satisfy IIA are those that were ruled out long ago (for instance, dictatorships and imposed rule) due to other serious deficiencies.

The *coup de grace* of the chapter is Arrow's theorem, a seminal result in voting theory that asserts the nonexistence of voting systems for elections with more than two candidates that satisfy IIA and four other desirable properties: **universality**, **monotonicity**, **citizen sovereignty**, and **nondictatorship**. Each of these properties is defined precisely, as is Arrow's notion of **voting systems** as functions. Readers investigate the role of transitivity in the definition of voting system and consider examples that illustrate why cyclic societal preferences are problematic. Finally, a precise statement of Arrow's theorem is given, followed by a discussion of how Arrow interpreted his result.

The chapter concludes by investigating a modification of Arrow's theorem that replaces the monotonicity and citizen sovereignty conditions with the **Pareto con-**

dition, also called **unanimity**. Sequential pairwise voting is then used to construct an example of a particularly grievous violation of unanimity.

Typographical Error

Chapter 4 contains a typographical error. The correct name of Arrow's second condition, which is equivalent to monotonicity, is **Positive Association of *Social and* Individual Values**. In the text, the italicized words are omitted.

Learning Objectives

After completing Chapter 4, the reader should be able to ...

- ... define Black's system, and identify which desirable properties are satsified and which are violated by Black's system.
- ... define the independence of irrelevant alternatives (IIA) criterion, and identify which systems satisfy and which systems violate IIA.
- ... state Arrow's definition of a voting system, and discuss the role of transitivity in this definition.
- ... state and define each of Arrow's five conditions, and explain how these conditions are related to the criteria studied in previous chapters.
- ... state Arrow's theorem, and discuss its implications for elections with more than two candidates.
- ... define the Pareto condition, and give examples of systems that satisfy and systems that violate the condition.
- ... state the strong form of Arrow's theorem, and compare and contrast the two different versions of the theorem.

Teaching Notes

This chapter is one of the most important in the book, mainly because it introduces students to Arrow's theorem, which is not only the punchline to the first three chapters, but also the single most significant and well-known result in all of social choice theory. If students take nothing else away from a course on mathematical voting theory, they ought to at least gain an appreciation for the central idea behind Arrow's theorem.

So what is that central idea? It is not that their are no perfect voting systems, although that is arguably true. It is not that there are no good voting systems, because that is arguably false. The best interpretation of Arrow's theorem is simply that certain desirable criteria are incompatible with each other. As a consequence, compromise is always necessary.

It's also important to point out that Arrow's theorem is a *mathematical* result, and it is limited by the model on which it is based. You may want to ask your students not only to critique Arrow's theorem, but also to discuss the strengths and weaknesses of the theoretical framework on which it depends. Is Arrow's definition of a voting system realistic? What are its strengths, and what are its limitations? The fact that much of social choice theory rests on an Arrovian framework does not necessarily imply that such a model is the only way to go. In fact, mathematicians can often gain different and sometimes conflicting insights by exploring non-traditional models. Encourage your students to do so, and to think critically about the observations that follow.

Reading Quiz Questions

1. Fill in the blank: The voting system studied in the opening example of Chapter 4 is named after Duncan **Black**.

2. **True** or false: Black's system satisfies the Condorcet winner criterion.

3. What does the acronym IIA stand for? **Independence of irrelevant alternatives**

4. True or **false**: Black's system satisfies IIA.

5. True or **false**: Instant runoff satisfies IIA.

6. Fill in the blank: The theorem studied in Chapter 4 is named after Kenneth **Arrow**.

7. Which of the following conditions are not part of the definition of a voting system?

    (a) **Includes a mechanism for breaking ties**

    (b) Is a function

   (c) Receives as input a collection of transitive preference ballots
   (d) Produces as output a transitive societal preference order
   (e) All of the above
   (f) None of the above

8. Fill in the blank: Preferences that are not transitive are also called **cyclic**.
9. Which of the following conditions are not part of Arrow's theorem?
   (a) Universality
   (b) **CWC**
   (c) IIA
   (d) Nondictatorship
   (e) All of the above
   (f) None of the above
10. Fill in the blank: The unanimity condition used in the strong form of Arrow's theorem is named after Vilfredo **Pareto**.
11. True or **false**: Sequential pairwise voting satisfies unanimity.
12. Which of Arrow's conditions is most closely related to anonymity? **Nondictatorship**
13. Which of Arrow's conditions is most closely related to neutrality? **Citizen sovereignty**

Questions for Class Discussion

1. Apart from the flawed voting systems we have considered so far, what other options would Americans have in deciding who their political leaders should be?
2. Is IIA a reasonable criterion? Is it as important as the others in Arrow's theorem?
3. Dictatorships and imposed rule both satisfy IIA. Are there ever situations in which these systems would be desirable?

4. Of all the fairness criteria we have considered so far, which do you think is the most important for a voting system to satisfy? Which is the least important?

5. How do you think Black's system compares to the other systems we have studied? What are its positive and negative qualities?

6. Even if we can't find a perfect voting system, could we find one that is "best?" If so, what qualities would define "best?"

7. What is your emotional reaction to Arrow's theorem? Does it seem discouraging to you? Why or why not?

8. What are some negative consequences of cyclic societal preferences? Are there ever circumstances in which a cyclic societal preference order would be reasonable or desirable?

9. Does it ever make sense for individual preferences to be non-transitive? Why or why not?

10. Is it reasonable to restrict voters' choices of preference orders in order to avoid a non-transitive societal preference order? If so, who should decide which preference ballots are permissible?

11. Do you think we will ever see an actual election that demonstrates a violation of unanimity? (Follow-up: If actual elections are unlikely to violate unanimity, why is the condition important?)

Discussion of Selected Questions

Warmup 4.1. Black's system satisfies all of the criteria listed.

Question 4.5. The Borda count does not satisfy IIA. The example from Question 4.2 illustrates this fact, since Black's system simply reverted to the Borda count in that case.

Question 4.6. Plurality does not satisfy IIA. To see this, simply consider the results of the 2000 U.S. Presidential election in Florida. It is likely that most of the Nader voters had preferences of $N \succ G \succ B$. If these voters had swapped the positions of Nader and Gore on their preference ballots, then Gore almost certainly would have won the state (and therefore the entire election). Thus, the societal preference between Gore and Bush would have been reversed in spite of the fact that none of the individual rankings between Gore and Bush had changed.

Question 4.7. With an agenda of D, P, W, Paul would defeat Dale and then Wayne would defeat Paul, leaving Wayne as the overall winner. If, however, the voters in the third column swapped the positions of Paul and Dale, then the overall winner would be Dale. Thus, the societal preference between Wayne and Dale would be reversed, even though no individual voter changed their relative ranking of Wayne and Dale. This example illustrates that sequential pairwise voting does not satisfy IIA.

Question 4.8. With the original preferences, instant runoff would produce a societal preference order of $P \succ K \succ R$. If, however, the voters in the first column swapped the positions of Pam and Rachel, the societal preference would change to $R \succ K \succ P$. What is noteworthy here is the fact that, although no individual voter changed their relative ranking of Karen and Pam, the societal ranking between these two candidates was reversed. This example illustrates that instant runoff does not satisfy IIA.

Question 4.14. Arrow's nondictatorship condition is most closely related to anonymity, although it is a weaker condition. (In other words, a system could satisfy the nondictatorship condition and still not be anonymous.) Citizen sovereignty is most closely related to neutrality, but is again a weaker condition.

Question 4.15. With any number of voters distributed between the two ballots listed, candidate C will always be a Condorcet loser, and thus the societal preference order will be either $A \succ B \succ C$, $B \succ A \succ C$, or $A \approx B \succ C$ (depending on which of the corresponding ballots receives more votes, or in the latter case, if both ballots receive the same number of votes). Even with a third ballot ($C \succ B \succ A$) added, it is still impossible for the societal preference order to be cyclic. To see this, suppose that the preferences of the voters are as shown below, with x, y, and z representing arbitrary whole numbers.

	Number of Voters		
Rank	x	y	z
1	A	B	C
2	B	A	B
3	C	C	A

If the corresponding societal preferences are cyclic, then it must be the case that either $A \succ B \succ C \succ A$ or $A \succ C \succ B \succ A$. In the first case, $B \succ C$ implies that $x + y > z$, while $C \succ A$ implies that $z > x + y$. Since these two inequalities cannot both be true, it follows that a cycle of the form $A \succ B \succ C \succ A$ cannot occur. A similar argument establishes the impossibility of an $A \succ C \succ B \succ A$ cycle as well.

Question 4.16. Many would say that IIA is too strong, but good arguments can be made against some of the other conditions as well. It can be an interesting exercise to have students develop arguments against each of the conditions, including even the ones that seem nonnegotiable, such as nondictatorship.

Question 4.17. Since plurality satisfies universality, positive association of social and individual values, citizen sovereignty, and nondictatorship, by Arrow's theorem it cannot also satisfy IIA. The same argument could not be used for instant runoff, since instant runoff already violates positive association of social and individual values. It would not contradict Arrow's theorem for instant runoff to satisfy IIA, since instant runoff already violates one of Arrow's other conditions.

Question 4.18. As we suggested in Question 4.14, any anonymous voting system must also satisfy Arrow's nondictatorship condition. Likewise, any neutral voting system must satisfy Arrow's citizen sovereignty condition. Since monotonicity is simply another name for positive association of social and individual values, it follows that any voting system that satisfies IIA and is anonymous, neutral, and monotone must automatically satisfy Conditions 2 – 5 in Arrow's theorem. Since it is impossible for all five conditions to be satisfied, any such system must violate universality.

Question 4.21. Candidate D would win the election under sequential pairwise voting with the agenda C, B, A, D. This outcome illustrates a violation of unanimity, since every voter prefers C to D, and yet D wins outright, thus being preferred to C in the resulting societal preference order. This violation is more perverse than those that occur with plurality and instant runoff, since in those systems, it would only be possible for candidate D to be *tied* with candidate C in the resulting societal preference order.

Question 4.22. All of the systems listed are identical, and equivalent to majority rule, for elections with only two candidates.

Question 4.23. Arrow's theorem does not hold for elections with only two candidates, as majority rule (with two candidates) satisfies all of Arrow's conditions.

Question 4.24. The table below summarizes the properties of dictatorship, imposed rule, and minority rule:

	Dictatorship	Imposed Rule	Minority Rule
Anonymity	No	Yes	Yes
Neutrality	Yes	No	Yes
Monotonicity	Yes	Yes	No
Majority criterion	No	No	No
CWC	No	No	No
CLC	No	No	No
IIA	Yes	Yes	No
Universality	Yes	Yes	Yes
Citizen Sovereignty	Yes	No	Yes
Nondictatorship	No	Yes	Yes
Unanimity	Yes	No	No

Question 4.25.

- A **dictatorship** satisfies universality, IIA, and unanimity (but obviously not nondicatorship).
- **Imposed rule** satisfies universality, IIA, and nondictatorship (but not unanimity).
- The **Borda count**, **plurality**, **instant runoff**, and **Black's system** all satisfy universality, and unanimity, and nondictatorship (but not IIA).
- The method suggested in Question 4.15, known as **Condorcet's method**, satisfies IIA, unanimity, and nondictatorship. However, it does not satisfy universality, as voters' allowable preference ballots may need to be severely restricted in order to avoid cyclic societal preferences.

Question 4.30. Most Condorcet completion methods begin by narrowing the set of candidates down to some smaller subset, typically based on the outcomes of the various head-to-head contests. For instance, the Smith/IRV method applies instant runoff to the candidates in the *Smith set*, which is defined to be the smallest non-empty set for which every candidate in the set would defeat every candidate outside the set in head-to-head contests.

Question 4.31. Even with the modified definition of a voting system, Arrow's theorem would still apply. Choosing a set of winning candidates is equivalent to defining a societal preference order in which all of the winning candidates are tied with each other and preferred to all of the losing candidates, who are also tied with each other. For example, in an election with candidates A, B, C, D, and E, selecting A and C as winners would be equivalent to producing a societal preference order of $A \approx C \succ B \approx D \approx E$.

Question 4.32. The current system used to judge Olympic figure skating is the official International Skating Union (ISU) judging system, also known as the Code of Points (CoP) or the New Judging System (NJS). The ISU system has been in place since 2004, and it provides detailed protocols for judging both technical elements and presentation (now called program components). The system mainly works by averaging point scores assigned according to specific rules. In 2008, the number of judges was reduced from 12 to 9, with the highest and lowest scores dropped, as well as the scores of two randomly selected judges. Thus, a skater's overall score is determined by the marks of 5 judges.

Question 4.33. One of the more interesting violations of IIA occurred at the 1995 World Figure Skating Championships, where Michelle Kwan's better-than-expected performance in the free skate reversed the rankings of Surya Bonaly and Nicole Bobek, even though both Bonaly and Bobek had finished skating before Kwan took the ice. Before Kwan skated, Bobek was ranked third and Bonaly second. However, after Kwan skated, these rankings were swapped. Incidentally, Kwan placed fourth and was never a serious contender for a spot in the top three.

Question 4.34. In the PU election, Melissa would win with 3 points. (The societal preference order would be $M(3) \succ G(2) \succ D(1) \succ S(0)$, with the number of points received by each candidate specified in parentheses.) In the CVAAB election, the societal preference order would be $G(2) \approx H(2) \succ F(1) \approx I(1)$.

In general, the number of pairwise comparisons is given by

$$\binom{n}{2} = \frac{n \times (n+1)}{2}.$$

The method of pairwise comparisons, also known as Copeland's method, satisfies anonymity, neutrality, monotonicity, universality, unanimity, citizen sovereignty, nondictatorship, CWC, CLC, and the majority criterion. The only criterion we have studied so far that is not satisfied by Copeland's method is IIA.

Question 4.36. The voting system used for American Idol is similar to instant runoff, with the caveat that voters' preferences may change between rounds of voting. Each week, votes are cast via unique telephone numbers assigned to each contestant. The contestant who receives the fewest number of votes (that is, the fewest phone calls to their number) is eliminated, unless the judges unanimously agree to "save" that contestant, in which case two contestants are eliminated the following week. The question of which fairness criteria are satisfied is dependent on how the system is modeled. Since voters are allowed to vote multiple times, and since voter preferences can change between rounds of voting, some effort is required to fit the American Idol system into the traditional Arrovian definition of a voting system. With that said, however, there are still several conclusions we can make. For instance, it is conceivable that two voters switching ballots could affect the outcome of the vote, especially if one of the voters called in significantly more times than the other. Thus, the system is not anonymous. Likewise, violations of monotonicity can occur just as they do in instant runoff. On the other hand, the system is clearly nondictatorial and non-imposed. The system also satisfies a somewhat weakened version of unanimity; in particular, if every caller prefers contestant A to contestant B, and contestant A receives at least one vote, then contestant B will receive no votes and will eliminated prior to contestant A. (This is, of course, assuming that the judges don't step in and "save" contestant B. Do you see how the fine print here complicates things?)

Question 4.37. The voting system for Last Comic Standing was similar to that used by American Idol, with a few exceptions. First, the initial eliminations (until five comics remained) were decided by a studio audience, with subsequent eliminations decided by phone-in and/or e-mail votes from the television audience. Second, each phone number and/or e-mail address was allowed only a limited number of votes.

Question 4.38. Sen's work dealt with personal freedom, and he believed that everyone ought to be able to be decisive over certain aspects of a social state. For instance, Sen would argue that a voter, say Bob, should be able to decide what city he lives in. In other words, Bob's vote should be decisive between any two social outcomes that differ only on where Bob lives. Sen defined a *minimal liberalism* condition which required the electorate to contain two individuals, each of whom was "decisive" over some pair of alternatives (meaning that the societal preference on that pair was determined by the decider's individual vote). Sen's theorem essentially showed that it is impossible for a voting system to satisfy minimal liberalism, universality, and unanimity.

The Gibbard-Satterthwaite Theorem deals with manipulation of voting systems. Loosely paraphrased, it states that no voting system for three or more candidates can be non-dictatorial, non-imposed, and non-manipulable. In the context of the theorem, a system is said to be manipulable if some voter, having full knowledge of the other voters' preferences, can yield a better outcome for himself by voting insincerely – that is, by submitting a ballot that does not reflect his true preferences.

Supplementary Questions

Question 4.39. In Chapter 3 (Question 3.14), we argued that sequential pairwise voting does not satisfy neutrality. Does sequential pairwise voting satisfy citizen sovereignty? Give a convincing argument or example to justify your answer.

Question 4.40. Find a system that satisfies Arrow's nondictatorship condition but that is not anonymous, or explain why no such system exists.

Question 4.41. Research each of the following voting systems, giving a brief description of the system and describing which fairness criteria are satisfied by the system:

(a) The Kemeny-Young method

(b) The minimax method (also called Simpson's method)

(c) The Schulze method (also called Schwartz sequential dropping)

(d) The ranked pairs method (also called Tideman's method)

What do all of these systems have in common?

Question 4.42. Research each of the following fairness criteria, and relate them to the fairness criteria we have discussed up to this point. Which of the criteria seem most important to you?

(a) The Smith criterion

(b) Reinforcement

(c) Reversal symmetry

(d) Independence of clones

(e) Consistency

(f) Participation

Question 4.43. Consider an election with the following preference ballots:

$A \succ B \succ C \approx D \succ E$	$A \succ E \succ D \succ C \succ B$	$A \approx E \succ B \approx D \succ C$
$A \succ D \approx B$	$A \succ E \succ D \succ C \succ B$	$A \succ C \succ D$
$A \succ B \succ C \succ D \succ E$	$C \approx B \succ E \succ A$	$E \succ A \succ B \succ D \succ C$
$A \approx B \succ D \approx C \succ E$	$A \succ B \approx D \succ E \succ C$	$B \succ A \succ C \succ D \succ E$
$B \succ C \succ D$	$A \succ B$	$A \succ E \succ C \succ B$
$A \approx C \succ D \approx B$	$A \succ B \succ C \succ D \succ E$	$A \succ D \succ B \succ E$

Who do you think should win the election, and what should the resulting societal preference order be? Use a variety of methods and arguments to justify your answer. Note that you will have to adopt some conventions about how to deal with ties and incomplete ballots.

Question 4.44. Consider an election with the following preference ballots:

$E \succ B \succ C \succ D \succ A$	$B \succ A \succ C \succ D \succ E$	$A \succ E \succ D \succ B \succ C$
$D \succ E \succ C \succ B \succ A$	$C \succ D \succ E \succ A \succ B$	$D \succ E \succ A \succ C \succ B$
$A \succ D \succ E \succ B \succ C$	$B \succ A \succ C \succ D \succ E$	$C \succ B \succ D \succ A \succ E$
$E \succ A \succ D \succ B \succ C$	$C \succ B \succ D \succ A \succ E$	$E \succ D \succ A \succ B \succ C$
$A \succ E \succ C \succ D \succ B$	$E \succ A \succ D \succ B \succ C$	$B \succ E \succ A \succ D \succ C$
$E \succ C \succ A \succ D \succ B$	$B \succ E \succ D \succ C \succ A$	$C \succ B \succ D \succ E \succ A$
$A \succ D \succ B \succ E \succ C$	$C \succ B \succ A \succ D \succ E$	$A \succ B \succ C \succ D \succ E$
$B \succ E \succ A \approx C \approx D$	$A \succ E \succ D \succ B \approx C$	

Who do you think should win the election, and what should the resulting societal preference order be? Use a variety of methods and arguments to justify your answer. Note that you will have to adopt some conventions about how to deal with ties and incomplete ballots.

Chapter 5

Explaining the Impossible

Chapter Summary

Chapter 5 focuses on the proof of Arrow's theorem and on several potential resolutions to the theorem. The chapter begins by considering a simple example that illustrates the general proof strategy to be used. The proof of Arrow's theorem then proceeds by showing that in elections with more than two candidates, any voting system that satisfies universality, IIA, and unanimity must be a a dictatorship. The assumptions of universality, IIA, and unanimity are used to identify a **pivotal voter** that turns out to be a **dictator**, thus establishing the strong form of Arrow's theorem. The proof concludes by showing why the strong form of Arrow's theorem implies the weaker version originally stated in Chapter 4.

After the proof of Arrow's theorem, three potential resolutions are considered. The first option is a slight weakening of the Pareto condition. The second option considered is **approval voting**, which readers discover can be modeled as an Arrovian voting system that violates universality, but that satisfies IIA and all of Arrow's other conditions. Finally, as a third potential resolution, readers investigate Saari's **intensity of binary independence** (IBI) condition and its relationship to the Borda count. The Borda count, which violates IIA, is shown to satisfy IBI and all of Arrow's other conditions. The chapter concludes with a summary discussion of Arrow's theorem and its various interpretations.

Error in Question 5.26

In the text, Question 5.26 (and the discussion surrounding it) asserts that there cannot exist a voting system for three or more candidates that satisfies Arrow's conditions 1, 3, and 5, and the modified Pareto condition. This assertion is in error, and the argument outlined in Question 5.26 suffers from a subtle flaw.[1]

First, note that a system in which all candidates tie, regardless of votes, does in fact satisfy Arrow's conditions 1, 3, and 5, and the modified Pareto condition. (Call this system V.) If we apply the construction from Question 5.26 to this example, by breaking ties in favor of the unanimously preferred candidate, we will see that the resulting system (call it V') in fact violates universality. To illustrate, consider the following preference schedule:

	Number of Voters	
Rank	1	1
1	A	C
2	B	A
3	C	B

With system V, the resulting societal preference order would be $A \approx B \approx C$, since by definition all candidates must tie regardless of the votes. However, system V' requires $A \approx C$ (since A is not unanimously preferred to C), $C \approx B$ (since C is not unanimously preferred to B), and $A \succ B$ (since A is unanimously preferred to B). This, however, is a violation of transitivity, since in a transitive order, $A \approx C$ and $C \approx B$ would together imply that $A \approx B$.

Given the invalidity of the argument in Question 5.26 and the demonstrated existence of a voting system that does satisfy Arrow's conditions 1, 3, and 5, and the modified Pareto condition, the question remains: What other systems also satisfy these conditions? This question seems to be open, and a complete characterization of all such systems would be an interesting result.

Learning Objectives

After completing Chapter 5, the reader should be able to ...

- ... give a thorough summary of the main ideas and strategies behind the proof of Arrow's theorem.

[1] Many thanks to Alexander Woo for pointing out this error.

- . . . define the modified Pareto condition, and explain its relation to resolving Arrow's theorem.
- . . . describe approval voting, and explain which of Arrow's conditions are satisfied and which are violated by approval voting.
- . . . define the intensity of binary independence condition (IBI), and give an example of a voting system that satisfies IBI.
- . . . explain how replacing IIA with IBI in the statement of Arrow's theorem provides a resolution to the theorem.

Teaching Notes

Since Arrow first published his now famous theorem, a variety of alternative proofs of the theorem have been discovered.[2] Some of these proofs, like Arrow's original proof, are fairly cryptic. Others are not actually proofs, in that they make simplifying assumptions that, while convenient, take away from the generality of the argument. We think that the proof presented in this chapter strikes a nice balance. It is fully general and mathematically correct, but avoids overly complicated notation and can be understood (with some effort) even by mathematical amateurs. It is one of the best and easiest proofs of Arrow's theorem that we have seen.

With that said, it is important to note that some students (in particular, those with less extensive mathematical backgrounds) will still struggle with the length and depth of the argument. We recommend allowing plenty of class time to deal with these struggles.

If necessary, the proof of Arrow's theorem can be omitted without any loss of continuity. However, we have found that a thorough treatment of the proof is well worth the time (and sometimes frustration) that it entails. Arrow's theorem is, after all, one of the most significant results in all of mathematical economics, and it is empowering for students to be able to understand and even help construct its proof.

If time is of the essence and it is only possible to cover a small portion of Chapter 5, we recommend the section on approval voting. Not only is approval voting quite different from the ranked systems covered in previous chapters, but students are also likely to encounter it at some point in their lives. Approval voting is, in some sense, the simplest improvement to plurality. This simplicity, however, comes at a price – namely, the violation of universality that results from forcing

[2]The question of whether new mathematics is created or discovered is a philosophical one, but we lean toward the latter.

voters to express dichotomous preferences (that is, preferences that include the $\succ$ symbol only once). Students should be invited to discuss the pros and cons of approval voting in light of this conflict.

Reading Quiz Questions

1. Fill in the blank: The modified Pareto condition says that if every voter prefers candidate A to candidate B, then it should not be the case that $\underline{\ \mathbf{B} \succ \mathbf{A}\ }$ in the resulting societal preference order.

2. **True** or false: There exists a voting system that satisfies universality, monotonicity, IIA, and the modified Pareto condition.

3. True or **false**: Because approval voting does not use ranked ballots, Arrow's theorem does not apply to it.

4. Which of the following conditions are not satisfied by approval voting as it was modeled in Chapter 5?

   (a) **Universality**

   (b) Unanimity

   (c) IIA

   (d) Nondictatorship

   (e) All of the above

   (f) None of the above

5. What does the acronym IBI stand for? **Intensity of binary independence**

6. **True** or false: There exists a voting system for elections with more than two candidates that satisfies universality, unanimity, IBI, and is not a dictatorship.

Questions for Class Discussion

1. How does the mathematical proof of Arrow's theorem compare to the way that theorems and laws are established in the sciences? In other words, what is the difference between the notions of "proof" in mathematics and in science?

2. Why can't Arrow's theorem by proved by simply listing all of the possible voting systems and showing that none of them satisfy all of Arrow's conditions?
3. How many different voting systems are there? Are there finitely many?
4. Is IBI a more reasonable condition than IIA? Is IBI as important as IIA?
5. How do you think the use of approval voting would affect the way candidates campaigned for an election?
6. In deciding how many candidates to approve of in an election decided by approval voting, which strategies seem reasonable, and which seem unreasonable? Which strategies would have the greatest impact on the outcome of the election?
7. How do you feel about the violation of universality exhibited by approval voting (the way we have modeled it)? Should this violation cause us to discard approval voting?
8. If you were trying to manipulate an election whose outcome was decided by approval voting, what strategy would you use, and why?
9. Given a choice between violating IIA and allowing the possibility of cyclic societal preferences, which would you choose, and why?
10. Suppose that in a particular election, you don't like any of the candidates? If the election is decided by approval voting, should you choose to not approve of anyone? Why or why not?
11. What are some of the practical advantages and disadvantages of approval voting?

Discussion of Selected Questions

Question 5.2. The version of the theorem stated at the end of Chapter 4 says that, for elections with three or more candidates, it is impossible for a voting system to satisfy the four conditions specified. The version stated here says that if three of the conditions are satisfied, then the fourth cannot be. These two formulations are simply two different ways of saying the same thing.

Question 5.5. Since $A \succsim B$ and $B \succsim C$, transitivity requires $A \succsim C$.

Question 5.7. Unanimity requires that $C \succ A$ in the resulting societal preference order.

Question 5.9. Since B is the last choice of every voter, unanimity requires B to be ranked last in the societal preference order produced by V.

Question 5.12.

(a) In both Table 5.2 and S', each of voters v_1, v_2, $\ldots$, v_{j-1} rank B first, and thus prefer B over A. Likewise, in both Table 5.2 and S', each of voters v_{j+1}, v_{j+2}, $\ldots$, v_n rank B last, thus preferring A over B. By our assumption, A is preferred over C by v_j. Thus, when v_j moves B between A and C to create S', the resulting preference between A and B is $A \succ B$, the same as v_j's preference in Table 5.2 (where v_j ranks B last, and thus $A \succ B$).

(b) As we argued in part (a), the relative rankings of A and B in S' are exactly the same as in Table 5.2. Given the fact that Table 5.2 yields a societal preference of $A \succ B$ (since B is ranked last), IIA requires S' to do the same.

(c) The argument that S' yields a societal preference of $B \succ C$ is nearly identical to that in parts (a) and (b).

(d) Since society prefers A over B and B over C, transitivity requires that $A \succ C$ as well.

Question 5.15. In Table 5.2, where v_j ranks B last, society ranks B last, and thus prefers C to B. But in Table 5.3, where v_j ranks B first, society ranks B first as well, and thus prefers B to C. Consequently, v_j changing his or her ranking of B (from last to first) causes a change in the resulting societal preference between B and C. Because v_j can affect the societal preference between B and C, and v_i completely controls the societal preference between B and C, it must be that v_j and v_i are the same voter.

Question 5.19. Lemma 5.18 implies that any voting system that satisfies the five conditions of the original form of Arrow's theorem must also satisfy the four conditions of the stronger form. But if the strong form of Arrow's theorem is true (which we have now proved), then no such system exists for elections with more than two candidates.

Question 5.21. Citizen sovereignty guarantees the existence of a preference schedule that yields a societal preference of $A \succ B$. If there were no such schedule, then it would always be the case that $B \succsim A$, regardless of what ballots were cast.

Question 5.22.

(a) In S'', A is always ranked at least as high as in S'. Although some voters may prefer B over A in S', this cannot happen in S''.

(b) To go from S' to S'', changes were made that were favorable to candidate A. Since S' produced a societal preference of $A \succ B$, monotonicity implies that S'' will yield the same $A \succ B$ preference.

(c) S and S'' are identical with regard to individual preferences between A and B. In both schedules, every voter prefers A over B.

(d) Since the societal preference produced by S'' is $A \succ B$, and since S and S'' have exactly the same relative rankings of A and B, IIA implies that $A \succ B$ in the societal preference order produced by S.

Question 5.25. If a voting system satisfies the original Pareto condition, then whenever every voter prefers A over B, society must also prefer A over B. But if society prefers A over B, society cannot also prefer B over A. Thus, whenever the original Pareto condition is satisfied, the modified condition must be satisfied as well.

Question 5.26. See the above note regarding the error in Question 5.26.

Question 5.38. There is indeed enough information to determine the societal preference between A and B. Note that if a voter prefers A over B with intensity k, then A will earn $k+1$ more points than B (from that voter) under the Borda count. Thus, for each voter that prefers A to B, we can add one to the intensity of their preference and sum these point differences among all of the voters who prefer A over B. The Borda count will produce a societal ranking of $A \succ B$ if and only if the number we obtain is greater than the corresponding sum for those voters who prefer B over A. Consequently, the only way the societal preference between a pair of candidates can change is if some voter changes either their relative ranking of the candidates or the intensity of their preference. As such, the Borda count does satisfy IBI.

Question 5.41.

(a) Lemma 5.3 and the result from Question 5.10 would still hold in elections with only two candidates. However, we needed a third candidate in order to argue that v_j was a dictator (on pages 84–86).

(b) We used transitivity in the proof of Lemma 5.3 – specifically, in Question 5.5. We also used transitivity in part (d) of Question 5.12.

(c) Throughout the proof, we constructed and modified preference schedules that would allow us to make the conclusions we desired. We never had to worry about whether our voting system could produce a societal preference order from the preference schedule we gave it. In other words, we assumed that as long as the preferences in our schedules were transitive, our voting system would be able to produce a transitive societal preference order from them. This assumption is exactly Arrow's universality condition.

(d) We used IIA in the proof of Lemma 5.3 (specifically, Question 5.6), and in parts (b) and (c) of Question 5.12.

(e) We used unanimity in the proof of Lemma 5.3 (specifically, Question 5.7), and in Question 5.10.

Question 5.42. In order to apply these criteria to Black's system, we would have to decide how a societal preference order would be produced in the event that there was a Condorcet winner. If, in this case, we adopted the same convention as with sequential pairwise voting (where the winner was preferred over all of the other candidates, who were tied with each other), then only the weaker version of unanimity (the modified Pareto condition) would be satisfied. With this convention, universality and IBI would also be satisfied.

Question 5.46. Voting for enshrinement into the National Baseball Hall of Fame is conducted by the Baseball Writers' Association of America (BBWAA). Each year, a screening committee of 6 BBWAA members prepares the ballot, which contains eligible players from previous years' ballots, in addition to newly eligible players who are nominated by at least 2 of the 6 screening committee members. Once the ballot is prepared, all members of the BBWAA who have been active baseball writers for at least 10 years are eligible to vote, and each such member may support up to 10 nominees. In order to be enshrined into the Hall of Fame, a player must obtain votes of support from at least 75% of the voters. Any player who receives at least 5% support is carried over to the next year's ballot, while players with less than 5% support are not eligible for subsequent votes. In addition, any player who remains on the ballot for 15 years but fails to be elected for enshrinement is automatically removed from all future ballots.

Question 5.47. Although certain assumptions need to be made in order to answer this question, it seems likely that, under approval voting, Al Gore would have won in 2000, and Norm Coleman would have won in 1998.

Question 5.48. If one lifts the restriction that the $\succ$ sign must appear only once in individual preferences, then we can model approval voting by assuming that each voter approves of exactly the candidates that he or she ranks in first place (with multiple approved candidates being indicated by a tie for first). With this model, approval voting can be shown to satisfy universality, monotonicity, the modified Pareto condition, citizen sovereignty, and nondictatorship, but not IIA.

Supplementary Questions

Question 5.49. Consider the following preference schedule for an election with 3 candidates:

	Number of Voters					
Rank	1	1	1	1	1	1
1	A	A	B	B	C	C
2	B	C	A	C	A	B
3	C	B	C	A	B	A

Suppose that approval voting is used to decide the outcome of the election, and suppose also that each voter approves of either one or two candidates. (Approving of no candidates or all three candidates is not a good strategic choice, since doing so has no impact on the outcome of the election.) Show that, depending on where each voter draws the line between approved and disapproved candidates, any societal preference order is possible.

Question 5.50. For an election with four candidates, consider a variation of the Borda count in which the points assigned are 5, 3, 1, and 0 (instead of the usual 3, 2, 1, and 0).

(a) Does this system satisfy IBI? Why or why not?

(b) In general, what conditions must be placed on the points associated with each ranking in order for the resulting system to satisfy IBI?

Question 5.51. Does approval voting satisfy CWC? Fully justify your answer.

Chapter 6

One Person, One Vote?

Chapter Summary

In Chapter 6, the focus shifts away from systems used to decide the winner of multiple candidate elections, and instead turns to systems used to make yes/no decisions. In particular, readers investigate systems in which certain voters are given more weight than others.

The chapter begins by considering an example involving the shareholders of a fictitious company. This examples serves to motivate the precise definition of a **weighted voting system**, as well as various concepts related to weighted voting systems, such as winning and losing **coalitions**. A notion of **isomorphism** is introduced to formalize what it means for two weighted voting systems to be essentially the same, and the terms **dictator**, **dummy**, and **veto power** are given precise meanings.

The second half of the chapter is devoted to the properties of **swap robustness** and **trade robustness**. These properties are formally defined and illustrated from within the context of both real-life and fictitious examples. Readers discover and prove that every weighted voting system is swap robust. This result is then used to explain why certain voting systems cannot possibly be weighted. Finally, readers investigate an important theorem that asserts the equivalence of trade robustness and weightedness. This theorem is used to provide an example of a voting system (specifically, the procedure used to amend the Constitution of Canada) that is weighted but not swap robust.

Learning Objectives

After completing Chapter 6, the reader should be able to . . .

- . . . define weighted voting system, and give several examples of weighted voting systems.
- . . . define coalition and its variations (*e.g.*, winning coalitions, losing coalitions), and give examples to illustrate these definitions.
- . . . define dictator, dummy, and veto power, and give examples to illustrate these definitions.
- . . . define what it means for two weighted voting systems to be isomorphic, and apply this definition to a variety of examples.
- . . . define swap robustness and trade robustness, and explain the relationship between these two properties.
- . . . describe in detail the relationship between the properties of swap robustness and weightedness, and use this relationship to prove that certain yes/no voting systems are not weighted.
- . . . describe in detail the relationship between the properties of trade robustness and weightedness, and use this relationship to prove that certain yes/no voting systems are not weighted.

Teaching Notes

This chapter has both a practical and a theoretical component to it, and some students may struggle with the latter. In particular, the difference between necessary and sufficient conditions, as illustrated by Theorems 6.19 and 6.26, may be worth an extended class discussion.

From a practical standpoint, it is important to note that the power held by each voter in a weighted voting system is not necessarily proportional to their weight. This fact is illustrated by several examples throughout the chapter, but Question 6.28 in the Questions for Further Study is a particularly interesting case.

Some students may wonder why we would even be interested in determining if a given voting system is weighted or not. One very practical reason is that it is relatively easy to determine the winner in a weighted voting system – all one must do is add the weights of the voters and see if this sum equals or exceeds the quota. This basic arithmetic problem is typically much simpler than navigating through a complex set of rules that specify the conditions for passage of a motion.

If your class contains mathematics majors or other mathematically inclined students, some may be interested in seeing the full proof of Theorem 6.26. It can be found in an article entitled "A characterization of weighted voting" by Alan Taylor and William Zwicker (*Proceedings of the American Mathematical Society*, 115:1089-1094, 1992).

Reading Quiz Questions

1. What was the name of the Vice President of Marketing from the opening example of Chapter 6? **Deanne Boomhauer**
2. Which one of the following is not a part of the definition of a weighted voting system?
   (a) A collection of voters
   (b) A collection of weights
   (c) A quota
   (d) **A collection of winning coalitions**
3. **True** or false: In a weighted voting system, a winning coalition is a coalition whose weight equals or exceeds the quota for the system.
4. **True** or false: Two different weighted voting systems can have the exact same winning coalitions.
5. When two weighted voting systems are essentially the same, they are said to be:
   (a) Homeomorphic
   (b) Isometric
   (c) **Isomorphic**
   (d) Diffeomorphic
6. True or **false**: A dummy can have veto power.
7. **True** or false: The system used to make decisions in the U.N. Security Council is a weighted voting system.

8. Which of the following are not states in the mystical land of Psykozia:

   (a) Ignorance

   (b) Confusion

   (c) Bliss

   (d) Disarray

   (e) All of the above

   (f) **None of the above**

9. True or **false**: The system used in the mystical land of Psykozia is a weighted voting system.

10. Fill in the blank: A one-for-one exchange of a single voter from each of two different coalitions is called a **swap**.

11. **True** or false: The system used to make decisions in the U.N. Security Council is swap robust.

12. True or **false**: The system used in the mystical land of Psykozia is swap robust.

13. **True** or false: If a yes/no voting system is weighted, then it must be swap robust.

14. True or **false**: If a yes/no voting system is swap robust, then it must be weighted.

15. **True** or false: Every swap is a trade.

16. True or **false**: Every trade is a swap.

17. True or **false**: If a yes/no voting system is swap robust, then it must be trade robust.

18. **True** or false: If a yes/no voting system is trade robust, then it must be swap robust.

19. **True** or false: If a yes/no voting system is weighted, then it must be trade robust.

20. **True** or false: If a yes/no voting system is trade robust, then it must be weighted.

21. True or **false**: The procedure used to amend the Constitution of Canada is weighted.

22. **True** or false: The procedure used to amend the Constitution of Canada is swap robust.

23. True or **false**: The procedure used to amend the Constitution of Canada is trade robust.

Questions for Class Discussion

1. Apart from the corporate world, can you think of any examples where a weighted voting system would be appropriate or desirable?
2. Would there ever be a reason for a dummy in a weighted voting system to bother voting? In other words, if a voter has no chance of influencing the outcome, is there any reason for them to vote?
3. In real life, how are the definitions of dictator, dummy, and veto power similar to and/or different from the definitions we used in this chapter?
4. Do you think the results of political elections would be different if each voter's ballot was weighted by how much they knew about the candidates, as measured by a pre-vote questionnaire? If so, how? What would be some other ways of weighting voter ballots, and what effect would these weightings have on voter behavior and election outcomes?
5. Would it ever make sense for a weighted voting system to have both positive and negative weights? If so, give a realistic example.
6. Consider a voting system in which some voters have positive weights, some voters have negative weights, and a motion passes if and only if the sum of the weights of the supporters is close to zero (that is, within some specified tolerance). Can you think of a situation in which such a voting system would be useful?
7. Which of the fairness criteria from Chapters 1 – 5 could be applied to weighted voting systems? What are some other criteria that might be useful to consider?

Discussion of Selected Questions

Question 6.3. In this example, the voters would be Doug, Nicholas, and Elisabeth, with weights of 101, 97, and 2, respectively. A reasonable choice for the quota would be one more than half of the sum of these weights, or 101.

Question 6.4. With a quota of 101, the motion would pass if and only if Doug voted for it, regardless of the votes of Nicholas and Elisabeth. With a quota of 103, Doug and one other voter (either Nicholas or Elisabeth) would have to vote in favor of the motion in order for it to pass. With a quota of 105, the only way the motion could pass is if both Doug and Nicholas voted for it. In this last case, Elisabeth's vote would be completely irrelevant.

Question 6.6.

(a) With a quota of 101, the winning coalitions would be {Doug}, {Doug, Nicholas}, {Doug, Elisabeth}, and {Doug, Nicholas, Elisabeth}. Of these, only {Doug} would be minimal. The losing coalitions would be {Nicholas}, {Elisabeth}, {Nicholas, Elisabeth}, and the empty coalition.

(b) With a quota of 103, {Doug} would go from winning to losing, and {Doug, Nicholas} and {Doug, Elisabeth} would both become minimal.

(c) With a quota of 105, the only winning coalitions would be {Doug, Nicholas} and {Doug, Nicholas, Elisabeth}. Only the former would be minimal, and all of the remaining 6 coalitions would be losing.

Question 6.7. With a quota of 150, the winning and minimal winning coalitions would be the same as with a quota of 105.

Question 6.11. Doug is a dictator with a quota of 101, and has veto power with a quota of 103 or 105. Nicholas has veto power with a quota of 105 (but not 101 or 103), and Elisabeth is a dummy with a quota of 101 or 105 (but not 103).

Question 6.12. Every dictator has veto power, but it is possible for a voter to have veto power and not be a dictator (for instance, Nicholas in part (b) of Question 6.4). A weighted voting system can have more than one voter with veto power, but can never have more than one dictator.

Question 6.13. The U.N. Security Council can be viewed as a weighted voting system by assigning each nonpermanent member a weight of 1, each permanent member a weight of 7, and a quota of 39. Each permanent member would have veto power, but there are no dictators or dummies.

Question 6.14. There are 23 different types of winning coalitions, 3 of which are minimal (indicated by an asterisk):

- 2 senators, 3 representatives, the president, and the vice president (*)
- 2 senators, 4 representatives, the president, and the vice president

- 2 senators, 5 representatives, the president, and the vice president
- 3 senators, 3 representatives, and the president (*)
- 3 senators, 3 representatives, the president, and the vice president
- 3 senators and 4 representatives (*)
- 3 senators, 4 representatives, and the president
- 3 senators, 4 representatives, and the vice president
- 3 senators, 4 representatives, the president, and the vice president
- 3 senators and 5 representatives
- 3 senators, 5 representatives, and the president
- 3 senators, 5 representatives, and the vice president
- 3 senators, 5 representatives, the president, and the vice president
- 4 senators, 3 representatives, and the president
- 4 senators, 3 representatives, the president, and the vice president
- 4 senators and 4 representatives
- 4 senators, 4 representatives, and the president
- 4 senators, 4 representatives, and the vice president
- 4 senators, 4 representatives, the president, and the vice president
- 4 senators and 5 representatives
- 4 senators, 5 representatives, and the president
- 4 senators, 5 representatives, and the vice president
- 4 senators, 5 representatives, the president, and the vice president

There are no dictators, dummies, or voters with veto power in Psykozia's federal system.

Questions 6.16 and 6.17. The voting system from part (d) is swap robust but not weighted. To see this, note that if the system was weighted, then any coalition containing Nicholas would have to be winning, since the coalition containing only Nicholas is winning. This, however, is not the case. The system from part (c) fails to be weighted for the same reason.

Question 6.22. Psykozia's federal system is not swap robust, and thus not weighted. Consider the following two winning coalitions, where S_i denotes Senator i, R_i denotes Representative i, and P denotes president:

$$C_1 = \{S_1, S_2, S_3, R_1, R_2, R_3, P\}$$
$$C_2 = \{S_2, S_3, S_4, R_3, R_4, R_5, P\}$$

Now swap S_1 in C_1 for R_5 in C_2. The resulting coalitions are the following, neither of which are winning:

$$C_1^* = \{S_2, S_3, R_1, R_2, R_3, R_5, P\}$$
$$C_2^* = \{S_1, S_2, S_3, S_4, R_3, R_4, P\}$$

Question 6.25. The argument here is similar to that in Question 6.18. Before the trade, the weights of C_1, C_2, ..., C_n all exceed the quota. After the trade, the sum of the weights must be the same as before, which means that at least one of C_1, C_2, ..., C_n must have a weight that exceeds the quota. Thus, every weighted voting system is trade robust.

Question 6.27. The minimal winning coalitions in this system fall into the following categories:

- Those that include Ontario and any six of the remaining provinces.
- Those that do not include Ontario, but do include Quebec, British Columbia and:
  - Alberta and any four of the remaining provinces
  - Manitoba, Saskatchewan, Nova Scotia, and two of Newfoundland, New Brunswick, and Prince Edward Island.

The voting system is swap robust. To see this, suppose C_1 and C_2 are both winning coalitions. Then both C_1 and C_2 contain at least 7 provinces and at least half of the total population. After any swap, both coalitions will still contain at least 7 provinces. Furthermore, by the same reasoning as in Question 6.18, at least one of C_1 or C_2 will contain at least half of the total population. Thus, at least one of C_1 or C_2 will remain winning.

Although the system to amend the Constitution of Canada is swap robust, it is not trade robust. Consider the following two winning coalitions:

$C_1 = \{$ON, BC, AB, MB, SK, NS, NL$\}$ (7 provinces, 73%)
$C_2 = \{$QC, BC, MB, SK, NS, NB, PE$\}$ (7 provinces, 50%)

Now trade Ontario in C_1 for New Brunswick and Prince Edward Island in C_2. The resulting coalitions are:

$C_1^* = \{$BC, AB, MB, SK, NS, NL, NB, PE$\}$ (8 provinces, 38%)
$C_2^* = \{$ON, QC, BC, MB, SK, NS$\}$ (6 provinces, 85%)

Neither of these coalitions are winning, and so the system is not trade robust.

Question 6.28. In the new system, the weights would be 100 (Doug), 98 (Nicholas), and 3 (Elisabeth). With a quota of 101, any two of the three voters would form a minimal winning coalition. Thus, all would have the exact same amount of power. This is in contrast to the original system, where Doug was a dictator, and Nicholas and Elisabeth were both dummies.

Question 6.29. Isomorphism is a transitive property. If V_1 and V_2 have the same winning coalitions, and V_2 and V_3 have the same winning coalitions, then V_1 and V_3 must have the same winning coalitions.

Question 6.30. There are many properties that any pair of isomorphic voting systems would necessarily have in common. The presence or absence of a dictator, the number of voters with veto power, and the number of dummies are three examples.

Question 6.31. If a yes/no voting system has exactly one minimal winning coalition, and that coalition contains only a single voter, then the unique member of the unique minimal winning coalition is called the dictator.

Question 6.32. If a yes/no voting system has a dictator, then every voter who is not a dictator is necessarily a dummy. However, a yes/no voting system can have dummies without having a dictator. For instance, in the example from Warmup 6.1, with a quota of 105, Elisabeth is a dummy, but neither Doug nor Nicholas is a dictator.

Question 6.33. A yes/no voting system is monotone if adding voters to a winning coalition (without removing any of the original voters) does not cause the coalition to become losing. (Similarly, removing voters from a losing coalition should not cause the coalition to become winning.) With nonnegative weights, every weighted voting system is monotone. However, if negative weights are allowed, then it is possible for a weighted voting system to be non-monotone. The procedure to amend the Constitution of Canada is an example of a system that is monotone and swap robust, but not weighted.

Questions 6.35 and 6.36. In the U.S. federal system, a bill must be passed by a majority of the House of Representatives (218 out of 435) and a majority of the Senate (51 out of 100) before being sent to the president for his signature. In the case of a tie in the Senate (50 out of 100), the vice president casts the tie-breaking vote. If the president signs the bill, it becomes law. If, however, the president vetoes the bill, it is sent back to Congress, where it must receive a two-thirds majority in each chamber (290 votes in the House and 67 in the Senate) in order to override the presidential veto and become law. The U.S. federal system is not swap robust, and is thus not a weighted voting system. An argument similar to that in Question 6.22 will suffice to establish this fact. Note that because Congress can override a presidential veto, the president does not truly have veto power in the sense that we have defined it.

Question 6.37. Such an argument can simply mirror that of Question 6.18, but applied to the particular example of Psykozia's federal system. For instance, suppose that each senator is given a weight of s, each representative a weight of r, the president a weight of p, and the vice president a weight of v. Suppose that a quota q can be established that makes the system weighted. Then, using the coalitions from the solution to Question 6.18 as a guide, we obtain:

$$\begin{aligned} 3s + 3r + p &\geq q \\ 2s + 4r + p &< q \\ 4s + 2r + p &< q \end{aligned}$$

Multiplying both sides of the first inequality by 2 yields

$$6s + 6r + 2p \geq 2q.$$

But adding together the second and third inequalities yields

$$6s + 6r + 2p < 2q,$$

a contradiction. Thus, the system cannot be weighted.

Question 6.38. The system used by Fresno's city council can be viewed as a $[5 : 2, 1, 1, 1, 1, 1, 1]$ weighted voting system. There are no dictators, dummies, or voters with veto power.

Question 6.39. The system used in Australia can be viewed as a $[5 : 3, 1, 1, 1, 1, 1, 1]$ weighted voting system. There are no dictators, dummies, or voters with veto power.

Question 6.40. In the Council of the European Union, each of the 27 member states is assigned a weight, with a majority of the total weight (currently 173 out of 345) required to pass a motion. Current weights are as follows:

- 29 votes: France, Germany, Italy, and the United Kingdom
- 27 votes: Poland and Spain
- 14 votes: Romania
- 13 votes: The Netherlands
- 12 votes: Belgium, the Czech Republic, Greece, Hungary, and Portugal
- 10 votes: Austria, Bulgaria, and Sweden
- 7 votes: Denmark, Finland, Ireland, Lithuania, and Slovakia
- 4 votes: Cyprus, Estonia, Latvia, Luxembourg, and Slovenia
- 3 votes: Malta

Supplementary Questions

Question 6.41. Craig and Alaine, a young couple, are adjusting to life with their newborn son, Addison. In particular, they have realized that going places and participating in various activities is not as easy as it used to be. If Craig and Alaine want to go somewhere without hiring a babysitter, then Addison must in some sense approve of the trip. For instance, if Addison is sleeping or ill or otherwise unable to leave the house, then at least one of Craig or Alaine must stay home with him. Of course, Addison is also incapable of leaving home by himself. Thus, Addison must have the approval (or, perhaps more appropriately, cooperation) of at least one of Craig or Alaine in order to go to his favorite toy store.

Consider this family's process of deciding whether or not to go somewhere as a yes-no voting system, with each member of the family being one of the voters.

(a) List all of the winning coalitions in this system.

(b) Are there any dictators, dummies, or voters with veto power in this system?

(c) Is this system weighted? Why or why not?

Question 6.42. Fred, Wilma, Pebbles, and Bam-Bam are trying to decide whether they should stay in and order pizza for dinner or go out to the Dino Rock cafe. (In case you're wondering, Barney and Betty left town for a romantic weekend getaway.) To decide this issue, they use a voting system that has the following winning coalitions:

$$\{F, W\}, \{W, P\}, \{P, B\}, \{F, B\}, \{F, W, P\},$$
$$\{F, W, B\}, \{W, P, B\}, \{F, P, B\}, \{F, W, P, B\}$$

(a) Is this voting system swap robust? Why or why not?

(b) What does your answer to part (a) allow you to conclude about whether the system is weighted or not?

(c) Suppose that, in addition to the winning coalitions specified earlier, {W, B} and {F, P} were also winning coalitions. Would the system then be swap robust?

(d) With the addition of these two winning coalitions, would your answer to part (c) allow you to conclude that the system is weighted?

(e) Is the revised system from part (c) weighted? If so, find weights and a quota for it. Otherwise, explain why it cannot be weighted.

Chapter 7

Calculating Corruption

Chapter Summary

Chapter 7 continues previous investigations of weighted voting by exploring ways to quantify the distribution of power within yes/no voting systems. The first **power index** considered is the **Banzhaf index**. Its calculation is based on the number of times voters are **critical**, or essential, to a coalition's winning status. The **Shapley-Shubik index** assumes that voters cast their ballots in a certain order, and that power should be attributed to **pivotal voters**, *i.e.*, those voters who first cause a coalition to become winning.

Readers calculate both the Banzhaf and the Shapley-Shubik indices for a variety of examples. These calculations introduce concepts and techniques from the field of **combinatorics** such as **binomial coefficients**, **permutations**, and **Pascal's triangle**.

Learning Objectives

After completing Chapter 7, the reader should be able to ...

- ... define critical and pivotal voters, and explain the distinction between the two.
- ... define the Banzhaf power index, and use it to calculate the distribution of power in a variety of yes/no voting systems.
- ... define the Shapley-Shubik power index, and use it to calculate the distribution of power in a variety of yes/no voting systems.

- . . . understand and apply basic concepts and techniques from combinatorics, including permutations, binomial coefficients, and Pascal's triangle.

Teaching Notes

This chapter is the first in the book that involves any kind of involved numerical calculations. Some students will enjoy this challenge, and others will not. Depending on the makeup of the class, it may make sense to omit certain sections, such as those on the Banzhaf and Shapley-Shubik indices for the Psykozia voting system. On the other hand, these more involved calculations do provide a way to introduce some interesting material on combinatorics. The text takes a very intuitive approach to binomial coefficients and Pascal's triangle, which allows students to gain an appreciation of some of the basic ideas behind the study of combinatorics without getting bogged down in tedious algebraic details.

Several online applets are available for calculating Banzhaf and Shapley-Shubik indices, and these can be useful for experimentation once students have mastered the basic ideas behind the calculations. One that is reasonably easy to use is available at the following link:

`http://www.math.temple.edu/ conrad/Power/powerindex.html`.

There are also some interesting theoretical questions pertaining to power indices, and in particular the ability to reverse engineer a voting system to have a particular, predetermined distribution of power. A well-written article related to this problem is Tolle's "Power distribution in four-player weighted voting systems," which appears in *Mathematics Magazine* (Vol. 76, No. 1., Feb. 2003, pp. 33–39.).

Reading Quiz Questions

1. Fill in the blank: The first power index considered in Chapter 7 is named after attorney John F. __**Banzhaf**__.

2. Fill in the blanks: The second power index considered in Chapter 7 is named after economists __**Shapley**__ and __**Shubik**__.

3. Which power index from Chapter 7 uses critical voters in its calculation? **Banzhaf**

4. Which power index from Chapter 7 uses pivotal voters in its calculation? **Shapley-Shubik**
5. Which power index from Chapter 7 considers ordered arrangements of voters? **Shapley-Shubik**
6. What f-word is used to describe the quantity $n \times (n-1) \times \cdots \times 2 \times 1$? **Factorial**
7. The branch of mathematics that deals primarily with counting is called:
   (a) Itemology
   (b) **Combinatorics**
   (c) Contingutorics
   (d) Combinatology
   (e) Permutography
   (f) All of the above
   (g) None of the above
8. Write the expression "5 choose 3" in symbolic form. $\boldsymbol{\binom{5}{3}}$
9. Fill in the blank: The famous triangle studied in today's reading was named after 17th-century mathematician Blaise **Pascal**.
10. For which power index or indices would one likely use quantities of the form "n choose k"?
    (a) Banzhaf
    (b) Shapley-Shubik
    (c) Neither
    (d) **Both**
11. For which power index or indices would one likely use quantities of the form $n \times (n-1) \times \cdots \times 2 \times 1$?
    (a) Banzhaf
    (b) **Shapley-Shubik**
    (c) Neither
    (d) Both

Questions for Class Discussion

1. Can you think of a real-world voting system in which it would make the most sense to measure power using critical voters? How about one in which pivotal voters would make more sense?

2. Which power index, Banzhaf or Shapley-Shubik, do you think most accurately represents the distribution of power in a yes-no voting system? Does your answer to this question depend on the context of the decisions being made or the system being used?

3. Can you think of examples of critical and/or pivotal voters in recent elections or electoral systems?

4. Would the concept of a pivotal voter (and consquently, the Shapley-Shubik index) ever make sense for voting systems in which all votes are are cast simultaneously? Why or why not?

5. How do you think that the original weights used by the Nassau County Board of Supervisors were chosen? Do these weights make sense in any intuitive way?

6. Apart from the Banzhaf and Shapley-Shubik indices, can you think of any other ways of measuring power in a yes-no voting system?

7. John Banzhaf was able to successfully communicate the mathematical ideas behind his power index to a non-mathematical audience. Have you ever had to do something similar? If you had to explain a result like Arrow's theorem to someone with no formal mathematical training, how would you do it?

Discussion of Selected Questions

Question 7.6. The Banzhaf index of a dictator will always be 1, since a dictator is the unique critical voter in every winning coalition. The Banzhaf index of a dummy will always be 0, since a dummy is never critical in a winning coalition. A voter with veto power is critical in every winning coalition, and thus must have a Banzhaf index that is at least as large as the index of any other voter in the system. For a system with n voters, this implies that the Banzhaf index of a voter with veto power must be at least $1/n$, although it could be higher.

Question 7.12. By arguments similar to those used in Question 7.6, the Shapley-Shubik indices of a dictator and a dummy must be 1 and 0, respectively. The

Shapley-Shubik index of a voter with veto power will again be at least as large as the indices of the other voters in the system, and thus at least $1/n$ in a system with n voters. To see this, let V be a voter with veto power, let v be any other voter. Suppose that in some ordered arrangement, v is pivotal. Then since V has veto power, V must precede v in the arrangement. Thus, moving V to the position immediately following v creates a new ordered arrangement in which V is pivotal. Thus, for every arrangement in which v is pivotal, there is another arrangement in which V is pivotal. In other words, a voter with veto power is pivotal at least as often as any other voter.

Questions 7.16 – 7.18. To calculate the Banzhaf index of each voter in the system, we can begin by identifying which voters are critical in each type of winning coalition. We have done so in the following list by underlining the critical voters. We have also indicated in parentheses the number of coalitions in each category.

- 2 senators, 3 representatives, the president, and the vice president (60)
- 2 senators, 4 representatives, the president, and the vice president (30)
- 2 senators, 5 representatives, the president, and the vice president (6)
- 3 senators, 3 representatives, and the president (40)
- 3 senators, 3 representatives, the president, and the vice president (40)
- 3 senators and 4 representatives (20)
- 3 senators, 4 representatives, and the president (20)
- 3 senators, 4 representatives, and the vice president (20)
- 3 senators, 4 representatives, the president, and the vice president (20)
- 3 senators and 5 representatives (4)
- 3 senators, 5 representatives, and the president (4)
- 3 senators, 5 representatives, and the vice president (4)
- 3 senators, 5 representatives, the president, and the vice president (4)
- 4 senators, 3 representatives, and the president (10)
- 4 senators, 3 representatives, the president, and the vice president (10)
- 4 senators and 4 representatives (5)

- 4 senators, 4 representatives, and the president (5)
- 4 senators, 4 representatives, and the vice president (5)
- 4 senators, 4 representatives, the president, and the vice president (5)
- 4 senators and 5 representatives (1)
- 4 senators, 5 representatives, and the president (1)
- 4 senators, 5 representatives, and the vice president (1)
- 4 senators, 5 representatives, the president, and the vice president (1)

To calculate the Banzhaf power of the president and vice president, we can simply add up the number of coalitions in which each is critical. This yields a Banzhaf power of $60 + 30 + 6 + 40 + 40 + 10 + 10 = 196$ for the president and $60+30+6 = 96$ for the vice president. To find the Banzhaf power of an individual senator or representative, we need to be a bit more careful. To illustrate, let S be a senator. Then S will belong to half of the coalitions that contain 2 senators, and three fourths of the coaltions that contain 3 senators. Thus, S is critical in exactly

$$\tfrac{1}{2}(60 + 30 + 6) + \tfrac{3}{4}(40 + 20 + 20 + 20 + 4 + 4 + 4) = 132$$

coalitions. Likewise, an individual representative, say R, belongs to three fifths of the coalitions containing 3 representatives, and four fifths of the coalitions containing 4 representatives. Thus, R is critical in exactly

$$\tfrac{3}{5}(60 + 40 + 40 + 10 + 10) + \tfrac{3}{5}(20 + 20 + 5 + 5) = 136$$

coalitions. Thus, the Banzhaf power of an individual senator is 132, and the Banzhaf power of an individual representative is 136. There are 4 senators, 5 representatives, 1 president, and 1 vice president. Thus, the total Banzhaf power in the system is

$$4 \cdot 132 + 5 \cdot 136 + 1 \cdot 196 + 1 \cdot 96 = 1500.$$

Thus, the Banzhaf indices are:

- $\frac{132}{1500} = 8.8\%$ for each senator
- $\frac{136}{1500} \approx 9.1\%$ for each representative
- $\frac{196}{1500} \approx 13.1\%$ for the president
- $\frac{96}{1500} = 6.4\%$ for the vice president

Question 7.24. Each row of Pascal's Triangle begins and ends with 1. This is due to the fact that $\binom{n}{0} = \binom{n}{n} = 1$ for all n. The other numbers in each row Pascal's Triangle can be found by adding together the two numbers in the previous row that are to the immediate left and right of the entry being calculated.

Questions 7.26 – 7.28. The cases used in Questions 7.16 – 7.18 can be used as a starting point for calculating the Shapley-Shubik index of each voter. Note that the pivotal voter in any ordered arrangement of voters will be critical in a coalition consisting of that voter and all of the voters preceding it in the arrangement. Thus, each coalition in which a voter is critical can be used to construct a set of ordered arrangements in which the voter will be pivotal. To illustrate, consider the first class of winning coalitions listed in the solution for Questions 7.16 – 7.18:

- 2 senators, 3 representatives, the president, and the vice president

An individual senator, say S, will be pivotal in any arrangement in which S is preceded by the other six voters in the coalition and followed by the remaining four voters that do not belong to the coalition. There are $\binom{4}{1} \cdot \binom{5}{3}$ ways to choose the one senator and three representatives that will precede S. (The remaining senators and representatives will follow S.) The six voters preceding S can be ordered in $6!$ ways, and the four voters following S can be ordered in $4!$ ways, which yields

$$\binom{4}{1} \cdot \binom{5}{3} \cdot 6! \cdot 4! = 691,200$$

arrangements (from this class of winning coalitions) in which S is pivotal. There are nine other classes of winning coalitions in which S is critical, and each of these yields a collection of ordered arrangements in which S is also pivotal. Similar calculations can be performed for each of the remaining voters in the system.

Question 7.29. The Banzhaf and Shapley-Shubik indices are often quite similar, but they can disagree as well. For instance, Taylor (reference [47] in the text) includes calculations of the Banzhaf and Shapley-Shubik indices for the full U.S. Federal System. In this system, the Banzhaf index gives the president approximately 4% of the total power, whereas the Shapley-Shubik index gives the president approximately 16% of the total power.

Question 7.31. A dictator is the unique critical voter in every winning coalition. A dummy is a voter who is never critical. A voter with veto power is critical in every winning coalition, but may not be the only critical voter.

Question 7.32. Both the Banzhaf and the Shapley-Shubik indices always sum to 1.

Question 7.33. In both of these examples, the Banzhaf and Shapley-Shubik indices are identical. With a quota of 101, Doug's index is 1, and Nicholas and Elisabeth's indices are both 0. With a quota of 105, Doug and Nicholas's indices are both $\frac{1}{2}$, and Elisabeth's index is still 0.

Question 7.34. The Banzhaf indices are approximately 0.308, 0.231, 0.231, 0.154, and 0.077. In the original system, the indices were 0.289, 0.250, 0.212, 0.173, 0.058, and 0.019. Thus, if the two smallest districts agree to always vote together, their total power will remain unchanged, but the first and third districts will become more powerful at the expense of the second and fourth districts.

Question 7.35. The Shapley-Shubik indices are 0.300, 0.250, 0.217, 0.167, 0.050, and 0.017.

Question 7.38. Binomial coefficients are named after their role in the expansion of binomial expressions. The Binomial Theorem states that if x and y are real numbers and n is a positive integer, then

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^{n-1}y + y^n.$$

In other words, the coefficients of the terms in the expansion of $(x+y)^n$ are exactly the binomial coefficients, which correspond to the entries of the appropriate row of Pascal's Triangle. For example,

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Question 7.39. Let's first observe that

$$\begin{aligned}\frac{n!}{(n-k)!} &= \frac{n \times (n-1) \times \cdots \times (n-k+1) \times (n-k) \times \cdots \times 2 \times 1}{(n-k) \times (n-k-1) \times \cdots \times 2 \times 1} \\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-k+1),\end{aligned}$$

which is exactly the quantity that one would calculate to find the number of ordered arrangements (permutations) of a set of k objects chosen from a set of n objects. For each choice of k objects, there are $k!$ different arrangements, so the total number of ways to choose k objects from a set of n objects is

$$\frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)}{k!} = \frac{n!}{k! \times (n-k)!}.$$

Question 7.40. Both sides of the equation calculate the number of subsets of a set containing n elements. The left side does so by adding together the number of subsets with 0 elements, the number with 1 element, the number with 2 elements, and so on. The right side views each subset as a sequence of choices: Should we include element 1? Should we include element 2? And so on... Since there are n objects, each of which may or may not be included in a given subset, there are

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ factors}} = 2^n$$

possible subsets.

Question 7.41. The Banzhaf index of each of the permanent members is 0.167. Each of the non-permanent members have a Banzhaf index of 0.0165. The Shapley-Shubik indices of the permanent and non-permanent members are 0.196 and 0.0019, respectively.

Question 7.42. There are two ways to approach this problem: through an extensive and often tedious case analysis, or by writing a computer program. The latter yields the following results:

Province	% of Population	Banzhaf Power	Banzhaf Index
Prince Edward Island	1%	68	0.090
New Brunswick	2%	68	0.090
Newfoundland	2%	68	0.090
Nova Scotia	3%	70	0.093
Saskatchewan	3%	70	0.093
Manitoba	4%	70	0.093
Alberta	10%	74	0.098
British Columbia	13%	82	0.108
Quebec	24%	82	0.108
Ontario	38%	104	0.138

Question 7.43. The Banzhaf indices are 0.28 and 0.12, and the Shapley-Shubik indices are 0.286 and 0.119.

Question 7.44. The Banzhaf indices are 0.455 and 0.091, and the Shapley-Shubik indices are 0.429 and 0.095.

Question 7.46. The original European Economic Community consisted of six member nations: France, Germany, and Italy (each having 4 votes), The Netherlands and Belgium (each having 2 votes), and Luxembourg (with 1 vote). Passage of a motion required 12 votes. The Banzhaf indices of this system are 0.238 (France, Germany, Italy), 0.143 (Belgium, The Netherlands), and 0 (Luxembourg). The Shapley-Shubik indices are 0.233, 0.150 and 0. Notice that in each case, Luxembourg is a dummy.

Questions 7.47 and 7.48. See Taylor (reference [47] in the text) for information regarding these questions.

Supplementary Questions

Question 7.49. Calculate the Banzhaf and Shapley-Shubik indices of the system from Question 6.41 (in Chapter 6 of this manual).

Question 7.50. Calculate the Banzhaf and Shapley-Shubik indices of the system from Question 6.42 (in Chapter 6 of this manual).

Question 7.51. Consider a weighted voting system with three voters, A, B, and C. For each of the following distributions of power, either (i) find weights and a quota for which the Shapley-Shubik index would yield the given distribution; or (ii) explain why it is impossible to find weights and a quota that would produce the given distribution.

(a) $(1, 0, 0)$

(b) $(\frac{5}{6}, \frac{1}{6}, 0)$

(c) $(\frac{4}{6}, \frac{2}{6}, 0)$

(d) $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$

(e) $(\frac{3}{6}, \frac{3}{6}, 0)$

(f) $(\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$

(g) $(\frac{2}{6}, \frac{2}{6}, \frac{2}{6})$

Question 7.52. Using the Shapley-Shubik index to measure power, find a weighted voting system with four voters (call them A, B, C, and D) for which:

- A is twice as powerful as B;
- B is twice as powerful as C; and
- C and D have the same amount of power.

Question 7.53. Show that the Banzhaf and Shapley-Shubik indices are always identical when there are only two voters.

Chapter 8

The Ultimate College Experience

Chapter Summary

Chapter 8 investigates the inner workings of the **Electoral College** and its role in **U.S. presidential elections**. The chapter begins with several actual and hypothetical examples involving data from the 2000 U.S. presidential election. After investigating the **winner-take-all rule** and its implications in the Electoral College, readers learn about the **history of the Electoral College** and use various **power measures** to explore potential **biases** toward larger or smaller states.

The chapter concludes by considering some examples of the **unpleasant and paradoxical outcomes** that can occur in U.S. presidential elections under the Electoral College. Particular attention is given to the potential for the Electoral College to be affected by a small number of **swing votes**, and to elect candidates who receive an very small percentage of the popular vote. Finally, four **alternatives to the Electoral College** are proposed, and readers are asked to investigate the pros and cons of each one.

Learning Objectives

After completing Chapter 8, the reader should be able to …

- … describe in detail the Electoral College and its role in U.S. presidential elections.
- … describe the winner-take-all rule, and discuss its implications for the Electoral College and U.S. presidential elections.

- . . . summarize some of the history of the Electoral College.
- . . . explain the distribution of power to the states in the Electoral College, and discuss any potential biases toward larger or smaller states.
- . . . give some arguments in favor of and in opposition to the Electoral College.
- . . . describe some unpleasant or paradoxical outcomes that can occur in U.S. presidential elections under the Electoral College.
- . . . describe some potential alternatives to the Electoral College, and discuss some pros and cons of each one.

Teaching Notes

Many students will come into this chapter with an opinion about the Electoral College already established. For most of these students, this opinion will probably be negative, although many students will be unable to adequately justify this opinion. In fact, many students who think that the Electoral College is antiquated would probably agree that it once served a legitimate purpose, even though they may have no idea what this legitimate purpose may be or why the Electoral College was created in the first place.

After working through this chapter, whether students change their opinion about the Electoral College or not, or create an opinion about the Electoral College for the first time, they should be able to justify their opinion about the Electoral College, and also understand why someone might hold an opinion different from their own. Given the disparity of opinions about the Electoral College, and the fact that arguments on both sides are not difficult to identify or understand, a formal debate between groups of students may be an appropriate way to conclude this chapter. Students could be allowed to select the side they would prefer to argue, or for a challenge the sides to be argued could be assigned to students immediately before the debate begins.

Also, at some point while working through this chapter, your students (or you) may point out that the Electoral College is technically not a weighted voting system according to Definition 6.2, since there are more than two candidates in U.S. presidential elections. This is a legitimate point, although in today's political climate, the winner-take-all rule makes it virtually impossible for a third-party candidate to actually receive any electoral votes. Also noteworthy is the fact that, in 2008, for the first time in history, Nebraska split their electoral votes, awarding 4 votes to John McCain and 1 to Barack Obama. Although such a split had always been possible (not only in Nebraska, but also in Maine), it had never actually occurred until Obama made the decision to campaign aggressively in key areas of the state – specifically, the 2^{nd} congressional district, which consists mainly of Omaha. Obama's strategy paid off; he won this district, and consequently its electoral vote.

Reading Quiz Questions

1. What is the total number of electors (or electoral votes) in the Electoral College? **538**

2. In order for the winner of a U.S. presidential election to be determined by the Electoral College, the winning candidate must receive at least how many electoral votes?

   (a) More than the number of electoral votes received by any of the other candidates.

   (b) 435

   (c) **270**

   (d) 11

3. Under the winner-take-all rule in the Electoral College, how is the candidate who will receive all of the electoral votes held by a state determined?

   (a) **The candidate must receive a plurality of the popular votes cast in the state.**

   (b) The candidate must receive a majority of the popular votes cast in the state.

4. List all of the states whose electoral votes are not automatically awarded using the winner-take-all rule. **Maine, Nebraska**

5. True or **false**: The number of electoral votes held by each particular state has been fixed by law since 1911.

6. **True** or false: Abolishing the Electoral College would require an amendment to the U.S. Constitution.

7. True or **false**: Abolishing the winner-take-all rule in the Electoral College would require an amendment to the U.S. Constitution.

8. True or **false**: Federal law mandates that all of the electors in the Electoral College from each particular state must vote for the candidate who won a plurality of the popular votes cast in the state.

9. As measured by number of voters per elector, is the Electoral College biased toward larger states or smaller states? **Smaller**

10. As measured by the Banzhaf power index, is the Electoral College biased toward larger states or smaller states? **Larger**

11. In a U.S. presidential election with only two candidates, how many popular votes must a candidate receive in order to have a theoretical chance of winning the election?

    (a) One more than exactly 50% of the votes cast

    (b) 40.9% of the votes cast

    (c) 37.45% of the votes cast

    (d) **Less than 1% of the votes cast**

12. Which of the following alternatives to the Electoral College was not presented in Chapter 8?

    (a) Plurality

    (b) Approval voting

    (c) **A runoff system**

    (d) A district system

    (e) None of the above

Questions for Class Discussion

1. Are there any dummies in the Electoral College? If not, is there any way that changes in a state's population could cause it to become a dummy?

2. Is the Electoral College anonymous? Why or why not?

3. From a practical perspective, what obstacles might stand in the way of making changes to the Electoral College?

4. What do you think the chances are that the Electoral College will some day be eliminated?

5. Should electors be required to vote according to the popular vote results in the state they represent? Why or why not?

6. Since third-party candidates have virtually no chance of winning under the Electoral College, should they simply be outlawed from the ballot? Why or why not?

7. What do you think the public perception of the Electoral College is? If you asked the average person on the street about the Electoral College, what kind of response do you think you would get?

8. What are some ways that the Electoral College could be modified to give third-party candidates more of a chance?

9. Is it a positive or negative feature of the Electoral College that third-party candidates are unlikely to be elected?

10. How do you think campaigns would be different if the Electoral College was replace by a national popular vote? How would this effect the outcome of U.S. presidential elections?

11. Does the Electoral College make a tie more or less likely than with plurality?

Discussion of Selected Questions

Question 8.3. For part (c), reminding students of Question 2.29(c) might lead to an interesting classroom discussion.

Question 8.4. Although Ross Perot received 18.91% of the popular vote, he didn't win a plurality in any individual state. Consequently, he failed to earn even one electoral vote. If electoral votes had been distributed proportionally instead of according to the winner-take-all rule, Perot would have fared much better in the Electoral College.

Question 8.6. Arizona was the biggest winner, with a 25% increase, from 8 votes to 10. Mississippi was the biggest loser, going from 7 votes to 6, a 14% decrease.

Question 8.8. Each of California's 55 electoral votes represents a population of approximately $654,472$. On the other hand, each of Wyoming's 3 electoral votes represents a population of only $167,853$. One could argue that the electoral votes in Wyoming carry more "bang for the buck," since far fewer voters must be persuaded to vote for a candidate in order for that candidate to receive electoral votes. On the other hand, the population density of Wyoming is much less than that of California, which makes campaigning in Wyoming less efficient.

Questions 8.9 – 8.10. For classes in which the material on combinatorics in Chapter 7 is too advanced, Questions 8.9 and 8.10 and the paragraphs immediately preceding and following them can be omitted without affecting the students' understanding of the remainder of the section in which these questions are contained or anything else in the remainder of this chapter.

Question 8.10. There are $51! = 1.55 \times 10^{66}$ possible arrangements. A computer forming one million such arrangements per second would take 4.92×10^{52} years to consider them all.

Question 8.12. The races in Iowa and Wisconsin were both very close, but pale in comparison to the race in New Mexico, whether you measure the difference as an actual number of votes or as a percentage. New Mexico was even closer than Florida if you measure the difference as an actual number of votes (although as a percentage Florida was much closer). It might be interesting to discuss with your class why Bush did not push for recounts in New Mexico while Gore was pushing for recounts in Florida.

Question 8.14. For part (a), if 538 of the non-Bush voters in Florida had changed their votes to Gore, then Gore would have received 2,912,791 votes in Florida and Bush would have only received 2,912,790. For part (b), if 3606 of the Bush voters in New Hampshire had changed their votes to Gore, then Gore would have received 269,954 votes in New Hampshire and Bush would have only received 269,953. Since New Hampshire had four electoral votes, this would have given Gore exactly the 270 electoral votes necessary to win the election. It might be worth discussing with your class that if New Hampshire only had three electoral votes, then this would have only given Gore 269 electoral votes, not enough to win the election (thanks only, by the way, to the elector from the District of Columbia who was pledged to Gore but abstained from voting in the Electoral College)[1], and the election would have been handed over to the U.S. House of Representatives for the first time since 1824.

[1]Full disclosure – if abstaining from voting in the Electoral College would have affected the outcome of the election, this elector would almost certainly have gone ahead and voted for Gore. While individual electors frequently violate the winner-take-all rule, no elector by doing so has ever actually affected the outcome of an election.

Question 8.15. If only one person showed up to vote in each of the 11 largest states and all voted for Kerry, and the entire population showed up to vote in each of the other states (and the District of Columbia) and all voted for Bush, then Bush would have won the popular vote by a margin of 125,884,431 to 11, but Kerry would have won the election in the Electoral College by a margin of 271 to 267. This assumes that the entire population of each of the 39 smallest states and the District of Columbia are all eligible voters. This is not correct, of course, nor is it in any way reasonable to assume that only one person would show up to vote in a state, or that the entire population of another state would show up to vote, even if they were all eligible voters. For part (c), the answer is yes. Remarkably, a candidate can win a U.S. presidential election without receiving a single popular vote. Recall that if no candidate receives a majority of the electoral votes cast, then the electoral votes are thrown out and the election is handed over to the U.S. House of Representatives.

Question 8.16. Remember that electors in larger states represent more people than electors in smaller states (see Questions 8.7 and 8.8). Thus if you wanted to construct a winning coalition for Kerry that contained the smallest possible number of popular votes, you should include more small states than large states. And in this winning coalition for Kerry, you would only have to assume that Kerry received one more popular vote than Bush, whereas in the states outside of Kerry's winning coalition, you could assume that Bush received all of the popular votes. Here is an example of this: Suppose that 40.9% of the population of CA, TX, NY, FL, IL, PA, OH, GA, NJ, VA, MA, and ID showed up to vote with all voters voting for Bush, and 40.9% of the population of each of the other states (and the District of Columbia) showed up to vote with one more than half of these voters voting for Kerry and the rest voting for Bush. Then with some assistance from your calculator you should be able to show that Bush would have won the popular vote by a margin of 78.012% to 21.988%, but Kerry would have won the Electoral College by a margin of 270 to 268.

Question 8.19. The answer to both questions is yes. Remember that electors in larger states represent more people than electors in smaller states (see Questions 8.7 and 8.8).

Question 8.23. In order to apply the criteria from Chapters 2 – 5, we must first decide how we are going to model the Electoral College – either as a system with 51 voters (the states), or as a system with around 210 million voters (the approximate voting-eligible population of the United States). Interestingly enough, in either model, the same criteria apply. In particular, the Electoral College satisfies neutrality, citizen sovereignty, monotonicity, nondictatorship, universality, and the modified Pareto condition. It violates anonymity, the majority criterion, CWC, CLC, IIA, and unanimity.

Question 8.25. In the 2000 presidential election, all of the states in Bush's winning coalition were critical. If Gore had won Florida instead of Bush in 2000, he would have won the presidency with 291 electoral votes. Thus, any state that Gore won with more than 21 electoral votes (FL, NY, and CA) would have been critical to his hypothetical coalition. In 2004, Bush won with 286 electoral votes. Thus, all of the states that Bush won with more than 16 electoral votes (TX, FL, and OH) were critical.

Question 8.24. Remember that while a single elector's abstention from voting could not cause a candidate destined for losing to receive 270 electoral votes, it could cause a candidate destined for winning to **not** receive 270 electoral votes.

Question 8.26. For part (d), what should strike you as being strange or unusual is that with the change to the election that you should have specified in part (b), Kerry would have won the election even though Bush would have received more than half of the popular votes. From a popular vote perspective, this would have been a violation of the majority criterion (which last happened in a U.S. presidential election in 1876).

Question 8.28. There have been four elections in U.S. history (1824, 1876, 1888, and 2000) in which the Electoral College winner did not receive a plurality of the popular vote. The 1824 election, however, could be disputed as falling into this category, since in several states, the electors were appointed by the legislature and not by a popular vote. The 1824 election was also the only presidential election in U.S. history in which no candidate received a majority of the electoral votes. Thus, the election was decided by the House of Representative, who chose John Quincy Adams (with 84 electoral votes), even though Andrew Jackson received more electoral votes (99).

Question 8.29. There has never been a unanimous Electoral College winner, although the elections of 1820, 1936, and 1984 came quite close.

Question 8.30. The 2000 election was especially close from the perspective that it would have only taken a switch of 269 popular votes from Bush to Gore to change the outcome (see Question 8.13). The 1880 election was especially close from the perspective that out of the 9,217,410 popular votes cast in the election, the two main candidates (Garfield and Hancock) were separated by only 9,070 popular votes, a difference of less than 0.098% (much smaller than the 0.516% of the popular votes that separated Bush and Gore in 2000). The 1960 election, in which the two main candidates (Kennedy and Nixon) were separated by less than 0.164% of the popular votes, was also especially close from the perspective that in only a very few states was there a dominant plurality winner.

Question 8.33. With only two candidates, a unanimous Electoral College winner would need to receive more than half of the popular vote in each state, and thus a majority nationwide. With more than two candidates, however, a candidate could win by receiving a small plurality (for instance, 21% with 5 candidates) in each of the 11 largest states, even if one of his or her opponents won the entire popular vote in the remaining 39 states and the District of Columbia. In such a situation, the winning candidate would fall far short of a plurality of the nationwide popular vote. (For a related question, see Supplementary Question 8.40 below.)

Question 8.39. The median voter theorem assumes that each voter and candidate can be placed on a linear issue space (for instance, a line segment with "liberal" on one end and "conservative" on the other) according to their positions and preferences. The conclusion of the theorem is that a candidate cannot lose an election if he or she takes a position that corresponds to the middle (or median) voter on this issue space. In other words, it is best for a candidate to take a centrist position instead of one at either extreme of the political spectrum. In spite of the popularity of the median voter theorem, its applicability is questionable, mainly because the model on which it is based is oversimplified. In reality, the views of both voters and candidates are not well-represented by positions on a one-dimensional issue space. Furthermore, analogs to the median voter theorem seem to break down in higher dimensions. For more information on the median voter theorem, see COMAP [20].

Supplementary Questions

Question 8.40.

(a) In a presidential election with only two candidates, would it be possible for a candidate to win unanimously in the Electoral College without receiving a majority of the nationwide popular vote? Give a convincing argument or example to justify your answer.

(b) In a presidential election with more than two candidates, would it be possible for a candidate to win unanimously in the Electoral College without receiving a majority of the nationwide popular vote? Give a convincing argument or example to justify your answer.

Question 8.41. In the 2000 U.S. presidential election, one tactic that was considered by some Nader supporters who thought that their votes for Nader might hurt Gore's overall chances in the election was swapping votes with some Gore supporters. More specifically, in close states that Gore might have lost if Nader supporters voted for Nader, these Nader supporters considered switching their votes to Gore if some Gore supporters in less contested states where Gore would almost surely have won would switch their votes to Nader. This would have left the popular vote totals for the two candidates essentially unchanged, thereby giving Nader the publicity he had earned, but might have allowed Gore to win in the close states he might otherwise have lost. Do you think this is a sensible way of dealing with some of the problems created by the Electoral College? Why or why not?

Question 8.42. According to the Banzhaf power percentages in Table 8.5, there is a clear bias in the Electoral College towards larger states. How would you explain then the fact that some of the strongest advocates of the Electoral College come from smaller states?

Chapter 9

Trouble in Direct Democracy

Chapter Summary

Chapter 9 focuses on **referendum elections** and the **separability problem**, which occurs when a voter's preferred outcome on a proposal in an election depends on the known or predicted outcome of other possibly related proposals. Readers consider examples of **paradoxical outcomes** such as an election in which the winning outcome is the last choice of every single voter. These examples motivate definitions of what it means for a voter's preferences to be **separable** and also reveal the importance of separability to the goal of ensuring election outcomes that accurately represent the will of the electorate.

After these initial examples and definitions, **binary preference matrices** are introduced as a way to model voter preferences in referendum elections. The notion of **symmetry** is defined from within this context and shown to be a necessary, but not sufficient, condition for separability. Readers are then asked to investigate **set theoretic results** pertaining to separability, and to explore the relationship between these results and similar results from economics.

To conclude the chapter, readers are asked to consider several **potential solutions** to the separability problem (including **avoiding nonseparable preferences**, **setwise voting**, **sequential voting**, and **contingent ballots**), and to propose their own solutions.

Learning Objectives

After completing Chapter 9, the reader should be able to ...

- . . . describe what a referendum election is, and give examples of situations in which a referendum election might be used.
- . . . give examples of paradoxical outcomes that can occur in referendum elections, and discuss why these outcomes occur.
- . . . define what it means for a voter's preferences to be separable, and discuss the role of separability in referendum election outcomes.
- . . . define what a binary preference matrix is, and explain how binary preference matrices can be used to model voter preferences in a referendum election.
- . . . define what it means for a binary preference matrix to be symmetric, and describe the relationship between symmetry and separability.
- . . . give examples of set theoretic results pertaining to separability, and use these results to explain how preferences in referendum elections behave differently than preferences in economic settings.
- . . . describe in detail some of the potential ways of solving the separability problem, and discuss the pros and cons of each of these potential solutions.

Teaching Notes

This chapter covers a relatively new area of mathematical research dealing with the separability problem in referendum elections. This problem was first noted in a 1997 paper by Brams, Zwicker, and Kilgour (see reference [10] on page 217 of the text), and it has been studied fairly extensively since then.

Your students might be interested in knowing that much of the recent work on the separability problem has been developed at least in part by undergraduates.[1] This observation provides an opportunity to highlight the fact that mathematics is a living and growing field. We do not know all there is to know about mathematics, and new mathematical results are constantly being discovered, developed, or applied in novel ways. Moreover, students just like yours are playing an important role in extending the boundaries of our current mathematical knowledge.

[1] A good example of some of this work is Hodge and Schwallier's paper entitled "How does separability affect the desirability of referendum election outcomes?" (Theory and Decision 61(3):251-276, 2006). This paper uses computer simulation to explore in a very practical way the effect of separability on the outcomes of referendum elections.

Most of the material in this chapter is fairly concrete, with the possible exception of the section entitled *Testing for Separability* on pages 177 – 180. This section can be omitted without any loss of continuity, and instructors wishing to focus more on the practical aspects of the separability problem may wish to do so.

Reading Quiz Questions

1. True or **false**: In a referendum election, if all but one voter's preferences are separable, then a Condorcet winning outcome will necessarily be selected by simultaneous voting.

2. **True** or false: In a referendum election, it is possible for the winning outcome to be the least preferred choice of all but one of the voters in the election.

3. **True** or false: In a referendum election, it is possible for the winning outcome to be the least preferred choice of every voter in the election.

4. Fill in the blank: A voter's preferences on a question are said to be **separable** if they do not depend on the outcome of other questions in the election.

5. Which of the following mathematical objects were used in Chapter 9 to represent a voter's preferences in a referendum election?

   (a) Binomial coefficients

   (b) Societal preference orders

   (c) **Binary preference matrices**

   (d) Social welfare functions

6. **True** or false: If a voter's preferences are separable, then they must be symmetric.

7. True or **false**: If a voter's preferences are symmetric, then they must be separable.

8. **True** or false: In an election with only two questions, the properties of separability and symmetry are equivalent.

9. True or **false**: If two sets of questions are both separable with respect to a particular voter, then their union must also be separable with respect to that voter.

10. **True** or false: If two sets of questions are both separable with respect to a particular voter, then their intersection must also be separable with respect to that voter.

11. **True** or false: In a referendum election in which every voter has separable preferences, it can never benefit a voter to vote insincerely.

12. Fill in the blank: The method of voting that views a vote of Y/Y/N not as separate votes on each of three proposals but as a single vote for the preferred outcome of the referendum election is called **setwise** voting.

13. True or **false**: Sequential voting always produces a more desirable outcome than simultaneous voting.

14. True or **false**: In a referendum election with only two proposals, the outcome produced by simultaneous voting can be preferred by a majority of voters to the outcome produced by sequential voting.

15. Fill in the blank: The type of ballot that mimics sequential voting is called a **contingent** ballot.

Questions for Class Discussion

1. What are some practical advantages and disadvantages of referendum elections? Should referendum elections be used more often or less often in U.S. politics?

2. What types of issues seem most appropriate for referendum elections?

3. Are referendum elections the best way to implement direct democracy? If so, why? If not, what other methods would be better?

4. How practical would it be to ask voters to rank the possible outcomes in a referendum election, instead of just specifying their first choice? Could such a modification, combined with one of the voting systems we have discussed in previous chapters, lead to improved outcomes?

5. Suppose that, as voters cast their ballots, the outcome based on the ballots cast so far was announced. Would this be a good way to alleviate the separability problem? What would be some advantages and disadvantages of this approach?

6. What are some real-life situations that might lead nonseparable preferences? Be specific.

Discussion of Selected Questions

Question 9.2. The outcome of the election, N/N, would be the same in each of the scenarios described in parts (a) and (b). In part (c), Logan is correct; if $10,968$ voters have preferences like Dave's, $10,968$ voters have preferences like Mike's, and 1 voter has preferences like Pete's, the outcome will still be N/N, the last choice of all but one of the voters.

Question 9.3. Suppose Dave's first choice is $Y/Y/N$, Mike's first choice is $N/Y/Y$, and Pete's first choice is $Y/N/Y$. Then the outcome will be $Y/Y/Y$, with each proposal passing by a 2-to-1 margin. This is conceivably the last choice of each voter, since it would involve spending more money than they have.

Question 9.7. None of the voter's preferences are likely to be separable. To illustrate, suppose Pete knew that both the first and second proposals were going to pass. Then he would likely vote no on the third proposal to avoid having all three proposals pass. (This, of course, assumes that Pete prefers an outcome of YYN to an outcome of YYY, the latter resulting in too much money being spent.) If, however, Pete knew that the first two proposals were going to fail, then he would certainly vote yes on the third proposal, since he prefers the outcome NNY to the outcome NNN. Because Pete's vote on the third proposal depends on the outcome of the first two, Pete's preferences on the third proposal are not separable.

Question 9.8. This voter's preferences are not separable. Although the voter always prefers Y to N on the first proposal, his or her preferences on the other proposals are not so clear-cut. For instance, if both the first and second proposals pass, then the voter prefers that the third proposal pass as well. (This is indicated by the preference $YYY \succ YYN$.) However, if both the first and second proposals fail, then the voter prefers that the third proposal fail as well (as indicated by the preference $NNN \succ NNY$).

Question 9.11. Only the third matrix is a binary preference matrix. The first contains the outcome 110 twice, which would suggest, nonsensically, that the voter prefers YN to YN. In contrast, the second matrix is missing the outcome 01. The third matrix is the only one that contains each of the four possible outcomes for a referendum election on two proposals.

Question 9.12. The corresponding binary preference matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 9.15. The complete binary preference matrix would be

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Question 9.21. If every collection of $n-1$ proposals is separable, then **every** collection of **any number** of proposals must be separable. This is because every collection of proposals can be formed by repeatedly intersecting pairs of larger collections of proposals, beginning with those that contain $n-1$ proposals.

Question 9.23. Theorem 9.22 does not *directly* imply that the winning outcome cannot be the least preferred choice of every single voter. In fact, Theorem 9.22 does not allow any conclusion at all to be made unless there is a Condorcet winning outcome. If no such Condorcet winner exists, then Theorem 9.22 is inconclusive.

With that said, note that in an election in which all voters have separable preferences, the most preferred and the least preferred choices for each voter are bitwise complements of each other (by symmetry). Thus, if an outcome x was the least preferred choice of every voter, then it's complement, say $\overline{x}$, would necessarily be the most preferred choice of every voter. Thus, $\overline{x}$ would in fact be a Condorcet winner, and would thus be selected according to Theorem 9.22.

Question 9.25. Both Dave's and Mike's preferences are separable, but Pete's are not. Although 00 is a Condorcet winning outcome, the overall winner of the election is actually 11. This example could be extended to one with an arbitrarily large number of voters by simply adding an equal number of voters with each of Dave's and Mike's preferences, while keeping only a single voter with Pete's preferences.

Question 9.26. The outcome of set-wise voting would be a three-way tie between 110, 101, and 011. Any of these outcomes would be a better choice than the proposal-by-proposal outcome of 111, which was the last choice of every voter.

Question 9.29. The outcome under simultaneous voting would be $Y/Y/Y$. For the sequence suggested in part (b), the outcome would be $Y/N/N$, the last choice of two of the three voters. (Note that after the first proposal passed and the result was announced, Voter 3 would vote N/N on the second and third proposals, since his or her first choice for the overall outcome, $N/Y/Y$, is no longer an option.) For the sequence in part (c), the outcome would be $Y/N/Y$. For the sequence in part (d), the outcome would be $Y/Y/N$.

Question 9.31. It would probably be best to vote last on the one proposal that seems most likely to be nonseparable for a large number of voters. Of course, such a proposal would not always be easy to identify.

Question 9.32. In a two-question election, the outcome of sequential voting will always be at least as desirable (to a majority of voters) as the outcome of simultaneous voting. This is due to the fact that each of the two possible ways of sequencing the election are equivalent to the method described in Theorem 9.30.

Question 9.33. Contingent ballots could be an ideal solution to the separability problem in the two-question case. With only two questions, the same logic used in Question 9.32 implies that contingent ballots could never produce an outcome that was viewed as worse (by a majority of voters) than the outcome of simultaneous voting. On the other hand, contingent ballots quickly become quite cumbersome in elections with more than two questions. And, as we have seen, in elections with three or more questions, the order in which questions are sequenced can have a dramatic impact on the outcome of the election. Contingent ballots, by mimicking sequential voting, would be susceptible to the same order dependencies.

Question 9.35. In an election with only two questions, it is not possible for the winning outcome to be the least preferred choice of every voter. Suppose, for instance, that the winning outcome is 11. For the first question to have passed, more than half of the voters must have voted either 10 or 11. Likewise, for the second question to have passed, more than half of the voters must have voted either 01 or 11. If nobody had voted 11, then more than half of the voters would have voted 10, **and** more than half of the voters would have voted 01. This is impossible, and so there must have been some voter who voted 11. The same reasoning applies to all of the other possible outcomes.

Question 9.36. It is not possible for a Condorcet losing outcome to be selected by simultaneous voting if all voters have separable preferences and vote sincerely. Suppose that an outcome x is a Condorcet loser, and let y be any other outcome. Also let $\overline{x}$ and $\overline{y}$ denote the bitwise complements of x and y, respectively. Since x is a Condorcet loser, a majority of voters must prefer y over x. However, since every voter has separable, and thus symmetric, preferences, it must also be the case that each voter who prefers y over x also prefers $\overline{x}$ over $\overline{y}$. Thus, $\overline{x}$ is preferred to $\overline{y}$ by a majority of voters. Since y was chosen arbitrarily (that is, y was chosen to represent *any* outcome), it follows that $\overline{x}$ would defeat *every* other outcome in a head-to-head contest. Thus, $\overline{x}$ is a Condorcet winner which, by Theorem 9.22, will be selected by simultaneous voting.

For part (b), even if one voter is allowed to have nonseparable preferences, it is still impossible for a Condorcet loser to be selected by simultaneous voting. To see this, suppose that n voters (say $v_1, v_2, \ldots, v_n$) have separable preferences, and one voter, v_{n+1}, has nonseparable preferences.

Suppose also that a particular outcome x is a Condorcet loser among these $n+1$ voters. For convenience in our argument, we will assume that $x = 111\cdots1$, although the same reasoning would apply for any other outcome. Since $111\cdots1$ is a Condorcet loser, the outcome $011\cdots1$ must be ranked higher than $111\cdots1$ by a majority of the voters. In order for this to happen, however, at least half of the n voters with separable preferences must rank $011\cdots1$ higher than $111\cdots1$.[2] But since each of these voters' preferences are separable, it must then be the case that each of them prefer 0 to 1 on the first proposal, regardless of the outcome of the remaining proposals. Thus, at least half of the n voters with separable preferences will vote no on the first proposal. By a similar argument, the same will be true for each of the other proposals. Thus, for each proposal, at least half of the n voters with separable preferences will vote no.

In the case that n is odd, each proposal will therefore receive at least $(n+1)/2$ no votes, and thus fewer than the $(n+1)/2+1$ yes votes required for passage. Consequently, the election outcome will be $000\cdots0$.

In the case that n is even, each proposal will receive at least $n/2$ no votes. If a proposal receives more than $n/2$ no votes, then it will not attain the majority $(n/2+1)$ required for passage. Thus, the only way for a proposal to pass is if

[2] Note that, with $n+1$ voters, the number of votes required for a majority depends on whether n is even or odd. If n is even, then $n+1$ is odd, and so a majority would consist of $n/2+1$ voters. If n is odd, then $n+1$ is even, and so a majority would consist of $(n+1)/2+1$ voters. In either case, at least half of of the n voters with separable preferences must be a part of any majority.

the one voter with nonseparable preferences, *and* exactly $n/2$ of the n voters with separable preferences, vote yes.

Recall that we are trying to argue that the Condorcet loser, $111\cdots1$, cannot be selected by simultaneous voting. So far, we have established that the only way $111\cdots1$ *could* be selected is if all three of the following conditions hold:

- n is even.
- On each proposal, exactly $n/2$ of the n voters with separable preferences vote yes.
- The one voter with nonseparable preferences votes yes on every proposal, and thus ranks the outcome $111\cdots1$ as his or her top choice.

These conditions, however, are incompatible with our assumption that $111\cdots1$ is a Condorcet loser. To see this, note that if exactly $n/2$ of the n voters with separable preferences vote yes on a given proposal, say Proposal 1, then each of these $n/2$ voters must prefer $111\cdots1$ to $011\cdots1$ (since their preferences are separable). But the one voter with nonseparable preferences also prefers $111\cdots1$ to $011\cdots1$, since $111\cdots1$ is his or her top choice. Thus, $n/2+1$ voters prefer $111\cdots1$ to $011\cdots1$, which means that $111\cdots1$ would defeat $011\cdots1$ in a head-to-head contest, a contradiction to the assumption that $111\cdots1$ is a Condorcet loser.

To summarize, if we assume that $111\cdots1$ is a Condorcet loser, then the only way for $111\cdots1$ to be selected by simultaneous voting is if a certain set of conditions are satisfied. However, these conditions will never be satisfied if $111\cdots1$ is in fact a Condorcet loser. Thus, assuming that all but one of the voters have separable preferences, it is impossible for $111\cdots1$ to be a Condorcet loser and still be selected by simultaneous voting.

On the other hand, if *two* voters are allowed to have nonseparable preferences, then a Condorcet loser can be selected by simultaneous voting. (To see this, try modifying exactly one of the preference matrices in Table 9.5 to make the corresponding preferences separable without altering the top choice.)

Question 9.37. All such preferences must be separable on each individual proposal. However, no additional separability is guaranteed. To illustrate, consider the two matrices shown below. The matrix on the left represents preferences that are separable on every possible set of proposals, while the one on the right represents preferences that are only separable on individual proposals (and, of course, $\emptyset$ and $\{1, 2, 3\}$).

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Question 9.38. There are 24 possible preference matrices for an election with two proposals. Of these, 8 are symmetric, and each of these 8 symmetric matrices correspond to voters with separable preferences. Thus, symmetry and separability are equivalent elections with only two proposals.

Question 9.39. For a referendum election with n proposals, there are 2^n possible outcomes, and thus $2^n!$ possible binary preference matrices. Of these, it can be shown that exactly

$$2^n \cdot (2^n - 2) \cdot (2^n - 4) \cdots 4 \cdot 2 = 2^{2^{n-1}} \cdot 2^{n-1}!$$

are symmetric. Since every separable preference matrix must be symmetric, it follows that the probability of a randomly selected matrix being separable is less than or equal to

$$\begin{aligned} \frac{2^{2^{n-1}} \cdot 2^{n-1}!}{2^n!} &= \frac{2^n \cdot (2^n - 2) \cdot (2^n - 4) \cdots 4 \cdot 2}{2^n \cdot (2^n - 1) \cdot (2^n - 2) \cdots 2 \cdot 1} \\ &= \frac{1}{(2^n - 1) \cdot (2^n - 3) \cdot (2^n - 5) \cdots 3 \cdot 1}, \end{aligned}$$

which clearly approaches 0 as n increases. Thus, for elections with numerous proposals (as in the 1990 California general election), it seems likely that most voters would have nonseparable preferences. In fact, subsequent research has shown that as the number of proposals increases, most randomly selected preferences become *completely* nonseparable, meaning that they are separable only on $\emptyset$ and the entire set of proposals.[3] Of course, preferences in actual elections may not be randomly distributed, and so the application of these results to real-life scenarios remains a subject of further investigation.

[3] See Hodge and TerHaar, Classifying interdependence in multidimensional binary preferences, *Mathematical Social Sciences* 55(2):190-204, 2008.

Question 9.40. Colorado's Amendment 36, which received only 34% support, would have changed Colorado's allocation of electoral votes from the winner-take-all rule to a proportional system like the one discussed in Chapter 8 (on page 162). Because the measure required electoral votes to be distributed in whole numbers, it seems likely that in most presidential elections, Colorado's 9 electoral votes would have been split 5–4. In other words, the prize for a candidate winning Colorado would have essentially been a net gain of 1 electoral vote, instead of the 9 up for grabs under the winner-take-all rule.

What makes Amendment 36 even more interesting is the fact that it would have been effective beginning with the 2004 presidential election, which was on the same ballot as the amendment. Early in the campaign, when it was expected that Colorado would lean toward Bush, most Democrats favored Amendment 36, while most Republicans opposed it. As the campaign progressed, however, Colorado began to look more like a swing state, and many Democrats withdrew their support for the amendment. Clearly, voters' preferences on Amendment 36 were not separable, since many wanted it to pass (and thus limit the effect of Colorado on the outcome of the election) only if their favored candidate was not expected to win the state. It is also interesting to note that had the amendment been in place for the 2000 U.S. presidential election, Al Gore would have won the electoral college and become president.

Question 9.42. It is possible for the preferences of a voter with a symmetric binary preference matrix to be completely nonseparable. The following is one example:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Supplementary Questions

Question 9.43 For each of the parts below, find a preference matrix corresponding to a voter (in an election with 3 proposals) whose preferences are separable on exactly the sets listed (and no others), or explain why no such matrix can be found.

(a) $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2, 3\}$

(b) $\emptyset$, $\{1, 2\}$, $\{3\}$, $\{1, 2, 3\}$

(c) $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 2, 3\}$

(d) $\emptyset$, $\{1\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3\}$, $\{1, 2, 3\}$

Question 9.44. Suppose, in a referendum election with four proposals, you know that the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$ are separable with respect to a particular voter. What other sets must also be separable with respect to this voter, and why?

Question 9.45. Suppose that the first (or last) two rows of a binary preference matrix are bitwise complements of each other. What can you conclude about the corresponding preferences, and why?

Question 9.46. Call a voter's preferences *monoseparable* if they are separable on each individual question, but not necessarily on larger sets of proposals. If a voter has monoseparable preferences, must their corresponding binary preference matrix be symmetric? Why or why not?

Question 9.47. Design a contingent ballot that could be used in a referendum election with three proposals. How many questions would such a ballot need to contain?

Chapter 10

Proportional (Mis)representation

Chapter Summary

Chapter 10 is a self-contained study of **apportionment** in the context of the distribution of seats in the U.S. House of Representatives to the 50 states. Readers learn about the **history of apportionment in the House**, and investigate **apportionment methods** attributed to **Alexander Hamilton**, **Thomas Jefferson**, **Daniel Webster**, **John Quincy Adams**, and **Joseph Hill**. Particular attention is given to the general principles behind **divisor methods**, of which the methods attributed to Jefferson, Webster, Adams, and Hill are examples. In addition, readers learn to evaluate apportionment methods by considering whether they satisfy the **quota rule**, and whether they favor smaller states, larger states, or neither.

The **Alabama**, **population**, and **new-states** apportionment paradoxes are also considered in Chapter 10, each from an historical perspective. The discussion of these apportionment paradoxes leads to a presentation of **Balinski and Young's theorem**, which states that it is impossible for an apportionment method to always satisfy the quota rule and also be incapable of producing paradoxical outcomes.

Learning Objectives

After completing Chapter 10, the reader should be able to . . .

- . . . describe the apportionment problem in the context of the distribution of seats in a legislative body.
- . . . summarize some of the history of apportionment in the U.S. House of

Representatives.

- . . . describe several different apportionment methods, and compare and contrast these methods.
- . . . explain what it means for an apportionment method to be a *divisor* method.
- . . . explain how certain apportionment methods favor smaller states, certain methods favor larger states, and certain methods are neutral.
- . . . describe the quota rule, and explain how certain apportionment methods can violate the quota rule while others cannot.
- . . . describe several different apportionment paradoxes, and summarize the historical context in which these paradoxes were first observed.
- . . . describe Balinski and Young's theorem, and discuss its implications to the problem of apportionment.

Teaching Notes

> *Apportionment basically deals with the problem of how to go about rounding off collections of numbers so that the sum of the rounded numbers is equal to the sum of the unrounded numbers.*

While we include the above statement in the introductory section of this chapter, you should make sure right away that your students are aware that apportionment has a very substantial application in the political world and is at the center of a fascinating part of American political history (as we note in the final paragraph of the introductory section of this chapter). In this chapter we basically tell this fascinating story, introducing the concepts of apportionment at appropriate times throughout. Your students may find it interesting that the method currently used to apportion the U.S. House was not chosen for any logical reason, but was a purely political choice, as we briefly discuss in the last paragraph in the section **Hill's Apportionment Method**. Your students may also find it interesting that the makeup of the Electoral College that resulted from the unconstitutional apportionment of 1872 directly affected the outcome of the 1876 U.S. presidential election, as we briefly discuss in the first full paragraph on page 204.

As we write in the footnote at the bottom of page 193, the population figures used by Congress for the 1794 apportionment were slightly different from the figures shown in Table 10.1. We chose to use the population figures shown in Table

10.1 not just in the interest of political correctness, but also because it makes the apportionments in Questions 10.3, 10.11, 10.14, 10.16, and 10.23 much more interesting to compare, since each of these apportionments is different from all of the others. This would not have been the case had we used the actual population figures that were used by Congress in 1794. Since the apportionments in Questions 10.3, 10.11, 10.14, 10.16, and 10.23 are the only "examples" of apportionments in the chapter, it might be worth emphasizing to your students that while it is possible for each of the five apportionment methods that we present in this chapter to give different apportionments with the same population figures, it is not necessary. We allude to this fact periodically throughout the chapter.

Reading Quiz Questions

1. True or **false**: The reason the writers of the Constitution did not specify exactly how the seats in the U.S. House of Representatives should be apportioned to the states was most likely because they could not agree upon a method.

2. In Hamilton's apportionment method, surplus seats are awarded one at a time to which states?

   (a) The states whose populations are the smallest

   (b) The states whose populations are the largest

   (c) The states whose standard quotas have the smallest decimal parts

   (d) **The states whose standard quotas have the largest decimal parts**

3. The first presidential veto in U.S. history was of a bill that would have approved which method for the initial apportionment of the House of Representatives?

   (a) Adams' method

   (b) **Hamilton's method**

   (c) Jefferson's method

4. True or **false**: The reason that Jefferson's apportionment method is called a *divisor* method is because it uses division to find the standard quota for each state.

5. Does Jefferson's apportionment method favor smaller states, larger states, or neither? **Larger states**

6. **True** or false: If Jefferson's method had continually been used to apportion the seats in the U.S. House of Representatives, then every apportionment of the House since 1852 would have violated quota.

7. Does Webster's apportionment method favor smaller states, larger states, or neither? **Neither**

8. True or **false**: Every apportionment method can violate quota.

9. **True** or false: If Webster's method had continually been used to apportion the seats in the U.S. House of Representatives, then no apportionment of the House would have ever violated quota.

10. Which of the following apportionment paradoxes was not presented in Chapter 10?

    (a) The Alabama paradox

    (b) **The redistricting paradox**

    (c) The population paradox

11. Which of the following does Hill's apportionment method use to determine the cutoff for rounding a quota up or down?

    (a) An arithmetic mean

    (b) **A geometric mean**

12. True or **false**: As a result of Balinski and Young's theorem, we know that every apportionment method is capable of producing paradoxes.

Questions for Class Discussion

1. Do you think that a particular apportionment method should have been specified in the U.S. Constitution? Why or why not?

2. Do you think Washington made the correct choice between Jefferson's and Hamilton's methods?

3. Should legislators be allowed to re-draw district boundaries in order to improve their chances of re-election? Why or why not?

4. Which apportionment method do you think is most fair, and why?

5. Should an apportionment method favor large states, favor small states, or be neutral? Why?

6. Do you think it is better for an apportionment method to violate quota or to produce the population paradox?

7. Do you think that a fixed House size of 435 seats is a good idea, or should the number of representatives be allowed to vary?

8. Can you think of any applications of apportionment apart from allocating seats to various states in a representative body?

9. Should the elected officials who are affected by the apportionment be deciding which method is used? If not, who should make this decision?

Discussion of Selected Questions

Question 10.3. The number of seats awarded to each of the states under Hamilton's method is shown in the following table.

State	Seats
Connecticut	6
Delaware	1
Georgia	2
Kentucky	2
Maryland	9
Massachusetts	13
New Hampshire	4
New Jersey	5
New York	9
North Carolina	11
Pennsylvania	12
Rhode Island	2
South Carolina	7
Vermont	2
Virginia	20
TOTAL	105

Question 10.5. The state whose decimal part makes up the largest percentage of its standard quota is Kentucky, and the state whose decimal part makes up the smallest percentage of its standard quota is Virginia. In the apportionment in Question 10.3, it can be argued that Pennsylvania was treated the best since the decimal part of its standard quota as a percentage of its entire standard quota is the smallest among the states that were awarded a surplus seat, and Delaware was treated the worst since the decimal part of its standard quota as a percentage of its entire standard quota is the largest among the states that were not awarded a surplus seat.

Question 10.7. In the apportionment in Question 10.3, it can be argued that Rhode Island was treated the best since the average number of residents represented by each of its representatives is the smallest among all the states, and Delaware was treated the worst since the average number of residents represented by each of its representatives is the largest among all the states.

Question 10.8. Delaware moves from being the state treated worst (from the perspective of the average number of residents represented by each of its representatives) to being the state treated best.

Question 10.12. Step 2 of Jefferson's method always gives away too few seats because the standard quotas are rounded down.

Question 10.13. The original standard quota for part (a) is 32.503, and the modified standard quota for part (b) is 34.305. With this modified standard quota, under Jefferson's method New York was given 34 seats. This can be viewed as unfair to New York's benefit, since according to its original standard quota it only deserved 32.503 seats, and so New York was given more than a full seat above the exact number it deserved based on its population. (This could more easily be described as violating the quota rule. However, we do not formally state the quota rule until page 202.)

Question 10.14. The four states that will have gained a seat are Connecticut, Delaware, Georgia, and Vermont, and the four states that will have lost a seat are Massachusetts, North Carolina, Pennsylvania, and Virginia.

Question 10.15. In Adams' method, the standard divisor always yields too many seats, as standard quotas are rounded up. Thus, the standard divisor will always need to be increased in order to yield a correct apportionment.

Question 10.17. With Webster's method, the standard quotas are rounded up in some cases and down in others. Thus, it is possible for the standard divisor in Webster's method to yield either too few seats (in which case the standard divisor would need to be decreased) or too many seats (in which case the standard divisor would need to be increased).

Question 10.18. By its very definition, Hamilton's method begins by rounding the standard quotas down, and then distributes at most one surplus seat to each state. Thus, each state receives either its standard quota rounded down, or its standard quota rounded up (if it is awarded a surplus seat).

Question 10.19. With a House size of 299 seats, Hamilton's method would have awarded 8 seats to Alabama, 18 seats to Illinois, and 9 seats to Texas. With a House size of 300 seats, Hamilton's method would have awarded 7 seats to Alabama, 19 seats to Illinois, and 10 seats to Texas. What should strike you as unusual or unfair about these two potential apportionments is that increasing the number of seats in the House from 299 to 300 caused Alabama to *lose* a seat.

Question 10.20. As the problem is originally stated, Maine would have experienced an Alabama paradox with a House size of 357, 382, 386, or 389. In each of these cases, Maine would be better off if the size of the House was decreased by one seat. With 390 seats, Maine would be better off if the size of the House was decreased by 2 seats. Likewise, Colorado would have experienced an Alabama paradox with a house size of 357. (A note on terminology: Here we are saying that a state *experiences an Alabama paradox with House size* n if the state's apportionment with a House size of n is worse than its apportionment with a House size of $n - 1$.)

We should note that there is a minor error in the data on which this problem is based. According to Balinski and Young (see [5]), Maine would have actually been awarded three seats for any House size between 350 and 382 seats, and for House sizes of 386, 389, and 390 seats. Thus, Maine would actually have only experienced an Alabama paradox for House sizes of 386 or 389. (Many thanks to Roger Nelson for pointing this out to us.)

Questions 10.21 – 10.22. It might be worth revisiting these questions after your students complete the sections on Hill's method and Balinski and Young's theorem. As we note in the paragraph following the statement of Balinski and Young's theorem on page 209, the population paradox cannot occur under any divisor method. Since Hill's method is a divisor method and was also used for the 2002 apportionment of the House, it is impossible for the population paradox to have occurred in either of these questions. Note that, for Question 10.22, Nevada's population increased by around 66%, while the population of Illinois grew only by about 8.6%.

Question 10.24. Hill's method is biased (albeit just slightly) in favor of smaller states. As we note in the last two sentences immediately preceding Question 10.23, under Hill's method a quota with a decimal part of .482 would be rounded up if it had a whole number part such as 5 (*i.e.*, under Hill's method the quota 5.482 would be rounded up to 6), but a quota with the same decimal part of .482 would be rounded down if it had a whole number part such as 15 (*i.e.*, under Hill's method the quota 15.482 would be rounded down to 15). More generally, under Hill's method a particular decimal part is more likely to be rounded up if it is part of a smaller quota than a larger quota. Since smaller states produce smaller quotas, this means that Hill's method is biased in favor of smaller states.

Question 10.26. If Maine had been considered a separate state for the 1794 apportionment, Hamilton's, Hill's, and Webster's methods would have essentially given the same results: Massachusetts would have received 10 seats, and Maine would have received 3. Since Massachusetts (with Maine included) would have received 13 seats with each of these apportionment methods, splitting Maine off into a separate state would have produced exactly what one would expect. This, however, is not the case with Adams' or Jefferson's methods. Viewing Maine as a part of Massachusetts, Adams' method would allocate 12 seats to Massachusetts. However, with Maine considered a separate state, Massachusetts and Maine would receive 10 and 3 seats, respectively, for a total of 13. This extra seat would come at the expense of Connecticut, who would lose one of its 7 seats. The opposite phenomenon happens with Jefferson's method – Massachusetts would receive 13 seats before the split, and 10 and 2 after. In this case, Rhode Island would pick up the extra seat, going from 1 seat to 2.

Question 10.27. It is more likely that original standard quotas rounded according to Condorcet's method would give away too many seats than too few seats, since it is more likely that a larger number of quotas would be rounded up than rounded down. As such, it is more likely that the standard divisor would need to be modified higher than lower. A standard divisor modified higher would cause the standard quotas in the system to become smaller. Since quotas for larger states change more quickly than quotas for smaller states when standard divisors are modified, a larger state is more likely to have its quota fall below the .40 cutoff when the corresponding standard divisor is modified higher. Therefore, Condorcet's method favors smaller states.

Question 10.30. Your students may find it easier to do the research for this question if they use the alternative names for the methods given in the last paragraph on page 210.

Questions 10.32. Mathematically advanced students may wish to try proving these results on their own. For Balinski and Young's proof, see [5].

Question 10.39. In 1872, the House size was initially chosen to be 283, so that Hamilton's and Webster's methods would agree. However, the final apportionment was based on a House size of 292 seats, and did not agree with either Hamilton's or Webster's method.

Questions 10.44 – 10.45. It is definitely best to approach these questions with a spreadsheet program such as Microsoft Excel to help with the calculations. Thus, your students probably ought to do some work on Question 10.46 before attempting either of these questions. For Question 10.44, Hamilton's method would have yielded one additional seat for California, taking one seat away from Utah. This would have made Bush's electoral vote total 270 instead of 271, but it wouldn't have changed the outcome of the election. For Question 10.45, although 13 states would have had different apportionments using Jefferson's method, the electoral votes received by each candidate would have been exactly the same. Gore would have gained votes from 4 states (CA – 2 votes, NY, IL, and MI) and lost the same number of votes from 5 states (MN, IA, NM, HI, RI). Bush would have gained votes from 2 states (TX and FL) and lost votes from 2 states (NE and WV).

Supplementary Questions

Question 10.48. In 1991, a lawsuit, *Commonwealth of Massachusetts* v. *Mosbacher*, was filed in Massachusetts District Court. Find out the details of this lawsuit, and write a summary of your findings. Include in your summary the outcome of the lawsuit, and your own personal feelings about what the outcome should have been.

Question 10.49. Find a copy of the article "Outcomes of Presidential Elections and the House Size" by Michael G. Neubauer and Joel Zeitlin in the journal *PS: Political Science & Politics*, and write a summary of what it states regarding the relationship between the apportionment of the U.S. House of Representatives and U.S. presidential elections.

Question 10.50. Since the Constitution guarantees that each state will have at least one seat in the U.S. House of Representatives, the problem of apportioning the seats in the House is often viewed in a slightly different way to how we viewed it in this chapter. This different way of viewing the apportionment problem is to first give each state its guaranteed initial seat, and then to apportion the remaining seats to the states based on their populations. (The reason for viewing the apportionment problem in this different way is to ensure that the issue of a state being apportioned zero seats will never arise.) For example, for the apportionment of 105 seats to the

15 states whose populations are given in Table 10.1, this different way of viewing the apportionment problem would be to initially give each state one seat, and then to apportion the remaining 90 seats to the 15 states based on their populations. Would any of the apportionments in Questions 10.3, 10.11, 10.14, 10.16, or 10.23 be changed under this different way of viewing the apportionment problem?