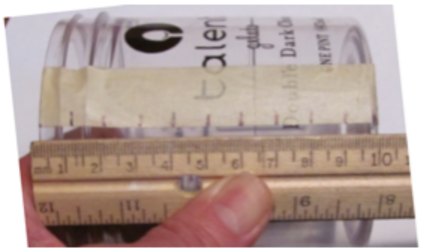
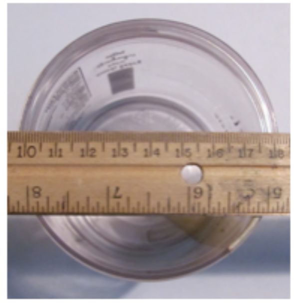
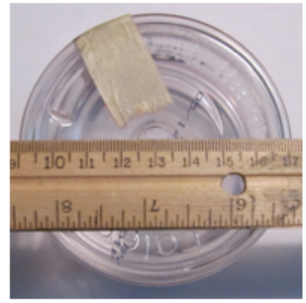
Testing Torricelli's Law

Prerequisite: Completion of Subsection 2.3.4 of *Differential Equations: Techniques, Theory, and Applications* by MacCluer, Bourdon, and Kriete

The container pictured below is a cylinder with radius 4 cm and height 10 cm. In its bottom, there is a hole of radius approximately 0.6 cm.

There is a dimple in the base of the container so that the hole in its bottom is about 0.8 cm above outer rim of its base. While its hole is plugged, the container is filled with colored water and then the plug is removed, allowing water to drain out. Watch the video of water draining from the container..

In this project you will use Torricelli's Law to model the draining of water from the container as shown in the video linked to above. The original video was captured at 30 frames per second and water heights (in centimeters above the hole in the container's bottom) were approximated at the times indicated in the table below.

Table I: Times to Height

Height (cm above hole)	Time (sec from start of video)	Time Elapsed after Plug Removal (sec)
9.2	1 + 19/30	$t_0 = 0$
8.2	2 + 3/30	$t_1 = 14/30 \approx .47$
7.2	2 + 18/30	$t_2 = 29/30 \approx .97$
6.2	3 + 7/30	$t_2 = 48/30 \approx 1.60$
5.2	3 + 27/30	$t_4 = 68/30 \approx 2.27$
4.2	4 + 18/30	$t_5 = 89/30 \approx 2.97$
3.2	5 + 11/30	$t_6 = 112/30 \approx 3.73$
2.2	6 + 6/30	$t_7 = 137/30 \approx 4.57$
1.2	7 + 11/30	$t_8 = 172/30 \approx 5.73$
0.2	9 + 15/30	$t_9 = 236/30 \approx 7.87$
0	10 + 22/30	$t_{10} = 273/30 \approx 9.1$

Let $h(t)$ be the height of water in the container t seconds after its plug is removed (these times appearing in the right-most column of the preceding table), where height is measured in centimeters above the hole through which water drains. Thus, e.g., $h(29/30) = 7.2$. In Subsection 2.3.4 of the text, we obtained the following differential equation for h based on Torricelli's Law:

$$\frac{dh}{dt} = \frac{-0.6\pi r^2\sqrt{2gh}}{A(h)}, \tag{1}$$

where r (in cm) is the radius of the circular hole in the bottom of the container, $g \approx 980$ cm/sec^2 is the acceleration due to gravity, and $A(h)$ is the area of a cross section of the container at height h. For our container, $A(h)$ does not vary with h because for $0 < h \leq 9.2$, the cross section of the container at height h is a disk of radius 4 cm.

Hence, for our container, the model of (1) takes the form

$$\frac{dh}{dt} = -k\sqrt{h} \quad \text{for some positive constant } k. \tag{2}$$

(a) Solve the following initial-value problem by separating variables, applying the initial condition, and then solving for h to obtain an explicit solution:

$$\frac{dh}{dt} = -k\sqrt{h} \quad h(0) = 9.2. \tag{3}$$

In your solution, h will depend on t and k. To emphasize the dependence of your formula on both t and k, label your explicit solution $h(t,k)$. You will be working with the function h in *Mathematica*. The accompanying notebook, "Torricelli.nb" shows how to define $h(t,k)$ as a function of t and k.

(b) You will be using *Mathematica* to complete the remaining parts of this project based on the notebook "Torricelli.nb". You'll use the given dimensions of the container, $g = 980$ cm/sec^2, and

$$k = \frac{0.6\pi r^2\sqrt{2g}}{A(h)}$$

to find a value for k, which you should label k_1. You'll substitute this value k_1 into your solution of part (a). You'll then have a single-variable formula $h(t,k_1)$ modeling the height of water in the tank at time t. According to the model, how long will it take for the water in the container to reach the level of the hole in its bottom? That is, when will $h(t,k_1) = 0$? *Mathematica* will produce a solution to this equation, but you may solve this equation by hand, with the assistance of a calculator. Let t_* be the "drain time" you just found using the model. Let t_e be the experimentally determined value for the drain time. From Table I t_e is about 9.1 seconds. Compute the relative percent error in the model's value t_* for drain time:

$$\frac{|t_* - t_e|}{t_e} \times 100.$$

(c) Return to your explicit solution "$h(t,k)$" from part (a), where k is not determined. Use $h(14/30) = 8.2$ (from Table I) to find a value for k, which you should label k_2. Substitute this value k_2 into your solution of part (a), obtaining a second model for water height. Solve $h(t,k_2) = 0$ to determine the drain time predicted by this model. Let t_{**} be the time you just found. Compute the relative percent error in this model's value t_{**} for drain time:

$$\frac{|t_{**} - t_e|}{t_e} \times 100,$$

where $t_e = 9.1$.

(d) Plot, over the interval $0 \le t \le 10$ on the same set of axes, the functions values $h(t,k_1)$, $h(t,k_2)$, as well as the tabulated water-height data on the preceding page. The *Mathematica* notebook "Torricelli.nb" illustrates how to do this. You'll see that neither plot fits the experimental data especially well. This does not mean that the model (2) is a mediocre one for the draining of water from our container. There is a value of k for which this model is quite good. In the next part of the project, you will both find such a k and assess how good it is both visually and through a "root-mean-square error computation".

(e) Return once more to your expression for h from part (a). We seek a value of k such that $h(t,k)$ provides a "best fit" for the experimental data tabulated above. The method of "least squares" is a common way for choosing such a k. Following the least-squares process, we choose k such that the following sum of squares assumes its minimum value:

$$S(k) = \left[\sum_{j=1}^{9}(h(t_j,k) - (9.2 - j))^2\right] + (h(t_{10},k) - 0)^2,$$

where t_j, for $1 \le j \le 9$, is the time (from Table I) when the value of h should be $9.2 - j$ and t_{10} is the time when h should be 0.[1] Note that the j-th term of the sum for $S(k)$, namely $(h(t_j,k) - (9.2 - j))^2$, is the square of the error that results when the model's value for height at time t_j is used in place of the actual (experimentally obtained) value. Using commands provided in "Torricelli.nb", define the sum $S(k)$ and plot it over the interval $0 \le k \le 1$. The plot will allow you to approximate the value of k that will minimize $S(k)$ and then you'll use calculus to find a better approximation for k. Call this "least-squares" value you obtain through calculus k_{LS}.

- Plot $h(t,k_{LS})$, $0 \le t \le 10$, along with the experimental data for water heights. Again, use"Torricelli.nb" to guide your work. You should get a very good fit!
- One way of quantifying the extent to which the fit obtained in Part (f) is good is to compute the root-mean-square error that results when $h(t,k_{LS})$ is used to approximate the experimentally obtained data for heights at the times given in Table I. This root-mean-square error is given by

$$\text{RMS Error LS} = \sqrt{\frac{1}{10}\left[\sum_{j=1}^{9}(h(t_j,k_{LS}) - (9.2 - j))^2 + (h(t_{10},k_{LS}) - 0)^2\right]}$$

Compute it. Then compute

$$\text{RMS Error 1} = \sqrt{\frac{1}{10}\left[\sum_{j=1}^{9}(h(t_j,k_1) - (9.2 - j))^2 + (h(t_{10},k_1) - 0)^2\right]},$$

[1] In the sum defining $S(k)$, we omit consideration of $(h(t_0,k) - 9.2)^2$ because at $t_0 = 0$, the function h takes the value 9.2, making $h(t_0,k) - 9.2) = 0$ independent of k.

and

$$\text{RMS Error 2} = \sqrt{\frac{1}{10}\left[\sum_{j=1}^{9}(h(t_j,k_2)-(9.2-j))^2 + (h(t_{10},k_2)-0)^2\right]}.$$

Interpretation: Focus on the first formula, that for RMS Error LS. Suppose, e.g., all the squared errors take the same value; that is, there is a constant c such that $(h(t_j, k_{LS}) - (9.2 - j))^2 = c$ for $1 \leq j \leq 9$ and $(h(t_{10}, k_{LS}) - 0)^2 = c$ as well. Then RMS Error LS reduces to c. Thus, if your RMS Error LS is, say, 0.1, you can view overall error to be comparable to a situation where each individual error, $(h(t_j, k_{LS}) - (9.2 - j))$, is 0.1. (Here the error 0.1 would have units of cm.)

Remark: The ideas used above to obtain the parameter k that provides the mathematical model that best fits the data are quite portable, allowing you to test other models and explore their possible predictive power. For example, the constants γ and σ given in part (c) of Problem 24 of the exercise set for Section 2.1 were obtained via a least-squares computation. The astonishing accuracy of the resulting model for Usain Bolt's record-breaking 100-meter dash at the 2008 Beijing Olympics raises many interesting questions.