February 25, 2010

Errata and Clarifications
Continuous Symmetry: Euclid to Klein

Particularly important corrections/revisions will be marked in red.

AMS Webpage for *Continuous Symmetry*: **www.ams.org/bookpages/mbk-47**

This errata sheet will be updated as additional items are brought to our attention. We appreciate receiving not only errors to correct, but also suggestions for revisions and more general comments about the text.

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I. Foundations of Geometry in the Plane.

We wish to thank **Robin Hartshorne** for pointing out certain errors and misstatements in Chapter I. Most of the revisions and corrections given below for this chapter are from his comments. *For convenience, a revised version of Chapter I incorporating all these changes will be posted on the AMS webpage for "Continuous Symmetry."*

Pages 17+, §3: *Distance and the Ruler Axiom.* The *Ruler Axiom* should be considered more properly as an assumption of additional *undefined objects* (*data*) for our system, i.e., the assumption of a *coordinate system* for each line in the plane. Hence there should be no "Ruler Axiom." It should be replaced by the assumption, for each line ℓ, of a one-to-one correspondence $\chi_\ell : \ell \to \mathbb{R}$. The *distance function* is then defined from this designated collection of coordinate systems $\{\chi_\ell\}_{\ell \in \mathcal{L}}$ in Definition 3.2.

The subsequent required changes throughout Chapter I are generally clear: appeals to the *Ruler Axiom* become appeals to the assumed collection of coordinate systems and the subsequent distance function on the plane. These changes are noted below and are designated by an asterisk (*).

***Page 17**, §3: *Distance and the Ruler Axiom.* Eliminating the *Ruler Axiom* requires the title of this section to be changed to *Distance and Coordinate Systems on Lines*. We then change the start of the section to read as follows:

> In Euclidean geometry there exists a *distance* between any two points in the plane. We build this into our model via the assumption of a *coordinate system* for each line.
>
> The desire is that each line ℓ appear to be a "copy" of the real number line $\mathbb{R}$, which at the very least requires the existence of a *one-to-one correspondence* χ from ℓ to $\mathbb{R}$. Such a mapping χ is termed a *coordinate system* on ℓ.
>
> **Definition 3.1**.
>
> A **coordinate system** χ on a line ℓ is a *one-to-one correspondence* $\chi : \ell \to \mathbb{R}$.
>
> We now assume, for each line ℓ in $\mathcal{E}$, the existence of a fixed coordinate system $\chi_\ell : \ell \to \mathbb{R}$. This adds the coordinate systems to our collection of *undefined objects* of the previous section; the properties of the coordinate systems will be specified by subsequent axioms.

> Notice a simple but important consequence of our assumption: *every line has an infinite number of distinct points*. For suppose ℓ is a line. Then the assumed coordinate system $\chi_\ell : \ell \to \mathbb{R}$ is a one-to-one correspondence between ℓ and $\mathbb{R}$. Since $\mathbb{R}$ is infinite, this means ℓ must also be infinite, as claimed. For this reason, the "three point geometry" of Exercise 2.3 will no longer satisfy the axiomatic system we are building.
>
> Given a line ℓ, any two points p, q in ℓ are identified with two points $\chi_\ell(p)$, $\chi_\ell(q)$ in $\mathbb{R}$. But points in $\mathbb{R}$ have a ***distance*** defined between them;...[*Now continue with the original text on line 3 of the second paragraph on page 18*.]

***Page 20**, Exercise 3.2a. Drop the second line "This shows that our model...."

***Page 21**, Exercise 3.3, line 3. Replace "the *Ruler Axiom*" with "the existence of coordinate systems."

lines 4-5. Replace "the *Ruler Axiom*" with "such that each line has a coordinate system."

part (c), line 2. Replace "the *Ruler Axiom*" with "a coordinate system for each line."

***Page 21**, Exercise 3.4, line 8. Drop the last line "Hence $\mathbb{MP}^2$ satisfies the *Ruler Axiom*."

***Page 22**, §4: *Betweenness*, first sentence. Replace "...we used the *Ruler Axiom* on each line ℓ in the plane to fix a coordinate system $\chi_\ell : \ell \to \mathbb{R}$ on ℓ and to use these coordinate systems to define a distance function..." with "...we used the coordinate systems $\chi_\ell : \ell \to \mathbb{R}$ for all lines ℓ in the plane to define a distance function...."

***Page 27**, §5: *The Plane Separation Axiom*, first sentence. Replace "three" with "two."

***Page 30**, paragraph following Figure 5.8. Replace "four" with "three."

Page 34, §6: *The Angular Measure Axioms.* The wording at the start of this section should more clearly state that we are assuming a new *undefined object*: an *angle measure function* $m : \mathcal{A} \to \mathbb{R}$ that is to satisfy the four *Angle Measure Axioms*. A more serious misstatement is found in the third paragraph: "...this list of axioms is sufficient to completely characterize angular measure..." should read "...this list of axioms, when combined with the other axioms for Euclidean geometry as summarized in §15, is sufficient to completely characterize angular measure...."

To handle both of these issues, here is a replacement for the first three paragraphs of §6:

> Let $\mathcal{A}$ denote the collection of all angles in the plane. In this section we assume a new *undefined object* for our axiomatic system: an **angle measure function** $m : \mathcal{A} \to \mathbb{R}$ that assigns to each angle $\angle A$ a real number $m\angle A$, understood to be its measure in degrees.
>
> The basic properties of this function are specified by four *Angle Measure Axioms*. None of these axioms will surprise you — they are all intuitively clear from our usual notion of angle measurement. What is less obvious is that this list of axioms, when combined with our other axioms for Euclidean geometry as summarized in §15, is sufficient to completely characterize angular measure, i.e., no other notion of angular measure can satisfy this complete set of axioms.

Page 46, §7: *Triangles and the SAS Axiom.* The first paragraph makes a seriously flawed claim. "None of our eight current axioms establish connections between distances measured along two *different* lines. ... There is — thus far — no axiom that requires any form of 'compatibility' between the coordinate systems chosen for different lines, and hence there is no relationship between distances measured along distinct lines." This is false: the *Plane Separation Axiom* postulates a powerful relationship between the notion of betweenness on different lines, creating a compatibility between coordinate systems on different lines. Fortunately we never used this incorrect statement. What follows is a replacement for the first two paragraphs of §7 that corrects our error.

> We assume a coordinate system for each line in the plane and use these coordinate systems to define a distance function for $\mathcal{E}$. However, among our current axioms, only the *Plane Separation Axiom* requires any "compatibility" between coordinate systems on different lines. This results from relationships established by the axiom linking *betweenness* on different lines. We thus have compatibility restrictions between the distance functions along distinct lines, even if the nature of these restrictions is somewhat obscure.
>
> In this section we add an axiom that explicitly involves the side lengths of triangles — it will result in greatly strengthen "global" distance relationships between coordinate systems on different lines.

***Page 47**, paragraph preceding the *SAS Axiom*. Replace "eight" with "seven."

Page 49, caption of Figure 7.10. Triangle $\triangle de_0 f$ should be $\triangle AB_0C$.

Page 54, Exercise 7.3. In addition to removing the *Ruler Axiom*, this exercise should be reorganized to start with coordinate systems on lines (postulated to exist in our axiomatic scheme) rather than a distance function (which we construct from the coordinate systems). Reword the initial paragraph and part (a) as follows:

> In this exercise you will show that the *SAS Axiom* is *independent* of the previous seven axioms, i.e., that there is a system in which the first seven axioms are valid but the *SAS Axiom* is false. We define such an "aberrant" system as follows. Suppose $\mathcal{E}$ is a set, $\mathcal{L}$ a collection of special subsets of $\mathcal{E}$ called *lines*, each line $\ell \in \mathcal{L}$ having a coordinate system χ_ℓ, and m an angle measure function. Let d be the corresponding distance function, and assume our axiomatic system satisfies all of the eight axioms up to and including *SAS*.
>
> We define a new axiomatic system by altering just one coordinate system as follows. Pick a specific line ℓ_0 in $\mathcal{E}$ with coordinate system $\chi = \chi_{\ell_0}$ and define a new coordinate system on ℓ_0 by $\eta(p) = 2\chi(p)$ for every point p on ℓ_0 (see Exercise 3.1a). Let ρ denote the new distance function that results on $\mathcal{E}$ by replacing χ with η. You first studied such a distance function alteration in Exercise 3.3.
>
> (a) Show that, for any two points p and q in $\mathcal{E}$, we have
>
> $$\rho(p,q) = \begin{cases} 2d(p,q) & \text{if } p \text{ and } q \text{ \emph{both} lie on } \ell_0, \\ d(p,q) & \text{in all other cases.} \end{cases}$$
>
> and that the first seven axioms are valid for our new axiomatic system where η replaces χ and consequently ρ replaces d.

***Page 59**, last sentence in proof of Corollary 8.11. Replace "*Ruler Axiom*" with "definition of distance in Definition 3.2."

Page 78, Exercise 10.2. Drop "distinct" in line one and add the following at the end: "if these two lines are distinct."

***Page 101**, line 10. Replace "*Ruler Axiom*" with "coordinate system on $L_1 = \overleftrightarrow{py_1}$."

***Page 116**, line -6. Replace "*Ruler Axiom*" with "existence of a coordinate system for each line."

***Page 118**, line 1. Replace "the *Ruler Axiom*" with "coordinate systems."

line 2. Replace "*Ruler Axiom*" with "*Ruler Placement Theorem* (Proposition 3.6)."

line 17. Replace "*Ruler Axiom*" with "*Ruler Placement Theorem* (Proposition 3.6)."

***Pages 119+**, §15: *Axioms for the Euclidean Plane.* Remove the *Ruler Axiom*.

II. Isometries in the Plane: Products of Reflections.

III. Isometries in the Plane: Classification and Structure.

Page 180, Exercise 1.8a. In the hint change Theorem 1.12b to Theorem 1.12a.

IV. Similarities in the Plane.

Page 235, Proof of Prop 4.1, line 6. $\overrightarrow{\phi(x)y}$ should be $\overrightarrow{\phi(p)y}$.

V. Conjugacy and Geometric Equivalence.

VI. Applications to Plane Geometry.

VII. Symmetric Figures in the Plane.

VIII. Frieze and Wallpaper Groups.

IX. Area, Volume, and Scaling.