HADAMARD: ELEMENTARY GEOMETRY
SOLUTIONS AND NOTES TO SUPPLEMENTARY PROBLEMS

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FOREWORD

This addendum to the Reader's Companion completes the solutions and annotations to the problems in Hadamard's elementary geometry text (*Lessons in Geometry. I. Plane Geometry*, Jacques Hadamard, Amer. Math. Soc. (2008)). Because of space constraints, they were not included in the printed volume containing solutions to most of Hadamard's problems (*Hadamard's Plane Geometry: A Reader's Companion*, Mark Saul, Amer. Math. Soc. (2010)).

Hadamard revised his geometry textbook a dozen times during his long life, and kept adding interesting problems to the miscellany he had provided in the original edition. So this collection of problems is the result of Hadamard's lifetime love of synthetic Euclidean geometry. The textbook itself is written in an abbreviated, almost telegraphic style. The problems offered for each chapter invite the reader to unfold the compressed exposition, discovering its implications through hands-on experience with geometry.

These problems have a somewhat different feel to them. Their results sometimes take us far from the main stream of the exposition in the text. Often, they require a deep, almost virtuosic control of the techniques and theorems offered in the text. Many of them generalize or build on problems mentioned earlier. While some of these problems fall more easily to advanced (i.e. analytic) methods, we have here provided solutions for the most part within the bounds of Hadamard's text and his synthetic approach.

As with most large efforts, much credit must be given to others. Behzad Mehrdad patiently reviewed each piece of the manuscript, offering valuable insights, correcting errors, and in some cases providing improved solutions to the problem. Alexei Kopylov likewise suggested significant corrections to and improvements on the original manuscript. I am also indebted to the Education Development Center (EDC) and Al Cuoco in particular for significant and sustained support in this translation project.

But the lion's share of the work was actually done by others. More than is the case with the problems in the textbook, these solutions are often based on those of D. I. Perepelkin and his colleagues, who prepared the Russian edition of Hadamard's book (*Gosudarstvenoye Uchebno-Pedagogicheskoye Izdatelstvo Ministerstva Prosveshcheniya RSFSR* Moscow, 1957). Perepelkin's solutions are written for a professional mathematical audience, and even where his ideas are central to the solution, the exposition has been reworked significantly. Of course, I take responsibility for any errors that have crept in during this process.

The work on Hadamard's *Elementary Geometry* was initiated under National Science Foundation grant ESI 0242476-03 to the Educational Development Center. Additional support was provided by the John Templeton Foundation through a grant to the Mathematical Sciences Research Institute, and the Alfred P. Sloan Foundation through a grant to the Courant Institute of Mathematical Sciences at New York University.

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August 2011

PROBLEMS, SOLUTIONS, AND NOTES

Exercises 349, 350, 353, 354,384, 386, 387, 393, 394, 400, 404, 413 are taken from the General Competition of Lycées and Colleges. Exercises 365, 374, 397, 406, 409, 412, 421 were taken from the contest of the Assembly of the Mathematical Sciences. We have not felt obligated to give these problems in the form in which they were originally proposed; we have, in particular, made certain changes in their formulation to correspond to other exercises given in the rest of this work.

Problem 343. If $A,\ B,\ C,\ D$ are four points on a circle (in that order), and if $a,\ b,\ c,\ d$ are the midpoints of arcs $\widehat{AB},\ \widehat{BC},\ \widehat{CD}, \widehat{DA}$, show that lines $ac,\ bd$ are perpendicular.

Solution. We have $\widehat{aB}= \frac{1}{2}\ \widehat{AB},\ \widehat{Bb}= \frac{1}{2}\ \widehat{BC},\ \widehat{cD}= \frac{1}{2}\ \widehat{CD},\ \widehat{Dd}= \frac{1}{2}\ \widehat{DA}$.

Thus $\widehat{ab} + \widehat{cd}= \frac{1}{2}\left(\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA}\right) = 180°$, and since the angle between lines ac and bd is measured by half this arc, it must be a right angle.

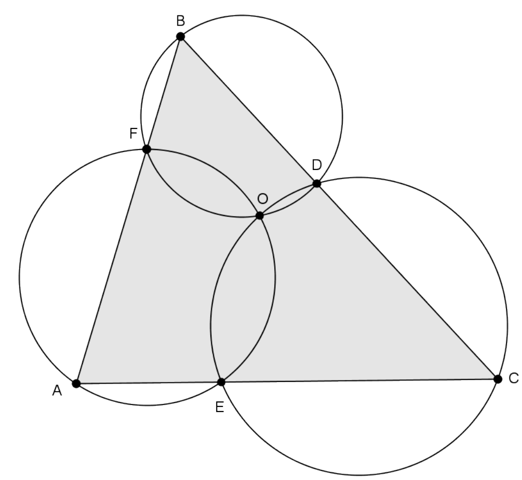

FIGURE t344a

Problem 344. We take points $D,\ E,\ F$ on sides $BC,\ CA,\ AB$ of a triangle, and construct circles AEF, BFD, and CDE. Prove that :

1^o. These three circles are concurrent at a point O;

2^o. If an arbitrary point P in the plane is joined to $A,\ B,\ C$, then the new points $a,\ b,\ c$ where $PA,\ PB,\ PC$ intersect these circles belong to a circle passing through O and P.

Solution. (1°) Suppose circles AEF and BFD intersect at point O (*fig.* t344a). We must show that O is on the circle through $C,\ E$, and D as well. We will do this by showing that quadrilateral $CEOD$ is cyclic, using the criterion of **80**.

Since quadrilateral $AEOF$ is cyclic, we know (**79**) that $\widehat{EOF} = 180^\circ - \widehat{CAB}$. Since quadrilateral $BFOD$ is cyclic, we have $\widehat{FOD} = 180^\circ - \widehat{ABC}$. By adding the angles about point O, we see that $\widehat{EOD} = 360^\circ - (\widehat{EOF} + \widehat{FOD}) = 360^\circ - (180^\circ - \widehat{CAB} + 180^\circ - \widehat{ABC}) = \widehat{CAB} + \widehat{CBA} = 180^\circ - \widehat{ACB}$ (this last because the three angles involved are the angles of triangle ABC). Thus the opposite angles of $CEOD$ are supplementary, and point O must lie on the circle through C, E, and D.

Note. This is a relatively easy problem. If students need a hint, it is usually enough to suggest using cyclic quadrilaterals and the criteria of **79** and **80**. Sometimes students have a slight problem imagining that O is the intersection of only two of the circles, and will make some assumption in their argument equivalent to the assumption that O is on the third circle as well. This error is worth examining in detail in the classroom.

In constructing a diagram with dynamic software, students may discover that the proof needs modification in certain cases. For some positions of D, E, and F, point O will end up outside the triangle, and the quadrilaterals referred to above won't lead to a solution. But other quadrilaterals will, and students can look at various such cases to convince themselves that the argument does not essentially change. Certain angles that are equal in some cases are supplementary in others, and vice versa.

In more advanced work, greater unity is achieved if we consider angles as oriented, defining equality of angles slightly differently. See the note below.

(2°) The solution to this exercise is again straightforward, but the relationship of the elements in the figure offer many different cases for exploration. We give just one here. We will show that the quadrilateral with vertices P, a, O, b (in some order) is cyclic. The argument can be repeated for the quadrilateral with vertices P, b, c, O, which proves the theorem.

In figure t344b, $\widehat{PaO}$ is supplementary to $\widehat{AaO}$, which in turn is supplementary to $\widehat{AFO}$ (because these are opposite angles in cyclic quadrilateral $AaOF$). Hence $\widehat{PaO} = \widehat{AFO}$.

Similarly, using cyclic quadrilateral $BbOF$, we see that $\widehat{BbO}$ is supplementary to both $\widehat{PbO}$ and $\widehat{BFO}$, so these last two angles are equal. But clearly $\widehat{AFO}$ and $\widehat{BFO}$ are supplementary, so $\widehat{PaO}$ and $\widehat{PbO}$ are also supplementary, showing that $PaOb$ is cyclic.

To show that $PcOb$ is cyclic, we note that $\widehat{PbO}$ supplements $\widehat{BbO}$, which is equal to $\widehat{BDO}$ (they both intercept arc $\widehat{BFO}$). And $\widehat{BDO}$ supplements $\widehat{CDO}$, so $\widehat{PbO} = \widehat{CDO}$. Finally, from cyclic quadrilateral $CDOc$, we see that $\widehat{CDO}$ supplements $\widehat{CcO} = \widehat{PcO}$, so $\widehat{PbO}$ also supplements $\widehat{PcO}$. This last statement shows that quadrilateral $PcOb$ is cyclic, and the argument is completed as indicated above.

Note. In other cases, we can make use of angles $\widehat{PaO}$, $\widehat{AaO}$, $\widehat{OFB}$, and so on. In each case, certain angles are equal, and others are supplementary. In advanced work, we consider oriented angles, and define 'equality' so that 'equal' oriented angles are either equal or supplementary, if orientation is not considered.

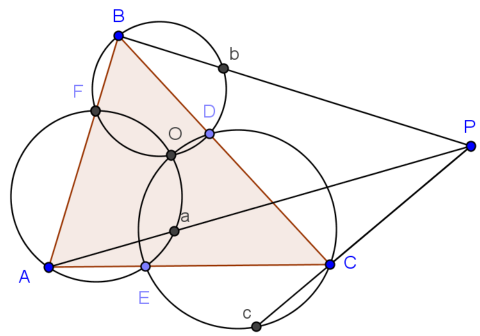

FIGURE t344b

The solution we have presented here can motivate this step, which students and teachers can take using other references. See, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, New York: Dover Books, 2007 (reprint of 1929 edition).

See also exercise 214.

Problem 345. With each side of a cyclic quadrilateral $ABCD$ as a chord, we draw an arbitrary circular segment. The four new points A', B' ,C', D' where each of these four circles S_1, S_2, S_3, S_4 intersects the next are also the vertices of a cyclic quadrilateral.

Solution. Since quadrilateral $ABB'A'$ (figure t345) is cyclic, we know that $\widehat{A'AB}+\widehat{BB'A'} = 180°$. Since quadrilateral $BCC'B'$ is cyclic, we know that $\widehat{BCC'}+\widehat{BB'C'} = 180°$. Adding, we find that $\widehat{A'AB} + \widehat{BB'A'} + \widehat{BCC'} + \widehat{BB'C'} = 360° = \widehat{BB'A'}+\widehat{A'B'C'}+\widehat{BB'C'}$, or $\widehat{A'AB}+\widehat{BCC'} = \widehat{A'B'C'}$. Similarly, using other cyclic quadrilaterals, we can prove that $\widehat{DAA'} + \widehat{DCC'} = \widehat{A'D'C'}$.

Adding again, we find $\widehat{A'AB} + \widehat{BCC'} + \widehat{DAA'} + \widehat{DCC'} = \widehat{A'B'C'} + \widehat{A'D'C'}$. But the first sum is equal to $\widehat{DAB} + \widehat{DCB} = 180°$, because these are the opposite angles of cyclic quadrilateral $ABCD$. Hence $\widehat{D'A'B'} + \widehat{D'C'B'} = 180°$, so that $A'B'C'D'$ is cyclic.

Note. The result is slightly more general than is indicated by the figure. The four points can be in any order on the circle, and the result will hold. That is, the given cyclic quadrilateral can have 'sides' which intersect (we don't usually consider these true quadrilaterals), and the result will hold.

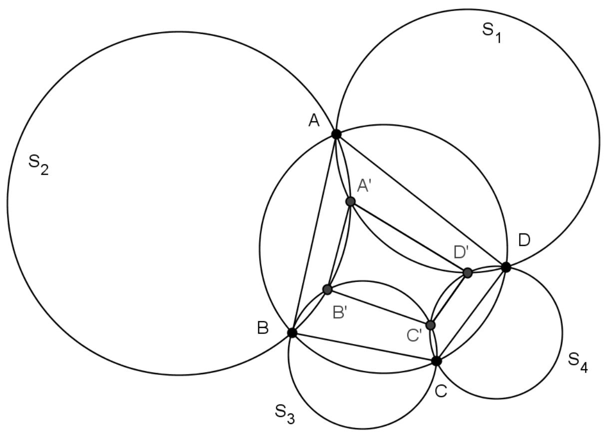

FIGURE t345

Students can be challenged to show that this is true. They will find that essentially the same proof holds, with certain equal pairs of angles replaced by supplementary pairs. In more advanced work with oriented angles, all these cases can be treated at once. (See the note to exercise 341)

Problem 346. Two circles S_1, S_2 intersect at A, A'; S_2 and a third circle S_3 intersect at B, B'; S_3 and a fourth circle S_4 at C, C'; and S_4, S_1 at D, D'. A condition for quadrilateral $ABCD$ (and, by the previous exercise, quadrilateral $A'B'C'D'$) to be cyclic is that the angle between S_1 and S_2, plus the angle between S_3 and S_4 (these angles being taken with an appropriate orientation[1]) be the same as the angle between S_2 and S_3 plus the angle between S_4 and S_1.

Solution. We will not give a general solution using oriented angles: this is beyond the scope of Hadamard's exposition. But we will give an indication, as in other solutions, of how some of the special cases can be treated more uniformly.

Lemma: The angle between two circles is equal to the sum of the inscribed angles in each circle which cut off the arc between the circles' points of intersection.

[1]One should try to orient the angles, using, as needed, the conventions of Trigonometry (*Leçons* de BOURLET, book I, chapter I) in order to give a proof which applies for all possible cases of the figure.

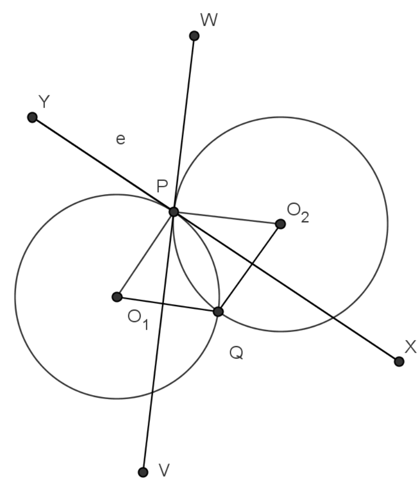

Figure t346a

Proof: In figure t346a, circles O_1 and O_2 intersect at points P and Q, and XY, VW are tangents at P. We take the angle between the circles to be $\widehat{YPW}$. Then, adding the angles about point P, we find $\widehat{YPW}+\widehat{WPO_2}+\widehat{O_2PO_1}+\widehat{O_1PY} = 360° = \widehat{YPW} + \widehat{O_2PO_1} + 180°$, since $PY \perp O_1P$ and $PW \perp O_2P$. Thus we have $\widehat{YPW} + \widehat{O_2PO_1} = 180°$, or

$$(1) \qquad 2\widehat{YPW} + 2\widehat{O_2PO_1} = 360°.$$

On the other hand, adding the interior angles of quadrilateral O_1PO_2Q, we have

$$(2) \qquad \widehat{O_1PO_2} + \widehat{PO_2Q} + \widehat{O_2QO_1} + \widehat{QO_1P} = \widehat{PO_2Q} + \widehat{QO_1P} + 2\widehat{O_1PO_2} = 360°,$$

since the quadrilateral is symmetric around line O_1O_2.

It follows from (1) and (2) that $2\widehat{YPW} = \widehat{PO_2Q}+\widehat{QO_1P}$, or $\widehat{YPW} = \frac{1}{2}\widehat{PO_2Q}+\frac{1}{2}\widehat{QO_1P}$. Since an inscribed angle is half of a central angle with the same arc, this proves the lemma.

We turn now to the proof of the main statement. Figure t346b shows the four circles and the quadrilateral in question. The interior angles of the quadrilateral are each broken, by the segments shown, into four smaller angles labeled a through p in the figure.

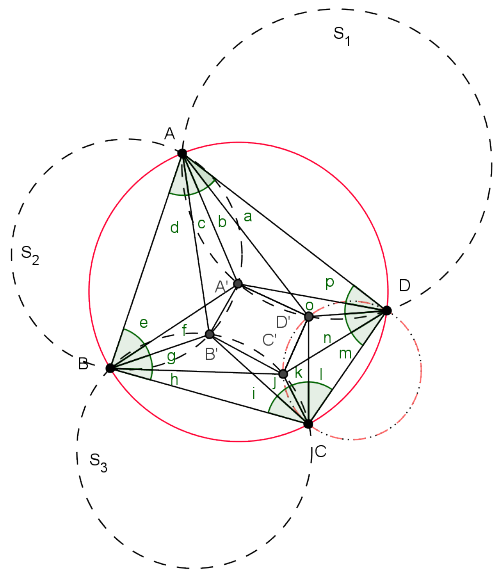

FIGURE t346b

Let the angle between S_1 and S_2 (not shown in figure t346b) be $\widehat{\alpha}$, the angle between S_2 and S_3 be $\widehat{\beta}$, the angle between S_3 and S_4 be $\widehat{\gamma}$, and the angle between S_4 and S_1 be $\widehat{\delta}$.

We must show that if $\widehat{\alpha} + \widehat{\gamma} = \widehat{\beta} + \widehat{\delta}$, then $\widehat{a} + \widehat{b} + \widehat{c} + \widehat{d} + \widehat{i} + \widehat{j} + \widehat{k} + \widehat{l} = \widehat{e} + \widehat{f} + \widehat{g} + \widehat{h} + \widehat{m} + \widehat{n} + \widehat{o} + \widehat{p}$

By our lemma, we know that:

$$\widehat{\alpha} = \widehat{e} + \widehat{p},$$
$$\widehat{\beta} = \widehat{d} + \widehat{i},$$
$$\widehat{\gamma} = \widehat{h} + \widehat{m},$$
$$\widehat{\delta} = \widehat{l} + \widehat{a},$$

Thus we have:

$$\widehat{e} + \widehat{p} + \widehat{h} + \widehat{m} = \widehat{d} + \widehat{i} + \widehat{l} + \widehat{a}. \tag{1}$$

We will build the required sum around this equality, using the cyclic quadrilaterals in the figure.

From cyclic quadrilateral $ABB'A'$, we have $\widehat{f} = \widehat{c}$. Similarly, using the other cyclic quadrilaterals, we have:

$$\widehat{g} = \widehat{j}$$
$$\widehat{n} = \widehat{k}$$
$$\widehat{o} = \widehat{b}$$

Adding these equalities to (1), we get the desired result

Note. This problem is not that difficult, but it is complicated. Students generally can get the result themselves, especially given the lemma (which is interesting in its own right).

As with other such problems, the result holds even when $ABCD$ is not a convex quadrilateral: it is true for any four points which are the intersections of pairs of circles from among four circles. Again, the general proof requires the use of oriented angles. See the note to exercise 344.

Problem 347. Consider four circles $S_1,\ S_2,\ S_3,\ S_4$, and pairs of common tangents of the same kind (that is, both internal or both external) $\alpha,\ \alpha'$ for $S_1,\ S_2$; $\beta,\ \beta'$ for $S_2,\ S_3$; $\gamma,\ \gamma'$ for $S_3,\ S_4$; $\delta,\ \delta'$ for $S_4,\ S_1$. If there is a circle tangent to $\alpha,\ \beta,\ \gamma,\ \delta$, then there is also a circle tangent to $\alpha',\ \beta',\ \gamma',\ \delta'$. A condition for the existence of such circles is that the sum of the lengths of two of these tangents (between their points of contact) be equal to the sum of the other two.

Solution. We give the solution for just one of the many possible cases described in the problem statement, a case involving pairs of common external tangents, leaving for students various other cases. The proof is not difficult in concept, although a written explanation is a bit complicated. For clarity, equal tangent segments in the diagrams bear the same lower-case letter.

Recall that a necessary and sufficient condition that a given quadrilateral have an inscribed circle is that the sums of the opposite sides of the quadrilateral are equal. In the present case, we assume (figure t347a) that we know this for quadrilateral $PQRS$: that

$$PS + QR = PQ + RS;$$

that is,

$$\delta + \beta = \alpha + \gamma. \tag{1}$$

We must prove that the corresponding sums are equal for quadrilateral $ABCD$. Using (1) and also the segments marked in figure 347a, we see that $(\alpha + p + q) + (\gamma + r + s) = (\delta + p + s) + (\beta + q + r)$. But $\alpha + p + q = \alpha'$, since they are the two common external tangents of circles $S_1,\ S_2$. And $\gamma + r + s = \gamma'$, since they are the common external tangents of circles $S_3,\ S_4$. Similarly, $\beta + q + r = \beta'$ and $\delta + p + s = \delta'$. Thus we have $\alpha' + \gamma' = \beta' + \delta'$. Adding to this equation pairs of equal tangent segments we get the desired result.

That is, we have $(\alpha' + a + b) + (\gamma' + c + d) = (\beta' + b + c) + (\delta' + a + d)$, and this is the statement required: that $AB + CD = AD + BC$.

Note. A subtle point in this problem is that there is much implicit in the diagram: more than just the equality of certain segments. For the problem statement does not tell use which pairs of the eight common external tangents form the quadrilateral which is assumed to have an inscribed circle.

For example, if we decide to choose $AB,\ BC,\ CD,\ DA$ as the original common tangents referred to in the problem statement, then (still using figure t347a) the given argument, traced backwards, assures us that $PQRS$ has an inscribed circle.

However, Figure t347b shows a case in which $PQRS$ is formed by common external tangents, but the 'wrong' pairs. In this case, $ABCD$ does not have an inscribed circle.

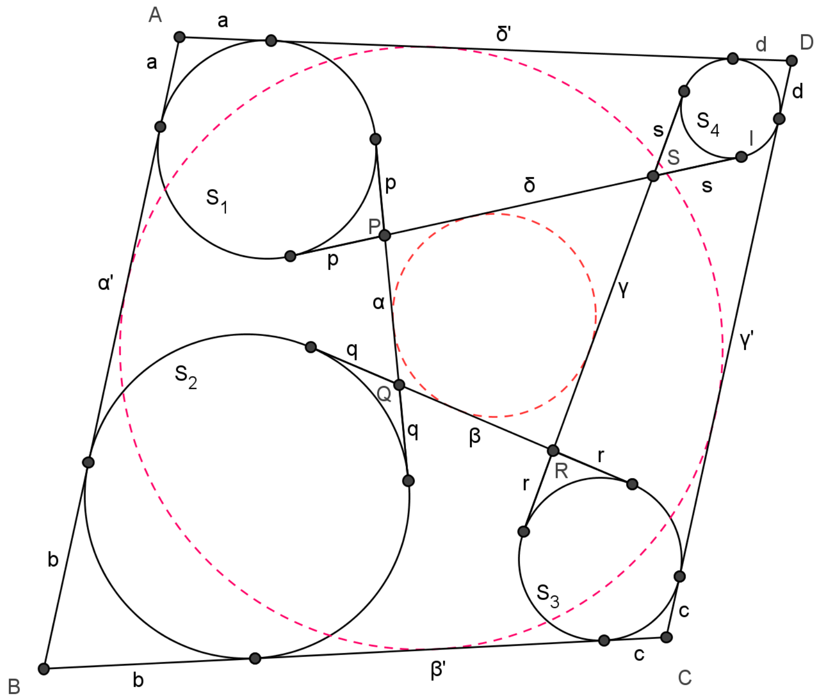

FIGURE t347a

On the other hand, the argument is valid for certain pairs of internal common tangents, and even for certain combinations of internal and external common tangents. It turns out that if an even number of pairs of external tangents are replaced by internal tangents, the result will hold.

Students can draw for themselves diagrams showing the situation when we use internal common tangents. Looking at such diagrams, or at figures t347a and t347b, they may be able to see that the trouble is that the tangent segments (p, q, etc.) that we want to add are in the 'wrong direction': We must sometimes subtract them instead of adding, and this doesn't give us the equality we want.

Using this as a hint, we can in fact make a more general argument by considering the given circles to be *oriented*. If we do this, then the orientation of a circle 'induces' an orientation of its tangents, and a line which touches the circle at just one point will be considered a tangent if and only if it also has the orientation of the circle. Likewise, four lines will be considered to form a quadrilateral if and only if their orientations allow one to trace the perimeter of the figure formed, moving in one direction only.

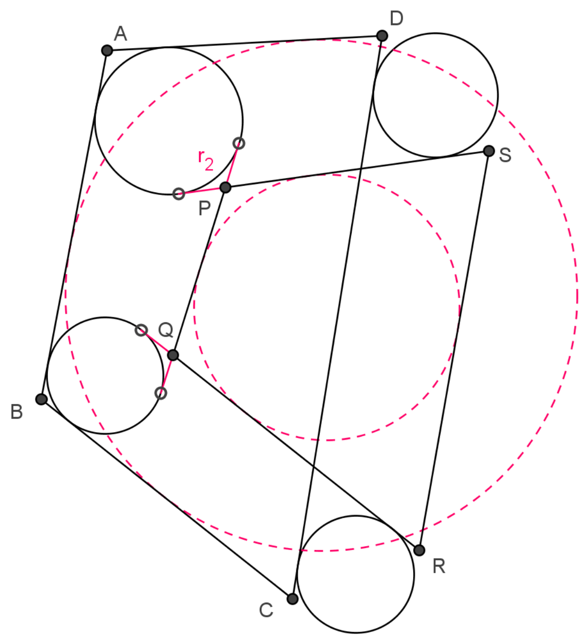

FIGURE t347b

Using these conventions, two circles have only two common tangents, not four. The common tangents will be external if the circles have the same orientation, and internal if they have opposite orientations. (The other two lines which 'look' tangent cannot be given the proper orientation for both circles.)

It turns out that this set of conventions eliminates exactly those cases for which the result of this problem are false, and allows for a simple general argument. Thus the best setting for this problem is one in which the given circles are oriented.

Problem 348. Given an arbitrary pentagon, we construct the circles circumscribing the triangles formed by three consecutive sides (extended if necessary). Show that the five points (other than the vertices of the pentagon) where each circle intersects the next are concyclic. (Exercise 106.)

Solution. Suppose the given pentagon is $ABCDE$, that the two sides adjacent to each side intersect at $KLMNP$ (figure t348), and that the circles described in the problem statement intersect again at A', B', C', D', and E'.

We will prove that these five points are concyclic by proving that points B', C', D', and E' are concyclic. Applying the same reasoning to A', B', C', and D',

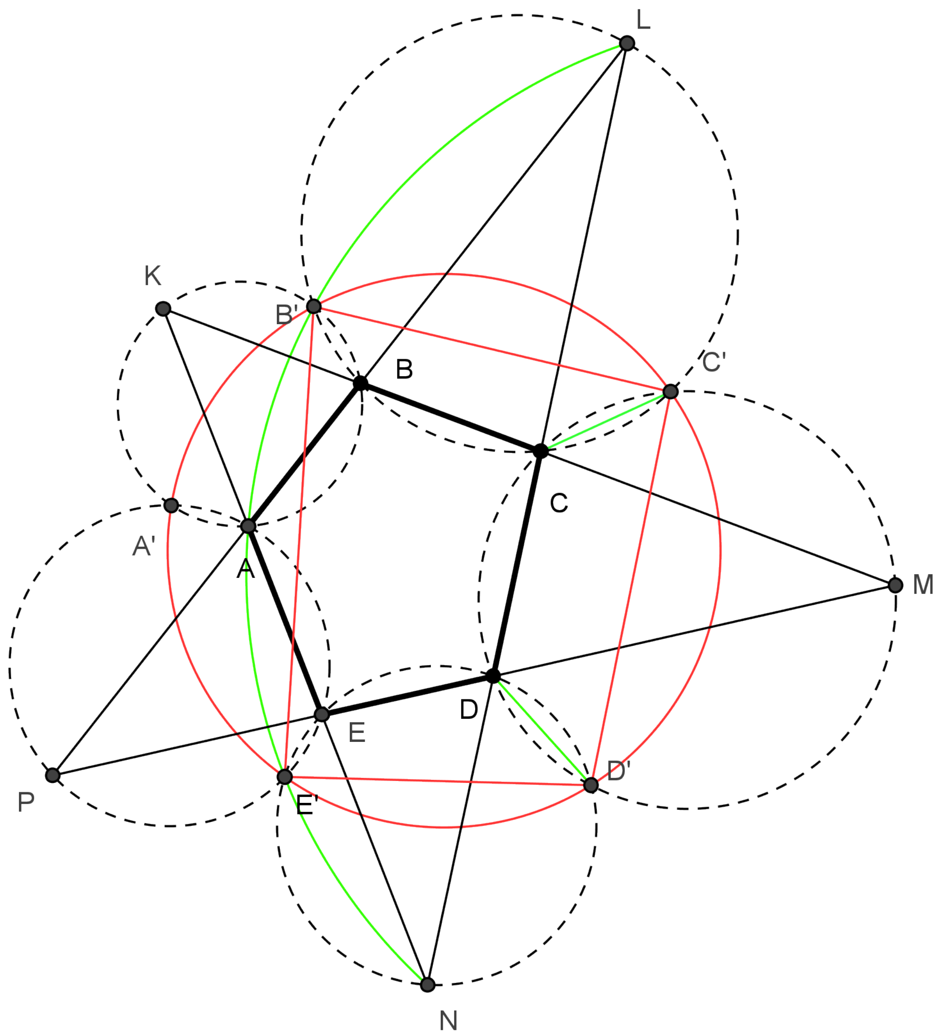

FIGURE t348

we see that these four points are concyclic. Then, since there is only one circle through points B', C', D', all five points lie on the same circle.

We will show that B', C', D', and E' are concyclic by showing that $\widehat{C'B'E'}$ is supplementary to $\widehat{C'D'E'}$. We examine this last angle by drawing segment DD', which splits it into two parts. Then $\widehat{C'D'D}$ supplements $\widehat{C'CD}$ (they are opposite angles in cyclic quadrilateral $CDD'C'$), which in turn supplements $\widehat{C'CL} = \widehat{C'B'L}$. It follows that $\widehat{C'D'D} = \widehat{C'B'L}$.

Now we look at $\widehat{DD'E'}$, which is the other piece of $\widehat{C'D'E'}$. But we cannot quite make use of similar arguments: they would get us results about angles with vertices at A'. We want to get results about angles with vertices at B', so our reasoning takes a slightly different course.

We have not yet used the hint in the problem, which suggests that we re-examine the situation of problem 106. That problem concerns four lines, in general position, which form four triangles. The first part of the problem tells us that the circumscribed circles of these four triangles pass through the same point.

To apply this result to the present problem, it is natural to try considering four out of five of the lines forming the given pentagon. For example, let us look at the diagram formed by all the lines except for DE. The triangles formed by these four lines are ABK, BCL, NCK, and ANL, and the result of problem 106 shows that their circumcircles pass through the same point, which must be point B' (since two of the circumcircles already intersect there). In particular, points $LB'AN$ lie on the same circle, (as do $KB'CN$, but in this proof we will need only the first four concyclic points).

Reasoning analogously, but starting with the diagram formed by all the lines except for BC, we can show that points $LNE'A$ lie on the same circle. Thus the five points L, N, A, B', E' all lie on the same circle.

Returning now to our examination of angles about point D', we see (from circle DEN) that $\widehat{DD'E'} = \widehat{DNE'}$. And from the previous paragraph, we know that quadrilateral $LB'E'N$ is cyclic, and $\widehat{DNE'}$ is the same angle as $\widehat{LNE'}$, which supplements $\widehat{LB'E'}$. It follows that $\widehat{DD'E'}$ supplements $\widehat{LB'E'}$.

Now we have:

$$\widehat{C'D'D} = \widehat{LB'C'}, \tag{1}$$

$$\widehat{DD'E'} + \widehat{LB'E'} = 180^\circ, \tag{2}$$

$$\widehat{LB'E'} = \widehat{LB'C'} + \widehat{C'B'E'}, \tag{3}$$

From (2) and (3) we have:

$$\widehat{DD'E'} + \widehat{LB'C'} + \widehat{C'B'E'} = 180^\circ \tag{4}$$

So from (1) and (4) we have $\widehat{DD'E'} + \widehat{C'D'D} + \widehat{C'B'E'} = 180^\circ$, or $\widehat{C'B'E'} + \widehat{C'D'E'} = 180^\circ$.

This last equation shows that quadrilateral $B'C'D'E'$ is cyclic, and the result of the problem follows from the comments in the second paragraph of this solution.

Note. We have drawn the diagram, and expressed the argument, for a convex pentagon $ABCDE$. But in fact the proposition holds true quite generally, for any five points on the plane, if we consider the circles circumscribing the triangles formed by three consecutive segments formed by the five points, connected in any order.

Students can experiment with a dynamic sketch, to test the truth of this assertion. They can even use the same five points, connected in different orders, to test the assertion. They will find that it is always true, and that the proof above 'almost' holds for any diagram. For some diagrams, equal angles become supplementary, and vice-versa.

A general argument can be constructed, as with many other such problems, by introducing oriented angles.

Problem 349. Two congruent triangles ABC, abc are given. Find the locus of points O with the following property: when triangle ABC is rotated about center O until AB occupies a new position $a'b'$ parallel to ac, the new position b' of B is on the line OC. Also find, under these conditions, the loci of the points a', b', c'.

Solution. Suppose (*fig.*349) O is a point with the property described. Then α, the angle of rotation, is one of the two supplementary angles formed by the lines of segments AB, ac. If $a'b'$ is the image of AB under a rotation through α about O, then by hypothesis, Ob' passes through point C. Thus either $\widehat{BOb'} = \widehat{BOC} = \alpha$ (they are the same angle), or $\widehat{BOb'}$, $\widehat{BOC}$ are supplementary and are equal to α, $180° - \alpha$. Thus O lies on a circle through B and C which is the locus of points at which BC subtends angle α or $180° - \alpha$. Conversely, if O' lies on such an arc, then $\widehat{BO'b}$ will also be α or $180° - \alpha$ (this is the angle of rotation), so point b' will lie on line OC.

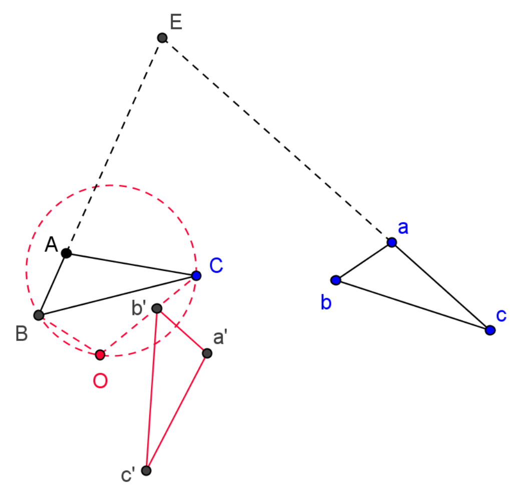

Figure t349

Now for any choice of point O on this locus, triangle AOa' will be isosceles. Since $\widehat{AOa'} = \alpha$, the angles of this triangle will be constant, and the ratio $AO : Aa'$ will likewise be constant. Thus the locus of points a' will be the image of the locus of point O under a homothecy with center A and ratio $AO : Aa'$ followed by a rotation. Thus the locus of a' is also a circle. Similar results hold for points b' and c'.

If we neglect the sense of the angle of rotation, there are two possible rotations (α and $180° - \alpha$). So there are two possible shapes for the isosceles triangle referred to, and thus two ratios of homothecy. Point a' can lie on either of two homothetic images of the circle along which O lies.

Notes. Constructing a diagram similar to figure t349, using geometry software, can be a challenge in itself. Some subtle points include determining the correct angle of rotation and constructing the circle which is the locus of O. Because it is difficult to find 'by hand' even one of the required positions of O, it is best to determine the locus of O first, then draw the figure. The loci of a', b', c' can then be investigated using a 'trace' command.

Very attentive students will note the special case when $AB \parallel ac$. In this case the loci in question are all lines. They may also note that the circles around which the images of C move must pass through point C itself.

Problem 350. Let A', B', C' be the reflections of the intersection point of the altitudes of a triangle in its sides BC, CA, AB. Let points M, N be the intersections of line $B'C'$ with AC and AB respectively; let points P, Q be the intersections of $C'A'$ with BA, BC; and let points R, S be the intersections of $A'B'$ with CB, CA. Show that lines MQ, NR, PS are concurrent. (Their point of intersection is just the point of intersection of the altitudes of the triangle ABC.)

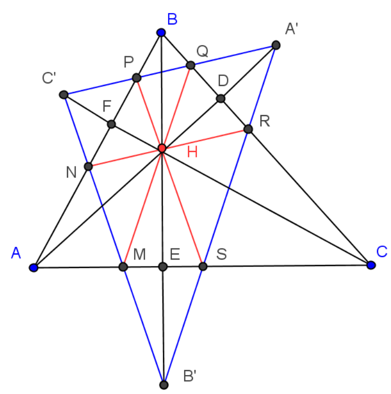

Figure t350

Solution. Suppose D, E, and F are the feet of the altitudes of triangle ABC (figure t350), and H is its orthocenter. Then the construction of triangle $A'B'C'$ described in the problem guarantees that $A'B'C'$, DEF are homothetic about center H (with ratio 2:1). The result of problem 71 tells us that DH, EH and FH are the angle bisectors of triangle DEF, and so, because the triangles are homothetic, these lines are also the angle bisectors of triangle $A'B'C'$.

In other words, an altitude of triangle $A'QR$ is also an angle bisector, so this triangle is isosceles (see exercise 5). Similarly, triangles $B'SM$ and $C'NP$ are isosceles, so $QD = DR$, $SE = EM$, $NF = FP$. Thus quadrilaterals $A'QHR$, $B'SHM$, $C'NHP$ are rhomboi, since their diagonals are perpendicular and bisect each other. It follows that QH and MH are both parallel to $A'B'$, and pass through H, so M, H, and Q must be collinear. In the same way, we can prove that points N, H, R and P, H, S are collinear as well.

Note. The hint in the problem statement guides us in the proof, because it gives us *a priori* the required point of intersection. The usual concurrence proof, without such a hint, would involve studying the intersection point of two of the given lines, to show that the third line passes through the point of intersection. Here, we've actually proved more than just the concurrence of three lines: we have proved that each of these three lines passes through the orthocenter of the triangle.

Problem 351. Inscribe a trapezoid in a given circle, knowing its altitude and the sum or difference of the bases.

Solution. Lemma 1: A trapezoid inscribed in a circle must be isosceles.

Proof: This statement follows from the second theorem of **64** and the theorem of **65**.

Lemma 2: In isosceles trapezoid $ABDC$, we draw altitude AH (figure t351). Then HD is half the sum, and CH is half the difference, of the bases.

Proof: If we draw altitude BJ, then $ABJH$ is a rectangle, so $AB = HJ$, and, by symmetry, $CH = JD$. Then $AB + CD = AB + CH + HJ + JD = HJ+JD+HJ+JD = 2(HJ+JD) = 2HD$, and $CD-AB = CH+HJ+JD-AB = CH + AB + JD - AB = CH + JD = 2CH$. This proves the lemma.

We turn now to the statement in the problem. We assume that we are given a circle, and the sum of the bases of the required trapezoid. Suppose this trapezoid is $ABDC$ (figure t351), and that we've drawn altitude AH. Then we know segments AH (the given altitude) and segment HD (half the sum of the bases, by lemma 2), so we can construct right triangle AHD. The perpendicular bisector of AD will pass through the center of the circle. The distance from this center O to point A will be the radius of the given circle, so we can find O by drawing a circle with the given radius about A, and finding its point of intersection with the perpendicular bisector of AD. We can then draw perpendicular OK to HD, and line OK will be an axis of symmetry for the required trapezoid. Since we know the positions of points A and D, we can reflect in line OK to get vertices B and C. For a solution to exist, AD cannot be greater than the diameter of the given circle; equivalently, $AO > AM$.

If we are given the difference of the bases of the trapezoid, we can construct triangle ACH, knowing AH (the given altitude) and CH (half the given difference, by lemma 2). As before, the perpendicular bisector of AC will be a centerline, and

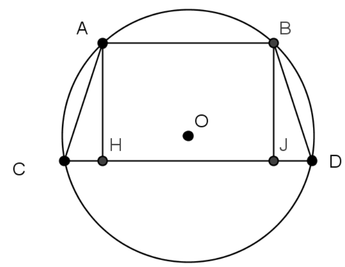

Figure t351

we can determine the location of point O by drawing a circle with the given radius, centered at A. Then we draw OK, which is again an axis of symmetry for the required trapezoid, and the construction can proceed as before.

Problem 352. Let AB be a diameter of a given circle. A circle CMD, with center A, intersects the first one at C and D, and M is an arbitrary point on this circle. Let points N, P, Q be the intersections of lines BM, CM, DM respectively with the original circle. Then:
1^o. $MPBQ$ is a parallelogram.
2^o. MN is the geometric mean of NC and ND.

Solution. (1°) We will show that the opposite sides of $MPBQ$ are parallel, by examining pairs of equal angles. Note first that $\widehat{ADB} = 90^\circ$, since it is inscribed in semicircle $\widehat{AB}$. Hence DB is tangent to circle A at D, and $\widehat{CDB} = \frac{1}{2}\,\widehat{DMC}$ (**74**). On the other hand $\widehat{CMD} = \frac{1}{2}$ (major) $\widehat{CD}$ (in circle A), so $\widehat{CDB}$ supplements $\widehat{CMD}$. But $\widehat{CMQ}$ also supplements $\widehat{CMD}$, so $\widehat{CDB} = \widehat{CMQ}$. Also, $\widehat{CDB} = \widehat{CPB}$, since the both intercept arc $\widehat{CB}$ in the original circle. So $\widehat{CMQ} = \widehat{CPB}$, and $MQ \parallel PB$.

We now show that $PM \parallel QB$. By symmetry about line AB, arcs $\widehat{BC}$ and $\widehat{BD}$ are equal, so $\widehat{DQB} = \widehat{CPB}$, since they intercept these equal arcs. Since $MQ \parallel PB$, we know that angles $\widehat{MQB}$, $\widehat{QBP}$ are supplementary. This means that angles $\widehat{MPB}$, $\widehat{QBP}$ are supplementary, and $PM \parallel QB$.

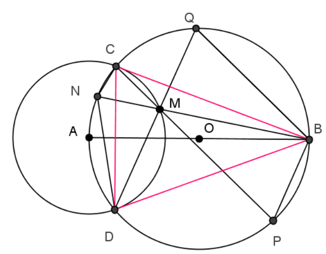

FIGURE t352

(2°) We will show that triangles CMN, MDN are similar. Indeed, triangle BCD is isosceles (it is formed by tangents BC, BD to circle M), so $\widehat{BDC} = \widehat{BCD}$. Also, $\widehat{BDC} = \frac{1}{2}\ \widehat{BQC} = \widehat{MNC}$, and $\widehat{BCD} = \frac{1}{2}\ \widehat{BPD} = \widehat{MND}$, so $\widehat{CNB} = \widehat{MND}$. Finally, $\widehat{NDQ} = \frac{1}{2}\ \widehat{NCQ} = \widehat{QBN} = \widehat{BMP}$ (from parallelogram $MQBP$), and $\widehat{BMP} = \widehat{CMN}$ (they are vertical angles), so $\widehat{NDQ} = \widehat{CMN}$. Thus triangles CMN, MDN have two pairs of common angles, so are similar. The required mean proportion follows immediately.

Note. One key to this diagram is to note that BC, BD are both tangent to circle M, so that these lines (and the arcs they intercept on cirlce O) are symmetric in line AB.

Problem 353. Given an isosceles triangle OAB (in which $OA = OB$), we draw a variable circle with center O, and two tangents from A, B to this circle which do not intersect on the altitude of the triangle.

1°. Find the locus of the intersection M of these two tangents.

2^o. Show that the product of MA and MB is equal to the difference of the squares of OM and OA.

3^o. Find the locus of point I on MB such that $MI = MA$.

Solution. Suppose P and Q are the points of contact of the two tangents from A and B to the circle in question (figure t353a), and let M be the intersection of lines AP, BQ.

(1°) We have $OA = OB$ and $OP = OQ$, so triangles OAP, OBQ are congruent (**34**, second case). Thus angles $\widehat{OAM}$, $\widehat{OBM}$ are equal. Let R be the intersection of lines BM, OA. Then $\widehat{MRA} = \widehat{QRB}$, and triangles ORB, MRA have two pairs of common angles, so their third pair of angles is equal, and $\widehat{AMB} = \widehat{AOB}$. But this last angle is constant, so M must lie on an arc of circle AOB.

Conversely, if M lies on arc $\widehat{AOB}$, then we can draw $OP \perp AM$, $OQ \perp BM$. Then $\widehat{OAM} = \widehat{OBM}$, and triangles OPA, OQB are congruent (**44**, **24**), so the distances OP, OQ from point O to lines AM, BM are equal. Thus these two lines are tangents to the same circle centered at O.

Thus the locus of point M is the circumcircle of OAB. (But see the note below.)

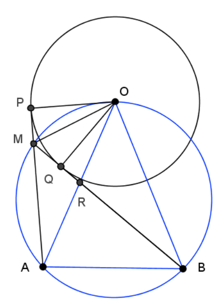

Figure t353a

Note. There are four tangents from A and B to any given circle, and so four points of intersection of pairs of these tangents. But two of these intersections lie along the altitude of the given triangle, and their locus is trivial. This exercise studies the locations of the other two intersections.

Figure t353a shows a case where these intersections P, Q are on the same side of line AB as point O. But if the given circle is large enough, these points will be on the other side of line AB. To get the full locus (the circumcircle of OAB), it is necessary to consider these positions as well.

The proof requires some adjustment in this situation. In particular, certain angles will be supplementary rather than equal. The proof will remain valid after these adjustments.

Note that the circle about point O cannot have a radius larger than OA, since there are no lines tangent to a given circle from a point inside. It is interesting to consider the limiting cases. When the circle centered at O passes through A and B, point M is diametrically opposite O on the circumcircle. And when the circle centered at O is tangent to AB, point M coincides with either A or B.

(2°.) From the congruent triangles identified in (1°), we have $PA = QB$. And $PM = QM$ (**92** so we have $MA \cdot MB = (PA - PM)(QB + QM) = PA^2 - PM^2$.

From right triangles OAP, OMP we have $PA^2 = OA^2 - OP^2$ and $PM^2 = OM^2 - OP^2$. Then we have $MA \cdot MB = (OA^2 - OP^2) - (OM^2 - OP^2) = OA^2 - OM^2$.

If point M is on the opposite side of AB from point O, we will have $MA \cdot MB = OM^2 - OA^2$.

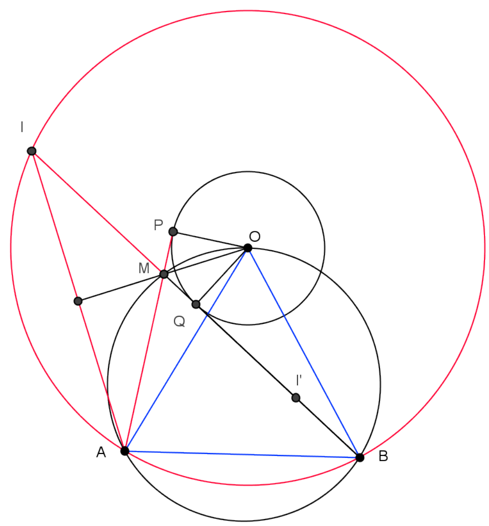

FIGURE t353b

(3°.) Suppose $MA = MI$, where I is on line MB (*fig.* t353b). Then triangle AMI is isosceles, and exterior angle $\widehat{AMB} = 2\widehat{AIB}$. But, since M is on circle AOB, $\widehat{AMB} = \widehat{AOB}$. Hence $\widehat{AIB}$ is constant, and I lies on an arc through A and B. In fact, the center of this arc is point O. Indeed, this center must lie on the perpendicular bisector of AB, and this segment must subtend an angle equal to twice $\widehat{AIB}$ at O. This describes point O and only point O.

Note. Figure t353b shows the case where I is not on ray MB. There is a second point (labeled I' in the figure), on ray MB, such that $AM = MI'$. It is not hard to see that this point also traces out a circular arc, but of a different circle.

Conversely, consider the circle centered at O and passing through A and B (so that at each of its points segment AB subtends an angle equal to $\frac{1}{2}\widehat{AOB}$). We can show that such a point is a position of point I. Indeed, if I is on this circle, we can draw IA and IB, then find a point M on IB such that $MI = MA$. (Point M is the intersection of IB with the perpendicular bisector of IA.) Then $\widehat{AMB} = \widehat{AIB} + \widehat{IAM} = 2\widehat{IAM} = \widehat{AOB}$, so M is on circle AOB. That is (by 2°), M is the intersection of two tangents to the same circle centered at O, one from B (which can only be BM) and one from A. Thus point I is on the tangent to this circle from B, and $MI = AM$.

In constructing a diagram like figure t353b, the whole locus is not traced out unless various pairs of tangents from A and B are considered. If point I is inside the angle vertical to $\widehat{AOB}$ (not shown), then the corresponding point M is the intersection of two tangents, but the tangent from A is the one 'below' the circle, not (as in the figure) the one 'above'. This fine point can be made clear with a dynamic sketch.

A similar proof, and similar notes, apply to point I' and its circle. A result close to this one is given in exercise 63.

Problem 354. On base BC of an arbitrary triangle ABC, we take any point D. Let O, O' be the circumcenters of triangles ABD, ACD.

1^o. Show that the ratio of the radii of the two circles is constant.

2^o. Determine the position of D for which these radii are as small as possible.

3^o. Show that triangle AOO' is similar to triangle ABC.

4^o. Find the locus of the point M which divides segment OO' in a given ratio; examine the special case when M is the projection of A onto OO'.

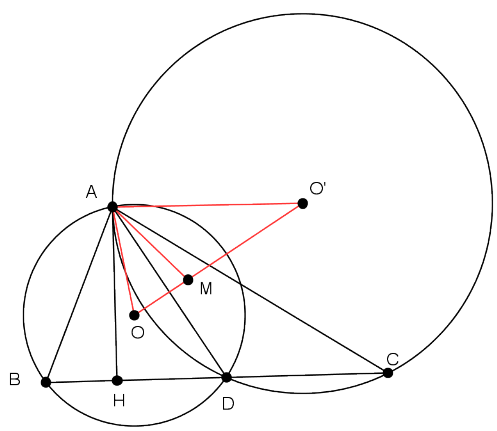

FIGURE t354

Solution. (1°) We use the result of **130b** (statement 4°) to examine the radii of the two circles referred to. Let H be the foot of the perpendicular from A to BC. Then we have $AO = \frac{AB \cdot AD}{2 \cdot AH}$, while $AO' = \frac{AC \cdot AD}{2 \cdot AH}$ (*fig* t354). The ratio of these two quantities is just $AB : AC$, which is constant, and does not depend on the choice of point D.

(2°.) In the expressions given above for $AO,\ AO'$, only AD varies, and it is not hard to see that each radius is smallest when the length of AD is smallest. This happens when $AD = AH$, or D coincides with H.

Note. Students can also see from this result when $AO,\ AO'$ are largest (if D is restricted to segment BC). (This occurs when AD coincides with the larger of $AB,\ AC$.)

Note that the wording of this question is very slightly ambiguous: the conditions for the minimum values of AO and AO' might be different. But because they remain in the same ratio, they are in fact minimal (or maximal) under the same condition.

(3°.) We have, from 1°, $AO : AO' = AB : AC$, so isosceles triangles $ABO,\ ACO'$ are similar. Thus $\widehat{OAB} = \widehat{O'AC}$, and therefore $\widehat{OAO'} = \widehat{BAC}$. It then follows from **118**, case II, that triangles $OAO',\ BAC$ are similar.

(4^o.) Suppose point M is such that $OM' : MO = k$, for some fixed ratio k. Then $OM = \frac{OO'}{1+k}$. We know (3°) that $AO : OO' = AB : BC$, a constant, and it is not hard to see that this implies that $AO : MO'$ is also constant. Now (also from 3°) $\widehat{AOM} = \widehat{AOO'} = \widehat{ABC}$, which is constant. Thus triangle AOM retains its shape as D varies: for any two positions of D, the two corresponding triangles AOM are similar (**118**, case 2). Hence the ratio $AM : AO$ is constant, as is angle $\widehat{OAM}$. It follows that the locus of M is a figure similar to the locus of O (see **150**, and also the solution to exercise 160). The locus of O is certainly the perpendicular bisector of AB, so the locus of M is also a straight line.

Notes. Students can fill in the algebraic details showing that the ratio $AO : OM$ is constant, and also the proof that any point on the line mentioned is a possible position of point M.

Now let P (not shown in the figure) be that position of M which is the projection of A onto OO'; that is, $AP \perp OO'$. Then, since $AO = OD$ and $AO' = O'D$, we see that OO' is the perpendicular bisector of AD, and in particular P is the midpoint of AD. The locus of P is thus a line parallel to BC, passing through the midpoints of $AB,\ AC$.

Note. To get the full loci of various points mentioned in this proof, we must consider positions of D *outside* segment BC. Students can explore this situation more fully.

Problem 355. An angle of fixed size rotates around a common point of two circles $O,\ O'$. Its sides intersect the two circles again at $M,\ M'$ respectively. Find the locus of points which divide MM' in a given ratio. More generally, find the locus of the vertex of a triangle with base MM', and similar to a given triangle.

Solution. We will use the result of exercise 162 to show that the vertex described moves along a figure similar to the two given circles: that is, along a third circle.

Suppose (*fig.* t355) that A is one of the intersections of given circles $O,\ O'$, and that $\widehat{MAM'}$ is the given angle. We will show that there is a similarity taking

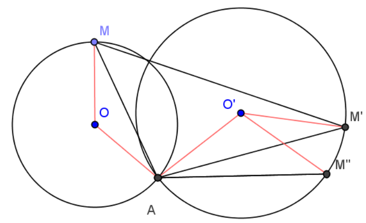

Figure t355

circle O onto circle O', and also taking M onto M'. This gives us the situation of exercise 162.

To this end, we construct angle $\widehat{MAM''} = \widehat{OAO'}$, and with the same orientation. Then it is not hard to see that angles $\widehat{OAM}$, $\widehat{O'AM''}$ are equal, and so isosceles triangles OAM, $O'AM''$ are similar. Thus the ratio $AM : AM''$ is constant, and if we rotate M about A through angle $\widehat{MAM''} = \widehat{OAO'}$, then dilate it about A in the ratio $AM : AM''$, it falls on M''. And if we do the same to circle O, it falls on circle O', with M, M'' a pair of corresponding points.

We now compose this with a rotation taking M'' onto M'. Indeed, angles $\widehat{MAM'}$, $\widehat{MAM''}$ are both constant, so their difference, or $\widehat{M''AM'}$, is also constant. Then $\widehat{M''OM'} = 2\widehat{M''AM'}$ is also constant, and M'' falls on M' when rotated through this angle about point O'.

Hence if we rotate circle O about A through $\widehat{OAO'}$, then dilate it about A in the ratio $AO : AO''$, then rotate it about O' through $\widehat{M''OM'}$ (which is constant), it will coincide with circle O', and points M, M' will correspond. The result of exercise 162 now implies that the required loci are all circles.

Problem 356. If five lines A, B, C, D, E are such that two of them, for instance A and B, are divided in the same ratio by the other three, then any two of them are divided in the same ratio by the other three. (The proof must distinguish two cases: of the two new line to which we want to apply it, one line may or may not be one of the original lines.)

Solution. We denote the given lines by a, b, c, d and e (rather than with the corresponding capital letters). Let P_{ab} denote the intersection of lines a, b, let P_{cd} the intersection of lines c, d, and so on (*fig.* t356). We will apply the Menelaus' Theorem (**192**) to various triangles.

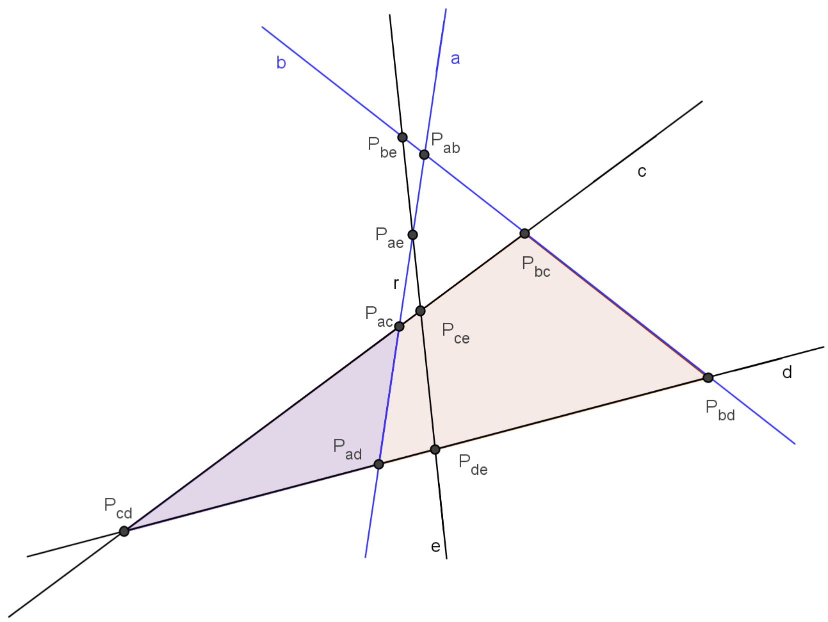

FIGURE t356

Applying this theorem to the triangle formed by lines $a,\ c,\ d$, and transversal e, we find:

$$\frac{P_{ae}P_{ad}}{P_{ae}P_{ac}} \cdot \frac{P_{ce}P_{ac}}{P_{ce}P_{cd}} \cdot \frac{P_{de}P_{cd}}{P_{de}P_{ad}} = 1.$$

Next we apply the same theorem to the triangle formed by $b,\ c,\ d$:

$$\frac{P_{be}P_{bd}}{P_{be}P_{bc}} \cdot \frac{P_{ce}P_{bc}}{P_{ce}P_{cd}} \cdot \frac{P_{de}P_{cd}}{P_{de}P_{bd}} = 1.$$

We equate the two left-hand members of these and simplify:

$$\frac{P_{ae}P_{ac}}{P_{ae}P_{ad}} \cdot \frac{P_{de}P_{ad}}{P_{ce}P_{ac}} = \frac{P_{be}P_{bc}}{P_{be}P_{bd}} \cdot \frac{P_{de}P_{bd}}{P_{ce}P_{bc}}.$$

Now if lines $c,\ d,\ e$ cut off proportional segments on $a,\ b$, then we have $\frac{P_{ae}P_{ac}}{P_{ae}P_{ad}} = \frac{P_{be}P_{bc}}{P_{be}P_{bd}}$.

These last two equations imply that $\frac{P_{de}P_{ad}}{P_{ce}P_{ac}} = \frac{P_{de}P_{bd}}{P_{ce}P_{bc}}$, or $\frac{P_{ce}P_{bc}}{P_{ce}P_{ac}} = \frac{P_{de}P_{bd}}{P_{de}P_{ad}}$, or that $a,\ b,\ e$ cut off proportional segments on $c,\ d$.

This shows that if two lines $a,\ b$ cut off proportional segments on three different lines $c,\ d,\ e$, then the same is true of the pair of lines $c,\ d$; that is, these two lines cut off proportional segments on $a,\ b,\ e$, which are three different lines. This reasoning applies also to the pairs of lines $c,\ e$ and $d,\ e$.

We must now look at a pair of lines which include a or b. For example, if $a,\ b$ cut off proportional segments on $c,\ d,\ e$, we must show that $a,\ c$ also cuts off proportional segments on $b,\ d,\ e$.

We can do this by applying the reasoning above twice in succession: The statement is true about $a,\ b$, and therefore also about $d,\ e$. And if it is true about $d,\ e$, it must also be true about $a,\ c$. The full statement is proved.

Notes. This proof is complicated, but not hard to scaffold. Students can be given the hint to use theorems on triangles and transversals, or even which triangles to single out. The algebraic part of the derivation is much simpler than the geometry which sets it up.

Details of the proof are tedious, but not difficult, except for the subtle argument in the last two paragraphs.

Problem 357. Let $a,\ b,\ c$ be the three sides of a triangle, and $x,\ y,\ z$ the distances from a point in the plane to these three sides. If this point is on the circumscribed circle, one of the ratios $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}$ is equal to the sum of the other two, and conversely.

Solution. Suppose M is a point on the circumcircle of triangle ABC. Without loss of generality, we assume that M is on the arc $\widehat{BC}$ not containing point A (*fig.* t357a). Let $MD = x,\ ME = y,\ MF = z$ be the perpendiculars from M to the sides of the triangle, and let $AB = c,\ BC = a,\ CA = b$.

We apply Ptolemy's theorem (**237**) to cyclic quadrilateral $ABMC$:

$$MA \cdot a = MB \cdot b + MC \cdot c. \tag{1}$$

Now $\widehat{ACM} = 180° - \widehat{ABM} = \widehat{MBF}$, so right triangles $MFB,\ MEC$ are similar, and $MB : MC = z : y = xz : xy$. In the same way (using similar triangles $MBD,\ MAE$), we have $MA : MB = y : x = yz : xz$. Hence we can write $MA = kyz,\ MB = kxz,\ MC = kxy$, for some constant k. Substituting these values in (1), we have $akyz = bkxz + ckxy$, and dividing by $kxyz$, we have $\frac{a}{x} = \frac{b}{y} + \frac{c}{z}$.

For M on the arc $\widehat{AC}$ not containing B, we have, similarly, $\frac{b}{y} = \frac{a}{x} + \frac{c}{z}$, and if M is on $\widehat{AB}$ we have $\frac{c}{z} = \frac{a}{x} + \frac{b}{y}$.

This theorem is stated elegantly if we use the convention established in the solution to exercise 301, about the signs of the distance of a point to the sides of a triangle. In this situation, one of these signs is positive and the other two negative, so we have, both in magnitude and sign:

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0. \tag{2}$$

It is convenient to use this convention also in proving the converse theorem. We will prove that if point M satisfies relation (2) (in both magnitude and sign, for the distances), then M lies on the circumcircle of triangle ABC.

Indeed, suppose M' is a point *not* on this circumcircle, and suppose $x',\ y',\ z'$ are its distances to the sides of the triangle (taken with the appropriate sign). Let M be the intersection of AM' with the circumcircle, and $x,\ y,\ z$ its (signed)

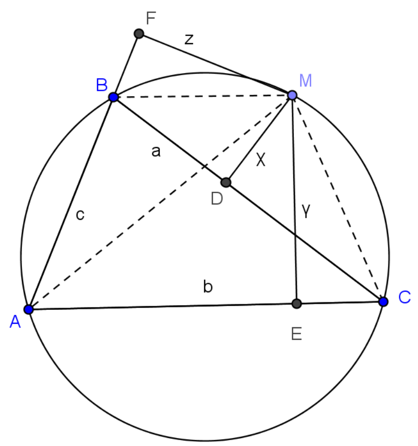

FIGURE t357a

distances to the sides of the triangle, as before. Let AM' and side BC intersect at P.

Using various similar triangles, and noting that $A,\ P,\ M,\ M'$ are collinear, we find: $y : y' = z : z' = AM : AM'$; $x : x' = PM : PM' = (PA+AM) : (PA+AM')$. Now it is not hard to prove (for example, by examining cross products) that if $m \neq n$ and $p \neq 0$, then $(m+p) : (n+p) \neq m : n$. Therefore $x : x' \neq AM : AM'$. In other words, $\frac{1}{y'} = \frac{AM}{AM'} \cdot \frac{1}{y}$; $\frac{1}{z'} = \frac{AM}{AM'} \cdot \frac{1}{z}$, but $\frac{1}{x'} \neq \frac{AM}{AM'} \cdot \frac{1}{x}$.

Since M is on the circumcircle, the original theorem assures us that

$$\frac{a}{x'}+\frac{b}{y'}+\frac{c}{z'} = \frac{a}{x'}+(\frac{b}{y}+\frac{c}{z})\cdot\frac{AM}{AM'} = \frac{a}{x'}-\frac{a}{x}\cdot\frac{AM}{AM'} = \frac{a}{x}\frac{x}{x'}-\frac{a}{x}\cdot\frac{AM}{AM'} = \frac{a}{x}(\frac{x}{x'}-\frac{AM}{AM'}) \neq 0.$$

This proves the converse theorem.

Notes. In the direct theorem, one central idea is that $MA : MB : MC = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}$. We have avoided the use of continued proportions, but students familiar with them may find it easier to understand the proof expressed in these terms.

In the converse theorem, we have omitted details (involving similar triangles) of the proofs of various proportions. Students can be asked to supply these.

The proof of the converse theorem needs some changes for certain positions of M'. Most obviously, M' might be inside the circumcircle. But there are also changes needed if AM' turns out to be tangent to the circumcircle, or parallel to BC. Students can be asked to explore these situations.

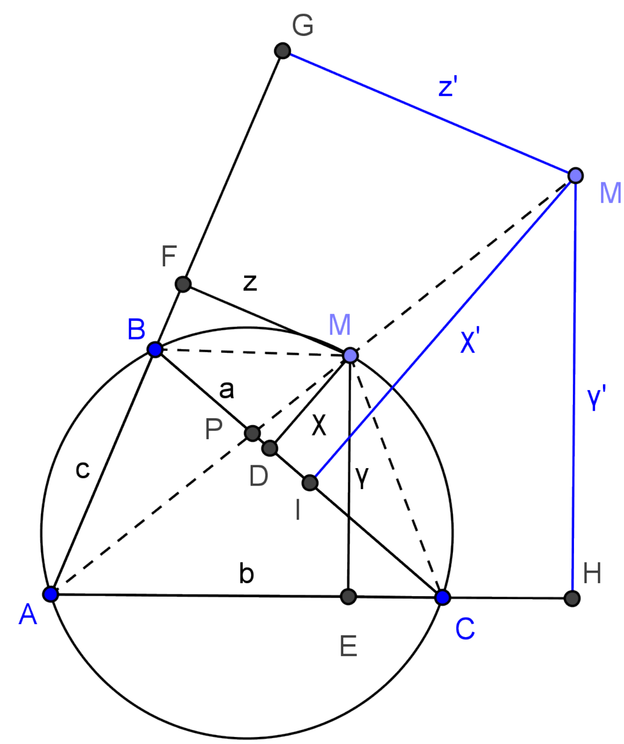

Figure t357b

The converse theorem is in fact not true unless we consider signed distances to the sides of the triangle. The following example shows this. If ABC is an equilateral triangle, and we take P on the exterior angle bisector at vertex A such that $CP \perp AC$, we have (in absolute value) $2x = y = z$, so that $\frac{a}{x} = \frac{b}{y} + \frac{c}{z}$, yet P certainly does not lie on the circumcircle of triangle ABC.

Problem 358. Given a line segment AB, and a point C on this segment, find the locus of the points of intersection of a variable circle passing through $A,\ B$ with a line joining point C to the intersection of the tangents at $A,\ B$ to this circle.

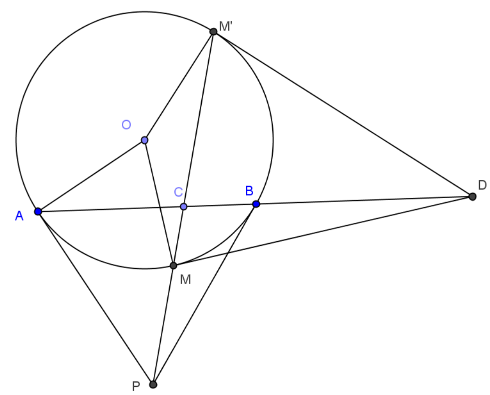

FIGURE t358

Solution. In figure t358, P is the intersection of the tangents to one of the circles referred to, and $M,\ M'$ are the intersections of line PC with the circle. We need the locus of M (or M') as the circle varies.

A diagram built around two tangents to the same point suggests consideration of poles and polars. Here, P is the pole of line AB, and if we draw the tangents to the circle at $M,\ M'$, then their intersection D is the pole of MM'. Since P is on the polar of point D, **204** shows that D must lie on AB. Then, by the definition of the polar, $C,\ D$ divide AB harmonically. Thus D does not depend on the circle, but only on the position of C on AB.

Now the length of tangent MD is $\sqrt{DA \cdot DB}$, which likewise does not depend on the particular circle drawn. Therefore $M,\ M'$ lie on a circle with this radius centered at D.

Conversely, any point on this circle is a possible position of M (or M'). Indeed, we let D be the harmonic conjugate of point C with respect to $A,\ B$, draw the circle centered at D with radius $\sqrt{DA \cdot DB}$, and pick a point M on this circle.

By **132** (converse), we can draw is a circle through $A,\ B,\ M$, which will be tangent to DM, then draw DM'. We must show that MM' is collinear with the intersection P of the tangents to this new circle from A and B. Again, we use polars. Line AB is the polar of P, and D is on AB. Therefore, by **204**, P is on the polar of D, which is line MM'. This concludes the proof.

Note. The converse statement may be difficult to formulate. The proof given here is not difficult: it is just the argument that led to the original conclusion, but in reverse.

A special case occurs when C is the midpoint of segment AB. In that case, point D recedes to infinity, and M, M' are diametrically opposite on the circle where it intersects the perpendicular bisector of AB.

Problem 359. On the extension of a fixed diameter of a circle O, we take a variable point M, from which we draw a tangent to the circle. Find the locus of the point P on this tangent such that $PM = MO$ (**92**).

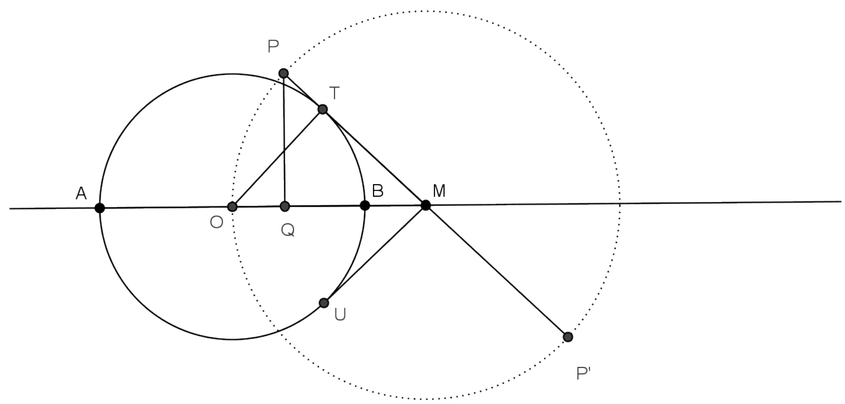

FIGURE t359

Solution. In figure t359, T is the point of contact of one of the tangents from variable point M to the circle. If P is the point described in the problem statement, we drop perpendicular PQ to OM. Then triangles MOT, MPQ are congruent (**44**, **24**), so $PQ = OT$. That is, the distance from P to line AB is constant and is equal to the radius of circle O, so P lies on one of the two tangents to the given circle which are parallel to OM.

Does every point on such a tangent belong to the locus required? After all, positions of point M are limited to those outside circle O. How does this constraint affect the locus of point P?

If we limit the discussion to a single tangent from M, and let M vary along ray OB, we find that the positions of P are limited to a line segment which can be described as half the side of a square tangent to circle O. If M varies along ray OA (the 'other side' of the given circle), P describes the rest of the side of this square.

But there is another point P' on MT such that $MP' = MT$, and there is another tangent MU from M to circle O. That is, for any position of M, there are actually four points fitting the description of P. These four points trace out a full locus of two tangents to circle O, both parallel to line AB.

Notes. Students can finish the proof that every point of the set described actually belongs to the locus.

This is not a difficult problem, especially if students draw the locus with dynamic software first, so that they know what they need to prove. A dynamic sketch will also make it clear why the fact that positions of M inside the circle lack tangents to the circle does not result in 'holes' or 'spaces' in the locus.

Problem 360. From a point M in the plane of a rectangle we drop perpendiculars to the sides, the first one intersecting two opposite sides at P, Q, and the second intersecting the other two sides at R, S.

1^o**.** For any M, show that the intersection H of PR and QS is on a fixed line, and the intersection K of PS and QR is on another fixed line.

2^o**.** Show that the bisector of angle $\widehat{HMK}$ is parallel to a side of the rectangle.

3^o**.** Find point M, knowing points H and K.

4^o**.** This last problem has two solutions M, M'. Show that the circle with diameter MM' is orthogonal to the circumscribed circle of the rectangle.

5^o**.** Find the locus of points M such that PR is perpendicular to QS.

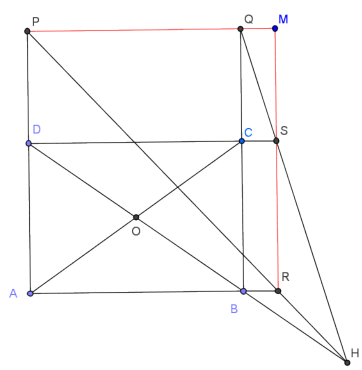

FIGURE t360a

Solution. (1°.) We will show that these fixed lines are the diagonals of the given rectangle. Suppose the rectangle is $ABCD$ (*fig.* t360a), O the intersection of its diagonals, and H the intersection of lines BD, PR. We apply Menelaus' Theorem (**192**) to triangle ABD with transversal PR:

$$\frac{HD}{HB} \cdot \frac{RB}{RA} \cdot \frac{PA}{PD} = 1.$$

Now suppose QS intersects BD at some point H'. We will show that H and H' in fact coincide, by applying Menelaus' Theorem to triangle BCD with transversal QS:

$$\frac{H'B}{H'D} \cdot \frac{SD}{SC} \cdot \frac{QC}{QB} = 1, \text{or equivalently, } \frac{H'D}{H'B} \cdot \frac{SC}{SD} \cdot \frac{QB}{QC} = 1.$$

But clearly $RB : RA = SC : SD$ and $PA : PD = QB : QC$ (these ratios are equal term-by-term), so it follows that $HD : HB = H'D : H'B$. All these relationships hold both in magnitude and sign, so points H and H' must coincide, and PR and QS intersect on line BC.

Similarly, we can prove that K, the intersection of PS and QR is on line AD.

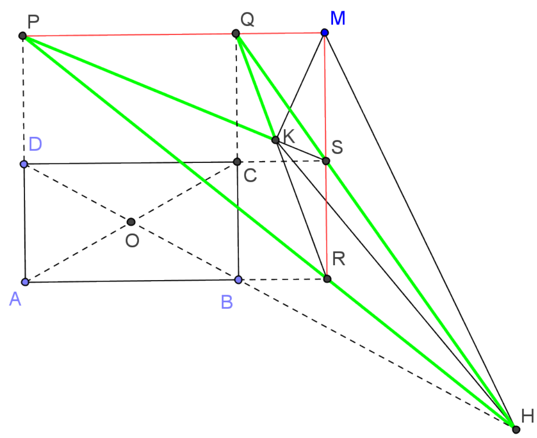

FIGURE t360b

(2°.) In fact, this bisector is just MS (or MP, depending on the position of point M). We can prove this by using results about a complete quadrilateral, applied to the quadrilateral formed by lines RQ, QS, PS, PR (*fig.* t360b). In this quadrilateral, diagonal HK is divided harmonically by diagonals PQ, RS (**202**). Therefore lines MQ, MK, MS, MH form a harmonic pencil (**201**). Then, since $MQ \perp MS$, lines MQ, MS bisect the angle between lines MH, MK (**201**, corollary 2).

Note. The argument holds, with slight adjustments, for other positions of M than those show in figure 360b. Note that this figure does not show the points of harmonic division along diagonal HK. But by construction (for example) P, K, and S are collinear, so RS is a diagonal of complete quadrilateral $PKQH$.

(3°.) We apply Menelaus' Theorem (**192**) to find the ratios $AP : PD$ and $AR : RB$. These ratios will locate points P and R, and therefore determine point M. First we apply the theorem to triangle ABD and transversal PR:

$$\frac{HD}{HB} \cdot \frac{RB}{RA} \cdot \frac{PA}{PD} = 1.$$

Applying the theorem to triangle ABC and transversal QR, we find:

$$\frac{KA}{KC}\cdot\frac{QC}{QB}\cdot\frac{RB}{RA}=1.$$

Multiplying these two equations together, and noting that $\frac{QB}{QC}=\frac{PA}{PD}$ (these ratios are equal term-by-term), we find:

$$\frac{HD}{HB}\cdot\frac{KA}{KC}\cdot\left(\frac{RB}{RA}\right)^2=1.$$

And dividing the first of the same two equations by the second, we find:

$$\frac{HD}{HB}\cdot\frac{KC}{KA}\cdot\left(\frac{PA}{PD}\right)^2=1.$$

It follows that $\frac{PA}{PD}=\pm\sqrt{\frac{HB\cdot KA}{HD\cdot KC}}$, $\frac{RB}{RA}=\pm\sqrt{\frac{HB\cdot KC}{HD\cdot KA}}$. The signs must be chosen so that the product $\frac{PA}{PD}\cdot\frac{RB}{RA}$ has the same sign as the ratio $HB:HD$.

The values of these two ratios (almost) determine the position of point M, but see 4° for further explanation. Point M will exist whenever $\frac{HB}{HD}$ and $\frac{KA}{KC}$ have the same sign (so that the expressions under the radical signs are not negative).

(4°.) The problem statement gives a hint: there are two positions of P and of R corresponding to each position of H and K. And, in fact (*fig.* t360c), there are two possible points P and P' dividing AB in the ratio found in 3°. Likewise, there are two points R, R', and these two sets of points determine points M, M'. We let O be the center of rectangle $ABCD$. Let ω be the midpoint of segment MM', and let O_1, O_2 and ω_1, ω_2 be the projections of O, ω on AB and AD. We make the following notes, leaving the easy proofs for the reader:

- Note 1: $\omega_1R=-\omega_1R'$.
- Note 2: $\omega_2P=-\omega_2P'$.
- Note 3: $O_1B^2+O_2D^2=OC^2$.
- Note 4: $\omega_1R^2+\omega_2P^2=\omega M^2$.
- Note 5: $O_1\omega_1^2+O_2\omega_2^2=O\omega^2$.

The result of **189** tells us that $O_1B^2=O_1R\cdot O_1R'=(O_1\omega_1+\omega_1R)\cdot(O_1\omega_1+\omega_1R')=(O_1\omega_1+\omega_1R)\cdot(O_1\omega_1-\omega_1R)=O_1\omega_1^2-\omega_1R^2$. In the same way, we have $O_2D^2=O_2P\cdot O_2P'=O_2\omega_2^2-\omega_2P^2$.

Adding, we find $O_1B^2+O_2D^2=O_1\omega_1^2-\omega_1R^2+O_2\omega_2^2-\omega_2P^2$. Using notes 3-5, we can derive from this the relation $OC^2+\omega M^2=O\omega^2$, in which OC and ωM are the radii of the circles in question, while $O\omega$ is the distance between their centers. This last relation shows that the two circles are orthogonal.

Notes. Students can be asked to verify the relationships in notes 1-5 above. They can also prove the following general statement:

Lemma: Two circles are orthogonal if and only if the sum of the squares of their radii is equal to the square of the distance between their centers.

(5°.) Suppose (*fig.*360d) that $PR\perp QS$. Since AM, PR are diagonals of rectangle $APMR$, $\angle RPM=\angle AMP$. Since CM, QS are diagonals of rectangle $QCSM$, we have $\widehat{QMC}=\widehat{MQS}=\widehat{HQP}$, and this last angle is complementary to $\widehat{RPM}$ (they are the acute angles of right triangle HQP). Therefore angles

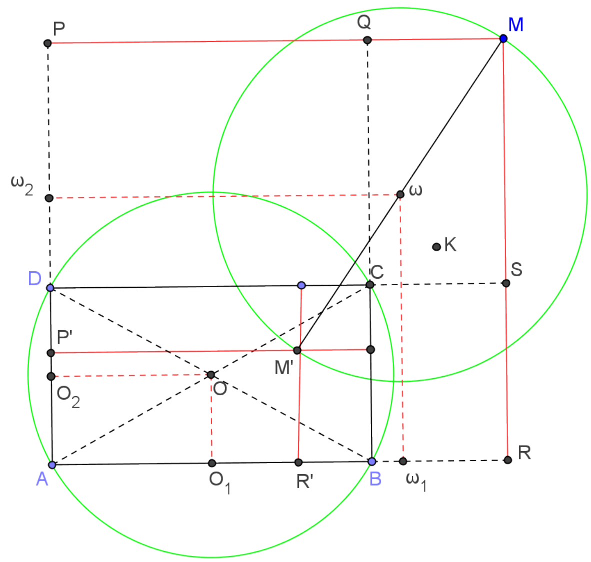

Figure t360c

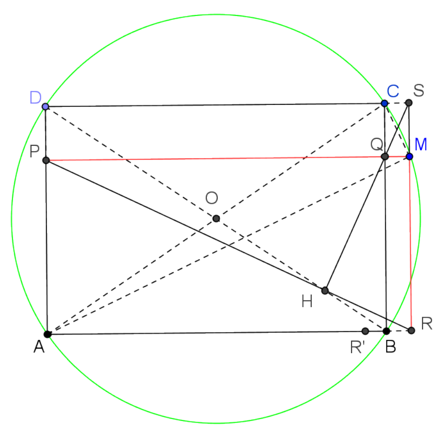

Figure t360d

$\widehat{AMP}$, $\widehat{QMC}$ are complementary, so $AM \perp MC$, and M lies on the circle with diameter AC.

Conversely, if M is on this circle, then $AM \perp MC$, and following our reasoning backwards, we find that $PR \perp QS$.

Notes. This part of the problem is difficult. One difficulty is the need to 'think out of the box'. Students must realize that point M is *outside* the given rectangle, and that R (in the case of figure t360d) divides the sides of the rectangle *externally* in the ratio required by the construction in 3°.

In problems where one must find a locus, software can often help, because the required locus can be traced. But in this case, it is difficult to find even one point on the locus without significant insight.

Problem 361. From vertices B and C of triangle ABC we draw two lines BB', CC' (where B' is on side AC and C' is on side AB), such that $BB' = CC'$. Show that the two angles $\widehat{CBB'}$, $\widehat{B'BA}$ into which BB' divides angle $\widehat{B}$ cannot both be less than or both be greater than the corresponding angles $\widehat{BCC'}$, $\widehat{C'CA}$ into which CC' divides angle C. (that is, we cannot at the same time have $\widehat{CBB'} > \widehat{BCC'}$ and $\widehat{B'BA} > \widehat{C'CA}$).

(Form parallelogram $BB'CF$, in which B, C are two opposite vertices, and, drawing $C'F$, compare the angles it determines at C' to those it determines at F.)

A triangle which has two equal angle bisectors is isosceles.

Solution. Suppose $\widehat{CBB'} > \widehat{BCC'}$ (figure t361). Let us compare triangles $C'BC$ and BCB'. They have a common side BC, their sides BB', CC' are equal, and these two pairs of equal sides include angles which are unequal in the order indicated in the first sentence of this paragraph. Therefore (**28**), $CB' > BC'$.

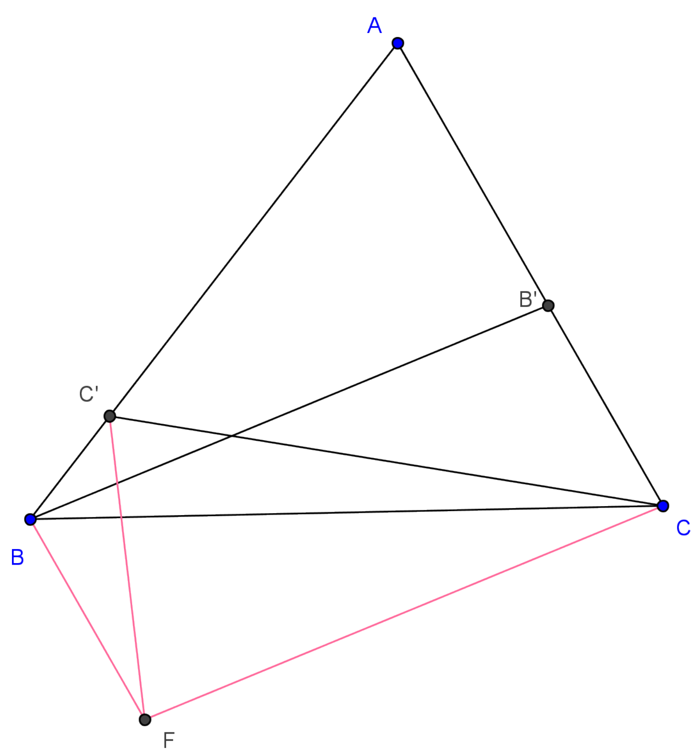

Figure t361

Let us draw line BF through B parallel to AC, and line CF through C parallel to AB. Then $BB'CF$ is a parallelogram, so $CB' = FB$. Since we know that $CB' > BC'$, it follows that $FB > BC'$, so in triangle $BC'F$, we have

$$\widehat{BC'F} > \widehat{BFC'}. \tag{1}$$

Now $CC' = BB'$ by hypothesis, and $BB' = FC$ by construction (of the parallelogram), so triangle $CC'F$ is isosceles, and

$$\widehat{FC'C} = \widehat{C'FC}. \tag{2}$$

Adding (1) and (2), we find that $\widehat{BC'C} > \widehat{BFC}$, And since $\widehat{BFC} = \widehat{BB'C}$, we have $\widehat{BC'C} > \widehat{BB'C}$. Now if two angles are unequal, their supplements are unequal in the opposite order, so $\widehat{AC'C} < \widehat{BB'A}$.

Thus we have $\widehat{ABB'} = 180° - \widehat{A} - \widehat{BB'A} < 180° - \widehat{A} - \widehat{AC'C} = \widehat{ACC'}$; that is, the condition that $\widehat{CBB'} > \widehat{BCC'}$ implies the condition that $\widehat{ABB'} < \widehat{ACC'}$, which is equivalent to the first assertion of the problem statement.

If we exchange the roles of points B and C, the same reasoning will show us that if $\widehat{BCC'} > \widehat{CBB'}$, then $\widehat{ACC'} < \widehat{ABB'}$.

We can now show that a triangle with two equal angle bisectors must be isosceles. Suppose BB' and CC' are these two equal angle bisectors. Then if $\widehat{ABC} > \widehat{ACB}$, we could divide by two to get $\widehat{CBB'} > \widehat{BCC'}$. But because we have angle bisectors, we could just as well write $\widehat{ABB'} > \widehat{ACC'}$. We have just shown that these two statements cannot both be true, so it cannot be true that $\widehat{ABC} > \widehat{ACB}$.

Analogous reasoning shows that we cannot have $\widehat{ABC} < \widehat{ACB}$, so these two angles must be equal, and the triangle must be isosceles.

Note. This is a classically difficult problem. Here it is solved by considering inequalities, one of a number of places where consideration of inequalities leads to the conclusion that certain objects are equal (in this case, two sides of a triangle).

Problem 361b. In any triangle, the greater side corresponds to the smaller angle bisector (take the difference of the squares of the bisectors given by the formula of **129**, and factor out the difference of the corresponding sides).

Solution. If t_a and t_b are the lengths of the angle bisectors of angles A and B of triangle ABC (with sides of length a, b, and c), the formula of **129** gives us:

$$t_a^2 = bc - \frac{a^2bc}{(b+c)^2},$$

$$t_b^2 = ac - \frac{b^2ac}{(a+c)^2}.$$

Then $t_a^2 - t_b^2 = c(b-a) + abc\left(\frac{b}{(a+c)^2} - \frac{a}{(b+c)^2}\right)$. If $b > a$, then clearly $\frac{b}{(a+c)^2} > \frac{a}{(b+c)^2}$, so $t_a > t_b$.

Note. Students can fill in the gap in this proof indicated by the term 'clearly': it involves simple algebra of inequalities.

They can also explain why the statement in the problem implies that a triangle with two equal angle bisectors must be isosceles.

Problem 362. Of all the triangles inscribed in a given triangle, which has the minimum perimeter?

Solution. We will consider triangle DEF to be *inscribed* in triangle ABC if points D, E, F lie respectively on sides BC, AC, AB of the triangle, or on their extensions.

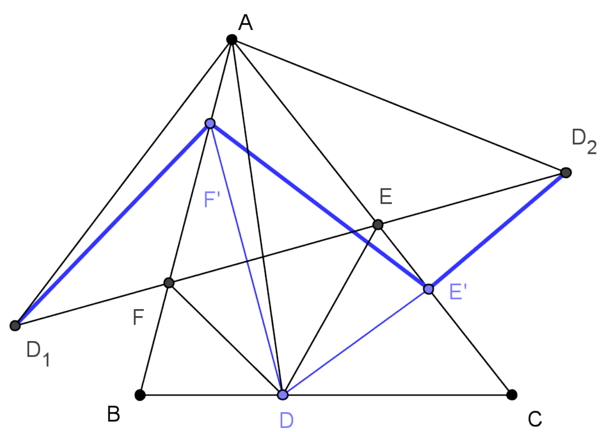

FIGURE t362a

Let us first fix point D on line BC, and find the triangle of least perimeter which is inscribed in ABC, with one vertex at point D. We construct points D_1, D_2 (figure t362a), the reflections of D in lines AB, AC respectively. The perimeter of any triangle $DE'F'$ which is inscribed in ABC will be equal to the broken path $D_1E'F'D_2$ (compare the solution to exercise 40). The triangle DEF of least perimeter will occur when the broken path is a line D_1EFD_2. Thus we can construct this triangle by reflecting D to get D_1 and D_2, then drawing line D_1D_2 to locate points E and F.

The more general problem will be solved if we can locate point D such that D_1D_2 is as short as possible. We note first that, from symmetry in lines AB, AC, we have $\widehat{D_1AB} = \widehat{BAD}$ and $\widehat{EAD_2} = \widehat{DAC}$, so $\widehat{D_1AD_2} = 2\widehat{BAC}$. That is, the measure of angle $\widehat{D_1AD_2}$ does not depend on the position of point D.

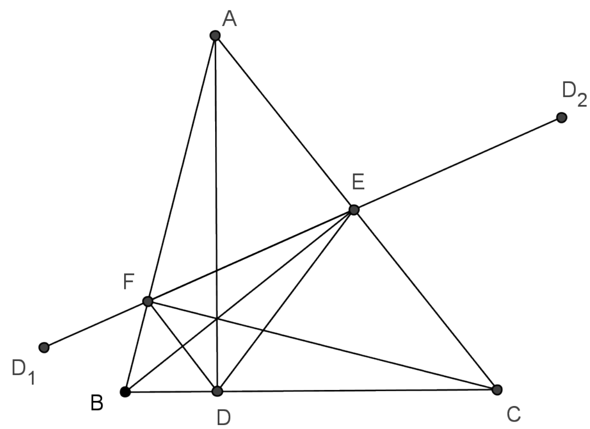

FIGURE t362b

Now point A is on line AB, so $AD_1 = AD$. And A is also on line AC, so $AD = AD_2$. Because the angle at A in isosceles triangle AD_1D_2 does not vary, the length of D_1D_2 depends only on the length of $AD_1 = AD = AD_2$, and is smallest when this length is shortest, which is when D is the foot of the altitude from A to BC.

The problem is solved, but we can characterize the required triangle of minimal perimeter more precisely: it is the triangle formed by the feet of the three altitudes of ABC. We have already seen that one vertex of the triangle of minimal perimeter is a foot of an altitude of triangle ABC. That is, if D is not the foot of an altitude, then DEF is not the minimal triangle sought. The same reasoning applies to vertices E and F: if they are not the feet of the altitudes of triangle ABC, then the inscribed triangle which includes them as vertices is not minimal. Thus the required triangle is indeed the one formed by the feet of the altitudes of the original triangle.

Problem 362b. In a quadrilateral $ABCD$, inscribe a quadrilateral $MNPQ$ with minimum perimeter. Show that the problem does not have a proper solution (that is, one which is a true quadrilateral) unless the given quadrilateral is cyclic.

But if $ABCD$ is cyclic, there exist infinitely many quadrilaterals $MNPQ$ with the same perimeter, which is smaller than, or equal to, the perimeter of any other quadrilateral inscribed in $ABCD$. This perimeter is the fourth proportional for the radius of the circle $ABCD$ and the two diagonals AC, BD.

What additional condition must $ABCD$ satisfy in order that the quadrilaterals $MNPQ$ found this way will also be cyclic? For this case, find the locus of the centers of their circumscribed circles.

Note. We break the problem statement into several parts:

1°: If there exists a quadrilateral of minimal perimeter inscribed in a given quadrilateral, then the given quadrilateral must be cyclic.

2°: Construct, in a given cyclic quadrilateral, an inscribed quadrilateral of minimal perimeter.

3°: Show that for a given cyclic quadrilateral, there are infinitely many inscribed quadrilaterals of minimal perimeter.

4°: The perimeter of the minimal inscribed quadrilateral is the fourth proportional for the radius of the circle $ABCD$ and the two diagonals AC, BD.

5°: For minimal inscribed quadrilateral $MNPQ$ itself to be cyclic, it is necessary and sufficient that the diagonals of $ABCD$ be perpendicular.

6°: In this case, find the locus of centers of the circles circumscribing $MNPQ$.

Solution. (1°.) Suppose quadrilateral $MNPQ$, inscribed in $ABCD$, has a perimeter smaller than that of any other inscribed quadrilateral (figure t362bi). Then (exercise 40) we must have $\widehat{AMQ} = \widehat{BMN}$: if point M did not have this property, then $QM + MN$ would not be minimal, and so the perimeter of $MNPQ$ would not be minimal. Similarly, the angles marked $\widehat{2}$, $\widehat{3}$ and $\widehat{4}$ must be equal.

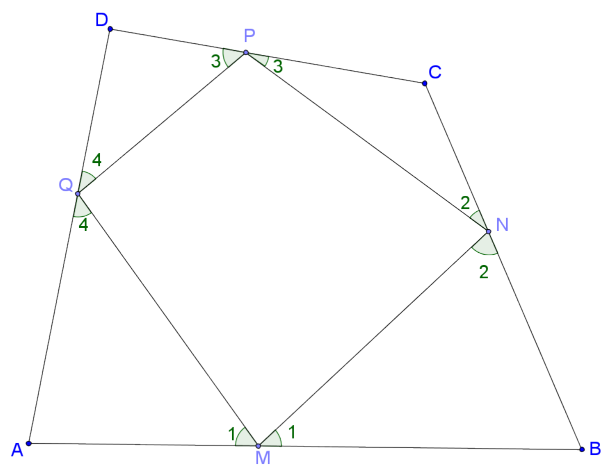

Figure t362bi

Then we have:

$$\widehat{A} + \widehat{1} + \widehat{4} = 180^\circ,$$
$$\widehat{C} + \widehat{2} + \widehat{3} = 180^\circ,$$
$$\widehat{B} + \widehat{1} + \widehat{2} = 180^\circ,$$
$$\widehat{D} + \widehat{3} + \widehat{4} = 180^\circ.$$

Adding, we have:

$$\widehat{A} + \widehat{C} + \widehat{1} + \widehat{2} + \widehat{3} + \widehat{4} = 360^\circ = \widehat{B} + \widehat{D} + \widehat{1} + \widehat{2} + \widehat{3} + \widehat{4},$$

or $\widehat{A}+\widehat{C}=\widehat{B}+\widehat{D}$. It follows (**80**) that $ABCD$ is cyclic.

(2°, 3°). We will use a proof based on geometric transformations. Figure 364bii shows the original quadrilateral $ABCD$ reflected four times: once in each of its sides. That is, we reflect first in line AB to get quadrilateral $ABC'D'$, then in line BC' to get quadrilateral $BA'D''C'$, then in line $C'D''$ to get quadrilateral $C'D''A''B'$, and finally in line $A''D''$ to get quadrilateral $D''C''B''A''$. Clearly these four new quadrilaterals are congruent to the original one. By **103**, we could also have gotten $BA'D''C'$ from $ABCD$ by rotating about point B through an angle equal to twice $\widehat{CBA}$. Likewise, we could have gotten $D''C''B''A''$ from $BA'D''C'$ by rotating about point D'' through an angle equal to twice $\widehat{ADC}$.

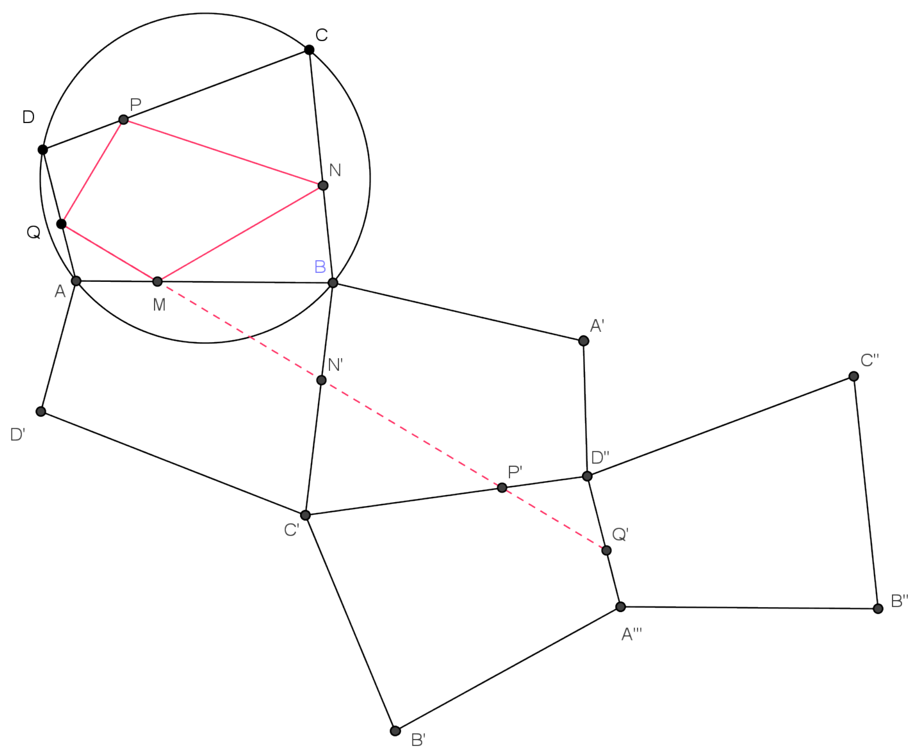

FIGURE t362bii

Suppose $MNPQ$ is a solution to our problem; that is, suppose it is a quadrilateral of minimal perimeter inscribed in $ABCD$. Then, by the results of 1°, it forms equal angles with the sides of $ABCD$. Let us look at the successive images of its vertices in the four reflections we consider above. Point N reflects into N', which is on segment BC'. Furthermore, Q, M, and N' are collinear, because $\widehat{AMQ}=\widehat{BMN}=\widehat{BMN'}$. Likewise, P, reflected first in AB, then in BC', goes to P', a point on $C'D''$ which is collinear with Q, M, and N'. Finally, reflecting Q in AB, BC', $C'D''$, we obtain Q', and Q, M, N', P' and Q' are all collinear.

Now we use the assumption that $ABCD$ is cyclic. If this is the case, then $\widehat{ABC}+\widehat{ADC}=180°$, so a composition of rotations through twice these two angles is a translation (**102**, **103**). So quadrilateral $A''B''C''D''$ is obtained from $ABCD$ by this translation. It follows that the sides of $D''C''B''A''$ are parallel to the corresponding sides of $ABCD$, and oriented in the same direction.

Thus, having chosen Q arbitrarily on segment AD, we can translate it by the distance and in the direction of AA'' to obtain Q'. Then M is the intersection of QQ' with AB, and N', P', determined likewise as intersections, can be reflected back to find N and P.

This completes the construction. We see that there are infinitely many minimal quadrilaterals, and that they all have the same perimeter, equal to the length of AA''.

Notes. For some positions of Q, and for some choices of cyclic quadrilateral $ABCD$, the construction of QQ' does not yield a corresponding minimal quadrilateral: the reflections of N' or P' may land outside segments BC or CD. But it is not difficult to see that there are still infinitely many positions of Q which yield a solution.

Students can think about how the proof fails if $ABCD$ is not to be cyclic. In that case, the composition of the four line reflections (or two rotations) will not be a translation.

Note that this solution can be seen as a generalization of the solution to exercise 362.

(4°.) This computation is a difficult one. Let R be the circumradius of $ABCD$. The product of the diagonals required in the final result suggests an application of Ptolemy's theorem:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC. \tag{1}$$

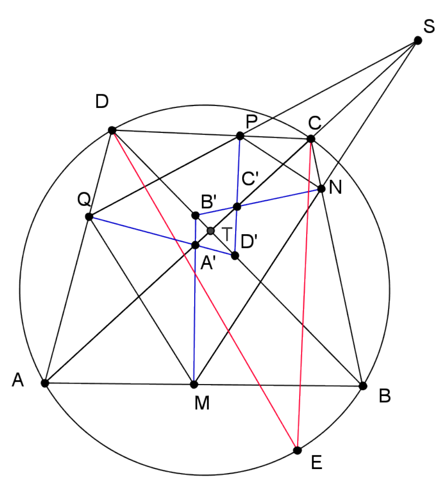

FIGURE t362biii

So we need to show that $R \cdot (MN + NP + PQ + QM) = AB \cdot CD + AD \cdot BC$. We do this through a rather artificial construction. We erect perpendiculars to the sides of $ABCD$ at points $M,\ N,\ P,\ Q$, and label the intersections of these perpendiculars as $A',\ B',\ C',\ D'$, as in figure t362biii. Note that, for example, MB' bisects $\widehat{NMQ}$, QD' bisects $\widehat{PQM}$, and so on.

We first show that $A,\ A',\ C',\ C$ are collinear. Indeed, point A', lying on the bisector MA' of $\widehat{NMQ}$, is equidistant from lines $MN,\ MQ$. Also, A' lies on QA', and so is equidistant from PQ and MQ. Therefore A' is equidistant from MN and PQ, and so lies on the bisector of the angle formed by those two lines. Let S be the intersection of lines $MN,\ PQ$. Then we have just shown that A' lies on the bisector of $\widehat{QSM}$. In the same way, we can show that C' is on this bisector. And point C is also on the bisector of $\widehat{PSN}$, since $\widehat{DPQ} = \widehat{CPN}$ and $\widehat{DPQ} = \widehat{SPC}$, so $\widehat{CPN} = \widehat{SPC}$. Likewise we can show that A is on the bisector of $\widehat{SQM}$, and therefore on the bisector of $\widehat{PSN}$

The argument of the preceding paragraph shows that $A,\ A',\ C',\ C$ are collinear. In the same way, we can show that $B,\ D',\ B',\ D$ are collinear.

Next we note that quadrilateral $MB'NB$ is cyclic, since $\widehat{B'MB} = \widehat{B'NB} = 90°$. Applying Ptolemy's theorem to this cyclic quadrilateral, we find:

$$(2). \qquad BM \cdot B'N + B'M \cdot BN = MN \cdot BB'$$

Let point E be diametrically opposite D on circle $ABCD$, and let R be the radius of this circle (*fig.* 362biii). Then triangles $BB'N,\ EDC$ are similar. Indeed, they are both right triangles: $\widehat{DCE} = 90°$ (since it is inscribed in a semicircle), and $\widehat{DEC} = \widehat{DBC}$ (they both intercept arc $\widehat{DC}$ on circle $ABCD$). Thus we have $BB' : B'N = ED : CD = 2R : CD$. Analogously, from similar triangles $BB'M,\ EDA$, we find $BB' : B'M = 2R : DA$. That is, the ratios $BB' : B'N : B'M,\ 2R : CD : DA$ are equal, and we can write, for some constant k, $BB' = 2R \cdot k,\ B'N = CD \cdot k,\ , B'M = DA \cdot k$, and rewrite equation (2) as:

$$BM \cdot k \cdot CD + k \cdot DA \cdot BN = MN \cdot k \cdot 2R,$$

or

$$BM \cdot CD + DA \cdot BN = MN \cdot 2R.$$

In the same way, we find that

$$CN \cdot DA + CP \cdot AB = NP \cdot 2R;$$
$$PD \cdot AB + DQ \cdot BC = PQ \cdot 2R;$$
$$AQ \cdot BC + AM \cdot CD = QM \cdot 2R.$$

These four (hard-earned) relationships involve pieces of the line segments we are interested in, so we add them:

$$\begin{gathered} 2R \cdot (MN + NP + PQ + QM) = AB \cdot (CP + PD) + BC \cdot (AQ + QD) + \\ + CD \cdot (AM + MB) + DA \cdot (BN + NC) = \\ = AB \cdot CD + BC \cdot AD + CD \cdot AB + DA \cdot BC, \end{gathered}$$

or

$$R \cdot (MN + NP + PQ + QM) = AB \cdot CD + AD \cdot BC,$$

and equation (1) shows that this is equivalent to the assertion of the problem.

Notes. In the course of this discussion, we have uncovered some interesting facts. For example, we find that opposite sides of quadrilateral $MNPQ$ intersect (at point S) on a diagonal of the original quadrilateral $ABCD$. In fact, this gives another construction of the 'minimal quadrilateral': Choose a point S on one of the diagonals of the original quadrilateral, and draw two rays (SM, SQ) which are symmetric in this diagonal. The intersection of these rays with the sides of the original quadrilateral are the vertices of a minimal inscribed quadrilateral.

(5°). We will show that $MNPQ$ is cyclic if and only if $AC \perp BD$. Indeed, a necessary and sufficient condition that $MNPQ$ be cyclic is that $\widehat{NPQ} + \widehat{QMN} = 180°$. A little algebra will show that this condition is equivalent (in figure t362biii) to $\widehat{BMN} + \widehat{CPN} = 90°$. But from cyclic quadrilaterals $BMB'N$, $CPC'N$ we have $\widehat{BMN} = \widehat{BB'N} = \widehat{TB'C'}$, $\widehat{CPN} = \widehat{CC'N} = \widehat{B'C'T}$. Looking at triangle $B'C'T$, we see that this condition is in turn equivalent to saying that $\widehat{B'TC'} = 90°$, or $AC \perp BD$.

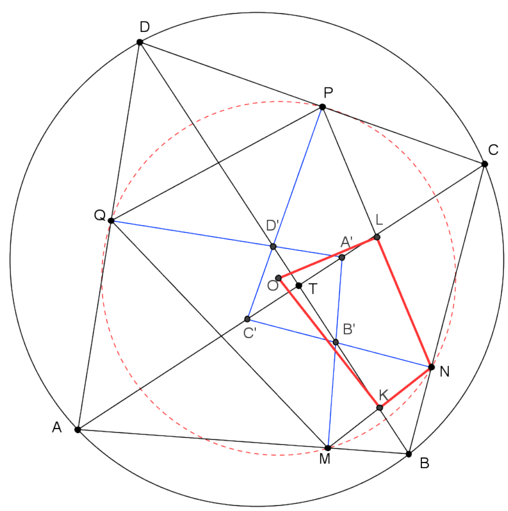

Figure t362biv

(6°). We base our solution on the result of exercise 193, which shows that if the sides of a variable polygon retain their direction, and all but one vertex slide along given lines, then the remaining vertex also slides along a line. We apply this result to quadrilateral $KNLO$ (*fig.* t362biv), where K, L are the respective midpoints of MN, NP and O is the center of the circle circumscribing $MNPQ$. Note that for any position of QQ' (that is, for any minimal inscribed quadrilateral), point O must lie on the perpendicular to MN through K and also on the perpendicular to NP through L.

The construction of $MNPQ$ shows us that its sides always retain their directions. Indeed, by this construction MN' (for example) is always parallel to AA'' (in figure t362bii), and so MN, its reflection in AB, must also retain its direction. For the same reason, NL retains its direction. And since KO, LO are perpendicular to these two segments, these two sides of $KNLO$ also retain their direction.

We now show that N, K, L slide along fixed lines. Point N certainly slides along line BC. Point K also slides along a fixed line. Indeed, if MN and $M'N'$ (not shown) are two corresponding sides of possible minimal quadrilaterals, then $MN \parallel M'N'$, and triangles MNB, $M'N'B$ are homothetic. This the midpoints K, K' of these segments lie along the median of any one of these triangles, so K slides along this median. Similarly, L slides along another line.

The result of exercise 193 then shows that point O slides along a fixed line.

Problem 363. Show that the point obtained in Exercise 105, if it is inside the triangle, is such that the sum of its distances to the three vertices is as small as possible (Exercise 269). Evaluate this sum. (Its square is half the sum of the squares of the three sides, plus $2\sqrt{3}$ times the area.)

What happens when the point is outside the triangle?

(This circumstance occurs when one of the angles, for instance $\widehat{A}$, is greater than 120°. Ptolemy's theorem gives the ratio of the sum $AB + AC$ to the segment AI intercepted by the circumscribed circle on the bisector of angle $\widehat{A}$, a ratio which is less than one. Applying the theorem of **238** to quadrilateral $AMBI$, it will be seen that the sum $MA + MB + MC$ is minimal when point M coincides with A.)

Note. We separate this problem into several statements:

1°. If it is inside the triangle, the point obtained in Exercise 105 is such that the sum of its distances to the three vertices is as small as possible

2°. The square of the minimal value of this sum is half the sum of the squares of the three sides, plus $2\sqrt{3}$ times the area of the triangle.

3°. The point obtained in Exercise 105 lies outside the triangle when one angle of the triangle is greater than 120°. In this case the point which minimizes the sum of the distances to the vertices of the triangle is the vertex of its obtuse angle.

Solution. (1°.) Suppose (*fig.* t363a) that point O is constructed as in exercise 105. That is, A', B', C' are the vertices of equilateral triangles constructed externally on the sides of a given triangle ABC, and O is the intersection of AA', BB', CC' (whose concurrence is proven in exercise 105).

Suppose M is any other point on the plane. We will show that $OA+OB+OC < MA+MB+MC$. Indeed, $AA' \leq MA+MA'$ (**26**), with equality if and only if M lies on segment AA'. The result of exercise 269 give us $MA' \leq MB+MC$, with equality if and only if M lies on arc $\overset{\frown}{BOC}$. It follows that $AA' \leq MA + MB + MC$, with equality if and only if M lies both on arc $\overset{\frown}{BOC}$ and segment AA'; that is, when M coincides with O. And, also from exercise 105, we know that $AA' = OA+OB+OC$. Thus $OA + OB + OC \leq MA + MB + MC$, with equality only when M and O coincide.

(2°). To evaluate this minimal sum, we reflect point A' in line BC to get point A'', and let D be the intersection of $A'A''$ with BC. Then AD is a median in triangle $AA'A''$, and also in triangle ABC. We apply **128** to both of these:

$$AA'^2 + AA''^2 = 2AD^2 + 2A'D^2; \tag{1}$$

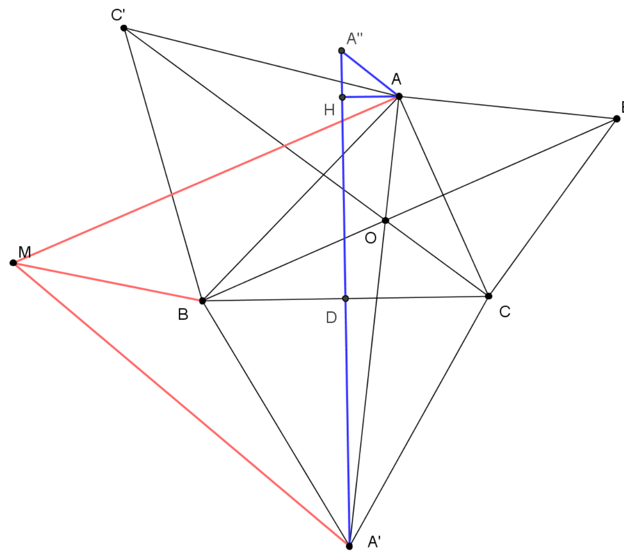

FIGURE t363a

$$AB^2 + AC^2 = 2AD^2 + 2CD^2 = 2AD^2 + \frac{1}{2}BC^2. \tag{2}$$

In equilateral triangle $A'BC$ we have altitude $A'D = \frac{\sqrt{3}}{2}BC$, so $2A'D^2 = \frac{3}{2}BC^2$, and we can write equation (1) as

$$AA'^2 + AA''^2 = 2AD^2 + \frac{3}{2}BC^2 \tag{3}$$

.

Now we can write equation (2) as $2AD^2 = AB^2 + AC^2 - \frac{1}{2}BC^2$. Using this last result, we can write equation (3) as

$$AA'^2 + AA''^2 = AB^2 + AC^2 + BC^2 \tag{4}$$

Let H be the foot of the perpendicular from A to $A'A''$. Then the result of **128b** gives us $AA'^2 - AA''^2 = 2A'A'' \cdot DH = 2\sqrt{3}BC \cdot DH = 4\sqrt{3}|ABC|$, where $|ABC|$ denotes the area of triangle ABC (since DH is equal to the altitude to BC in triangle ABC). Adding this last equation to (4), and dividing by 2, we find that $AA'^2 = \frac{1}{2}(AB^2 + BC^2 + AC^2) + 2\sqrt{3}|ABC|$, the required result.

(3°). (Proof due to Behzad Mehrdad.) Suppose point O lies outside the triangle (*fig.* 363b). In figure t363b $\widehat{CAB} > \widehat{COB} = 120°$. We draw the circumcircle of ABC, and also point I where angle bisector AI of $\widehat{BAC}$ intersects the circle. Note that $BI = CI$. We will show that the minimal value of $MA + MB + MC$ (where M is any point on the plane) occurs when M coincides with A; that is, that $AB + AC \leq MA + MB + MC$.

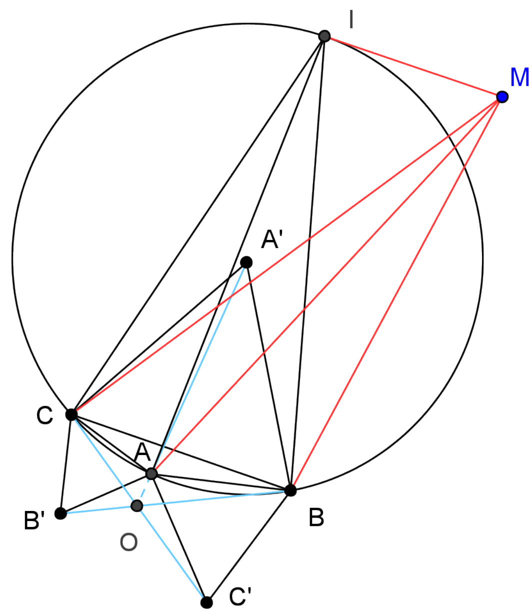

Figure t363b

Now $\widehat{CAB}$ is cut into two parts by AM, and without loss of generality, we assume that the larger part is $\widehat{CAM} \geq \widehat{BAM}$ (an analogous argument holds if the larger part is $\widehat{BAM}$). Since $\widehat{CAB} > 120°$, we must have $\widehat{CAM} \geq 60°$, which means that in triangle CAM, side AC is not opposite the largest angle, and so cannot be the largest side (**28**). That is, either (i) $AC \leq MC$ or (i) $AC \leq MA$.

In case (i), the triangle inequality (**26**) applied to triangle ABM, gives us $AB \leq MA+MB$. Adding these two results, we have $AB+AC \leq MA+MB+MC$. In case (ii), we note that $\widehat{CAB}$, an obtuse angle, is the largest angle in triangle ABC, so $AB < BC \leq MC+MB$ (this last from applying the triangle inequality to MBC). And $AC \leq MA$ by assumption. Adding, we have $AB + AC \leq MA + MB + MC$, with equality only when A and M coincide.

Note. Figure t363b shows only one possible position of point M. For other positions, the points B, M, I, C may form a quadrilateral in a different order. Nonetheless, the argument given above remains valid.

Students can show that O lies outside the triangle if and only if one of its angles is greater than 120°. One argument might come from noting that BB' and CC' meet at a 120° angle no matter what the angles of the original triangle are. If, say, the triangle's interior angle at A is greater than 120°, then point A must lie inside the circle through B, O, C, so that O lies outside the triangle.

Problem 364. More generally, find a point such that the sum of its distances to the three vertices of a triangle, multiplied by given positive numbers $\ell,\ m,\ n$, is minimal. We assume first that the three given numbers can represent the sides of a triangle.

(Let this triangle be T, and let its angles be $\alpha,\ \beta,\ \gamma$. At A, using sides $AB,\ AC$ respectively, we construct two angles $\widehat{BAC'},\ \widehat{CAB'}$ equal to α. Likewise, at B, using sides $BC,\ BA$ we construct angles $\widehat{CBA'},\ \widehat{ABC'}$ equal to β, and at C, using sides $CA,\ CB$, we construct $\widehat{ACB'},\ \widehat{BCA'}$ equal to γ. All of these angles are exterior to the triangle. Lines $AA',\ BB',\ CC'$ intersect at a point O, which is the required point if it is inside the triangle. If it is not, and also in the case in which the three given numbers are not proportional to the sides of a triangle, the minimum is achieved at one of the vertices of triangle ABC.

In the first case, where the minimum is not at a vertex, the square of the minimum can be expressed in terms of the sum

$$\ell^2(b^2+c^2-a^2)+m^2(c^2+a^2-b^2)+n^2(a^2+b^2-c^2)$$

and the product of the areas of triangles T and ABC.

Solution. We separate this problem into several statements:

1°. The construction described gives a single point (the lines mentioned are concurrent).

2°. If this point lies inside the triangle, it yields the required minimum.

3°. The square of the minimum is equal to

$$\frac{1}{2}\left[\ell^2(b^2+c^2-a^2)+m^2(c^2+a^2-b^2)+n^2(a^2+b^2-c^2)\right]$$

plus eight times the product of the areas of triangles T and ABC.

4°. If this point is not inside the triangle, or if the three given numbers are not proportional to the sides of a triangle, the minimum is achieved at one of the vertices of triangle ABC.

Proof of (1°): Figure t364a shows triangle T and the construction described in the problem statement. Let O be the intersection of circles $BA'C$ and CAB'. We will first show that point O lies on circle ABC' as well. Indeed, since $\alpha,\ \beta,\ \gamma$ are the angles of triangle T, we have $\alpha+\beta+\gamma=180°$. Thus $\widehat{AB'C}=\beta$, and from cyclic quadrilateral $AOCB'$, $\widehat{AOC}=180°-\beta$. In the same way, from cyclic quadrilateral $A'BOC$, we have $\widehat{COB}=180°-\alpha$. Then $\widehat{AOC}+\widehat{COB}=360°-\alpha-\beta$, and $\widehat{AOB}=360°-(\widehat{AOC}+\widehat{COB})=\alpha+\beta=180°-\gamma=180°-\widehat{AC'B}$. This means that quadrilateral $AOBC'$ is cyclic, so O lies on the circle through $A,\ B,\ C'$.

Next we show that $A,\ O,\ A'$ are collinear. Indeed, we saw above that $\widehat{AOC}=180°-\beta$. From cyclic quadrilateral $A'BCO$, we have $\widehat{CBA'}=\widehat{COA'}=\beta$, so angles $\widehat{AOC},\ \widehat{COA'}$ are supplementary, and points $A,\ O,\ A'$ are collinear.

In the same way (and using the fact that $AB'CO,\ A'BOC,\ BC'AO$ are all cyclic quadrilaterals), we can show that $B,\ O,\ B'$ and $C,\ O,\ C'$ are collinear. This proves that lines $AA',\ BB',\ CC'$ are concurrent at O.

(2°). Let M be any point on the plane (*fig.* t364b). We apply Ptolemy's theorem and its inequality generalization (**237** and **237b**, with special regard to the last paragraph of **237b**), to the quadrilateral formed by $M,\ A',\ B$, and C:

$$MA'\cdot BC\leq MB\cdot A'C+MC\cdot A'B,$$

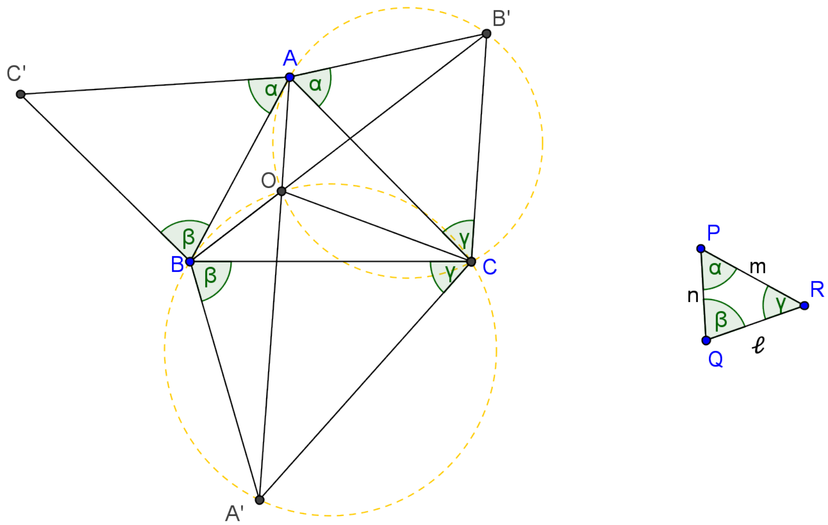

Figure t364a

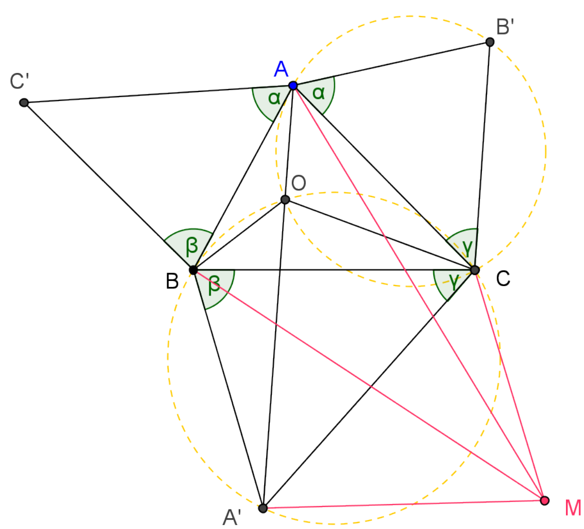

Figure t364b

with equality when M lies on minor arc $\widehat{BC}$ of circle $A'BC$. But by construction, the three triangles around ABC are similar to triangle T. In particular, triangles T, $A'BC$ are similar, and we have:

$$(1) \qquad BC : CA' : A'B = \ell : m : n,$$

so the previous inequality can be written as

$$(2) \qquad \ell \cdot MA' \leq m \cdot MB + n \cdot MC.$$

By the triangle inequality (**26**), we have $AA' \leq MA + MA'$, with equality if and only if M lies on segment AA'. Multiplying this inequality by ℓ, and combining it with inequality (2), we find $\ell \cdot AA' \leq \ell \cdot MA + \ell \cdot MA' \leq \ell \cdot MA + m \cdot MB + n \cdot MC$, with equality if and only if M lies on the intersection of circle $A'BC$ with segment AA'; that is, with point O.

Note. The argument here generalizes that of exercise 363.

An easy special case occurs when M and A' coincide. In that case, $A'A = MA$, and the required inequality is easy to see.

We have shown that $\ell \cdot AA'$ is a lower bound for the sum we are trying to minimize. But if we follow the same logic using BB' or CC', we will get $m \cdot BB'$ or $n \cdot CC'$ as a lower bound. Which one is actually the minimum?

In fact, all three ($\ell \cdot AA'$, $m \cdot BB'$, $n \cdot CC'$) are equal. This follows from pairs of similar triangles. For example, $AA' : BB' = m : \ell$ because triangles ACA', BCB' (in figure t364a) are similar: $\widehat{ACA'} = \widehat{ACB} + \gamma = \widehat{BCB'}$, and $\widehat{CA'A}$, $\widehat{CBB'}$ intercept the same arc on the circle through A', B, C.

This argument generalizes the result of exercise 105, where it was proven that $AA' = BB'$ (in figure t105). In both cases, the proof can be given in terms of congruent (or similar) triangles, or in terms of transformations (rotations and dilations).

3°. We now give a computation of the minimal value, which is equal to $\ell \cdot AA'$. We must express the length of AA' in terms of the lengths a, b, c of the given triangle and ℓ, m, n. We let A'' be the reflection in line BC of point A'. Let AA'' intersect BC at D (*fig.* t364c), and draw AA'' and AD.

From the Pythagorean theorem, applied to triangle $A'BD$ we have:

$$(1) \qquad A'D^2 = A'B^2 - BD^2.$$

In triangle $AA'A''$, segment AD is a median. Hence we have (**128**):

$$(2) \qquad A'A^2 + A''A^2 = 2AD^2 + 2A'D^2.$$

From Stewart's theorem (**127**) applied to triangle ABC and line AD we have:

$$(3) \qquad AD^2 = \frac{BD}{BC} \cdot AC^2 + \frac{DC}{BC} \cdot AB^2 - BD \cdot DC.$$

Noting that $DC = BC - BD$, we can rewrite this last product:

$$(4) \qquad BD \cdot DC = BD \cdot (BC - BD) = BD \cdot BC - BD^2$$

We now use (1), (3) and (4) to rewrite (2):

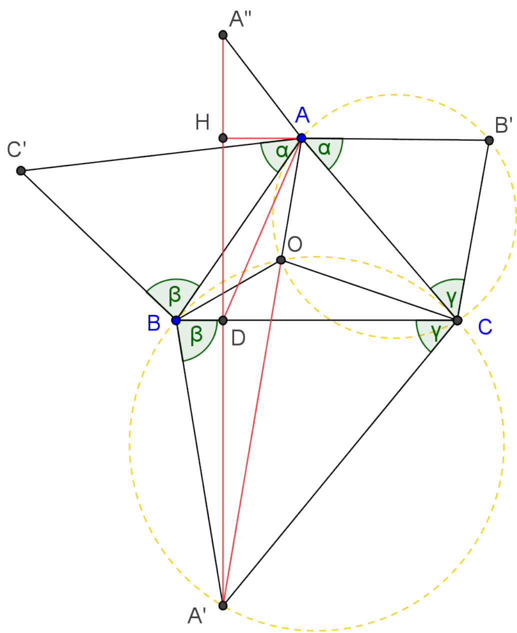

FIGURE t364c

$$A'A^2+A''A^2 = 2\left(\frac{BD}{BC}\cdot AC^2 + \frac{DC}{BC}\cdot AB^2 - BD\cdot BC + BD^2\right)+2\left(A'B^2 - BD^2\right)$$

$$= 2\frac{BD}{BC}\cdot AC^2 + 2\frac{DC}{BC}\cdot AB^2 + 2BC^2\left(\frac{A'B^2}{BC^2} - \frac{BD^2}{BC^2} - \frac{BD}{BC} + \frac{BD^2}{BC^2}\right)$$

$$= 2\frac{BD}{BC}\cdot AC^2 + 2\frac{DC}{BC}\cdot AB^2 + 2BC^2\left(\frac{A'B^2}{BC^2} - \frac{BD}{BC}\right). \tag{5}$$

We now rewrite this last equation in terms of the sides of triangle T. First we note that $A'D$ is an altitude in triangle $A'BC$. Hence we have (**126**):

$$BD = \frac{A'B^2 + BC^2 - A'C^2}{2BC};\quad DC = \frac{A'C^2 + BC^2 - A'B^2}{2BC}. \tag{6}$$

Noting that triangles $T,\ A'BC$ are similar, we can replace sides of the latter with proportional sides of the former in the expressions in (6), to get:

$$\frac{BD}{BC} = \frac{n^2+\ell^2-m^2}{2\ell^2};\quad \frac{DC}{BC} = \frac{\ell^2+m^2-n^2}{2\ell^2};\quad \frac{A'B}{BC} = \frac{n}{\ell},$$

where the constant of proportionality (not shown) or its square has canceled out in each fraction.

If we substitute these values into (5) and multiply by ℓ^2, we have:

$$\ell^2(A'A^2+A''A^2) = (n^2+\ell^2-m^2)AC^2+(\ell^2+m^2-n^2)AB^2+(2n^2-(n^2+\ell^2-m^2)BC^2$$

$$= (n^2+\ell^2-m^2)AC^2+(\ell^2+m^2-n^2)AB^2+(n^2+m^2-\ell^2)BC^2$$

(Note that the factors of 2 in the denominators have canceled out.)

We will get close to the expression given in the problem statement if we express this last relationship in terms of the sides of the original triangle. We have $AB = c,\ BC = a,\ CA = b$, and the last equation becomes:

$$\ell^2(A'A^2+A''A^2) =$$
$$\ell^2(AC^2+AB^2-BC^2)+m^2(AB^2+BC^2-AC^2)+n^2(AC^2+BC^2-AB^2)$$

$$= \ell^2(b^2+c^2-a^2)+m^2(c^2+a^2-b^2)+n^2(b^2+a^2-c^2). \tag{7}$$

Letting H be the foot of the perpendicular from A to $A'A''$, we apply the result of **128b** to triangle $AA'A''$:

(8). $$A'A^2 - A''A^2 = 2A'A''\cdot DH = 4A'D\cdot DH.$$

Since we need to involve the area of ABC, we take this opportunity to rewrite this last product in terms of areas of triangles. We have (using absolute value for area) $2|ABC| = DH\cdot BC;\ 2|A'BC| = A'D\cdot BC$, so $DH = \frac{2|ABC|}{BC};\ A'D = \frac{2|A'BC|}{BC}$, and

$$A'A^2 - A''A^2 = 4A'D\cdot DH = \frac{16}{BC^2}\cdot|ABC|\cdot|A'BC|.$$

Since triangles $A'BC,\ T$ are similar, we have (**257**) $|A'BC| : |T| = BC^2 : \ell^2$, and this last relationship can be rewritten as:

(9). $$\ell^2(A'A^2 - A''A^2) = 16\cdot|ABC|\cdot|T|$$

Now adding (7) and (9), and dividing by 2, and recalling that the desired minimum is given by $\ell\cdot AA'$, we get the required result:

$$\ell^2\cdot AA'^2 = \frac{1}{2}\left[(\ell^2(b^2+c^2-a^2)+m^2(c^2+a^2-b^2)+n^2(a^2+b^2-c^2)\right]+8\cdot|ABC|\cdot|T|.$$

Note. As is often the case when we use metric relationships in a triangle, several of the arguments above can be simplified using trigonometric, rather than purely synthetic methods.

4°. Suppose O lies outside triangle ABC, say on the extension of AA' past A (*fig.* t364d). We draw circle ABC, and locate point I on this circle such that $BI : IC = n : m$ (**116**). The theorem of **116** assures us that there are two such points, and we choose the one which does not lie on arc $\widehat{BAC}$.

We also find B'' on IB and C'' on IC such that $IB'' = n,\ IC'' = m$. Then $B''C'' \parallel BC$. Denoting the length of $B''C''$ as ℓ', and noting that triangles $IBC,\ IB''C''$ are similar, we have:

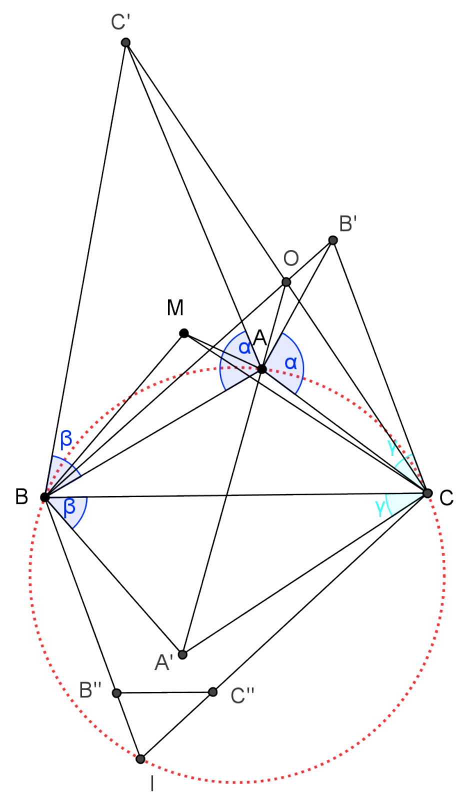

Figure t364d

(10) $$BC : CI : BI = \ell' : m : n$$

Now since O is outside triangle ABC, BB' must lie outside the triangle, which implies that the sum $\widehat{BAC} + \alpha > 180°$. But quadrilateral $ACIB$ is cyclic, so $\widehat{BAC} + \widehat{BIC} = 180°$. Therefore $\widehat{BIC} < \alpha$. Therefore in triangles T, $B''IC''$ we have (**28**)

(11). $$\ell' < \ell$$

We next apply Ptolemy's theorem (**237**) to quadrilateral $ABIC$: $AB \cdot IC + AC \cdot BI = BC \cdot AI$. If we replace the segments common to this equation and to (10) with their proportional quantities ℓ, m, n, and cancel out the constant of proportionality, we get

(12) $$m \cdot AB + n \cdot AC = \ell' \cdot AI.$$

Now let M be any point in the plane, other than A. Then quadrilateral $BMCI$ may or may not be cyclic, but in either case (**237, 237a**) we have $MI \cdot BC \leq MB \cdot IC + MC \cdot IB$. Again, using (10) to replace segments with proportional quantities, we have $\ell' \cdot MI \leq m \cdot MB + n \cdot MC$, with equality holding only if M lies on arc $\widehat{BAC}$. Now by (11), we have $\ell' \cdot MA \leq \ell \cdot MA$, and adding the last two inequalities we obtain:

$$\ell' \cdot MI + \ell' \cdot MA \leq \ell \cdot MA + m \cdot MB + n \cdot MC. \tag{13}$$

Equality holds if M is on arc $\widehat{BAC}$ and $MA = 0$. The last condition is stronger than the first, so equality holds only when M coincides with A.

From the triangle inequality (**26**), we know that $IA \leq MI + MA$, with equality when M is on segment IA. Together with (13), this implies:

$$\ell' \cdot IA \leq \ell \cdot MA + m \cdot MB + n \cdot MC, \tag{14}$$

with equality only if M coincides with A.

Finally, combining (12) and (14), we find that

$$m \cdot AB + n \cdot AC \leq \ell \cdot MA + m \cdot MB + n \cdot MC,$$

with equality if and only if M and A coincide. That is, the expression $\ell \cdot MA + m \cdot MB + n \cdot MC$ is minimal when M coincides with A.

Finally, we consider the (relatively easy) case when segments of length ℓ, m, n do not form a triangle. Suppose, for instance, that ℓ is the largest of these segments. Then we have $\ell > m + n$ (otherwise the three segments would form a triangle), and for any point M different from A, we have $\ell \cdot MA + m \cdot MB + n \cdot MC \geq (m+n) \cdot MA + m \cdot MB + n \cdot MC = m \cdot (MA + MB) + n \cdot (MA + MC) \geq m \cdot AB + n \cdot AC$. (**26**) Again, the given expression is smallest when M coincides with A.

Problem 365. We divide each side of a triangle into segments proportional to the squares of the adjacent sides, then join each division point to the corresponding vertex. Show that:

1^o. The three lines obtained in this way are concurrent;

2^o. That this is precisely the point that would be obtained in Exercise 197, taking the point O to be the center of mass of the triangle;

3^o. That this point is the center of mass of the triangle PQR formed by its projections on the sides of the original triangle.

Solution. ($1°$). This statement can be proven by applying Ceva's theorem (**198**). Suppose (*fig.* t365) $BD : DC = c^2 : b^2$; $CE : EA = a^2 : c^2$; $AF : FB = b^2 : a^2$. The product of these three ratios is 1, so Ceva's theorem shows that lines AD, BE, CF are concurrent. Let O' be their point of intersection.

($2°$). We first compute the ratio $O'P : O'Q : O'R$ (where P, Q, R are the points mentioned in $3°$), then compare this with the corresponding ratios for O. This will tell us about the relationship between the two points.

In fact, we will show that $O'P : O'Q : O'R = a : b : c$ (an interesting result in its own right), by looking at areas. Using absolute value to denote area, we have $|ABD| : |ABO'| = AD : AO' = |ADC| : |AO'C|$ (each pair of triangles has equal altitudes from B or C, so the ratio of their areas is the ratio of the corresponding

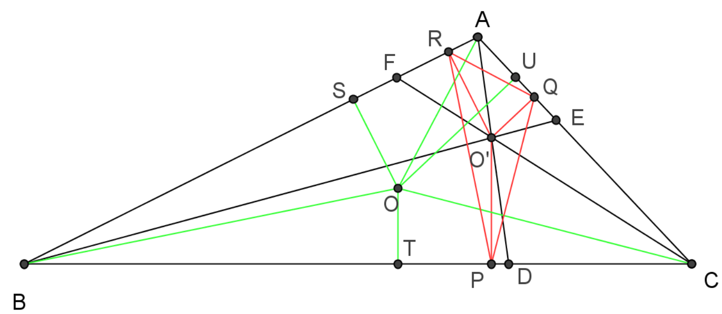

FIGURE t365

bases). We can write this proportion as $|ABD| : |ADC| = |ABO'| : |AO'C|$. But $|ABD| : |ADC| = BD : DC = c^2 : b^2$, since these triangles have equal altitudes from A.

Hence $|ABO'| : |AO'C| = c^2 : b^2$ as well, or $O'R \cdot c : O'Q \cdot b = c^2 : b^2$. It follows that $O'Q : O'R = b : c$. In the same way, we can show that $O'P : O'Q = a : b$, or

$$O'P : O'Q : O'R = a : b : c. \tag{1}$$

.

Now we make a similar, but easier, computation for O, which is the centroid of triangle ABC. In fact, in exercise 295 (letting $p = q = r = 1$ in that argument) it is shown that triangles OAB, OBC, OCA are equal in area. So, for example, $|AOB| : |BOC| = c \cdot OS : a \cdot OT = 1 : 1$ (where OS, OT, OU are the distances from O to AB, BC, AC). Hence $OS : OT = \frac{1}{c} : \frac{1}{a}$. Similarly, $OT : OU = \frac{1}{a} : \frac{1}{b}$. That is, $OS : OT : OU = \frac{1}{c} : \frac{1}{a} : \frac{1}{b}$.

It now follows from the second lemma in the solution to exercise 197 that lines AO', BO', CO' are symmetric to AO, BO, CO in the corresponding angle bisectors of ABC. This implies that O' is the point obtained from O by the construction of exercise 197.

(3°). Quadrilateral $ARO'Q$ has two opposite angles equal to 90°, so it is cyclic. Therefore angles $\widehat{BAC}$, $\widehat{RO'Q}$ are supplementary, and it follows from **256** that $|RO'Q| : |ABC| = (O'Q \cdot O'R) : bc$. Similarly, we can show that $|RO'P| : |ABC| = (O'R \cdot O'P) : ac$ and $|PO'Q| : |ABC| = (O'P \cdot O'Q) : ab$. From these proportions, and from (1), it follows that $|RO'Q| = |RO'P| = |PO'Q|$. For example, we have

$\frac{|RO'Q|}{|ABC|} : \frac{|RO'P|}{|ABC|} = \frac{|RO'Q|}{|RO'P|} = \frac{O'Q \cdot O'R}{bc} : \frac{O'R \cdot O'P}{ac} = \frac{O'Q}{b} : \frac{O'P}{a} = 1$. The result of exercise 295 then shows that O' is the center of mass (the intersection of the medians) of triangle PQR.

Problem 366. In a given triangle, inscribe a triangle such that the sum of the squares of its sides is minimal. (Assuming that this minimum exists, show that it can only be the triangle PQR of the preceding exercise.)

Conclude that the point O' (in the preceding exercise) is the one such that the sum of the squares of its distances to the three sides is the smallest possible (Exercises 137, 140).

More generally, in a given triangle, inscribe a triangle such that the squares of its sides, multiplied by given numbers, yield the smallest possible sum.

Solution. We divide the problem into three parts, and offer two complete solutions.

(1°). Triangle PQR of exercise 365 is the inscribed triangle such that the sum of the squares of its sides is minimal.

(2°). The centroid of that triangle (point O' in exercise 365) is the point such that the sum of the squares of its distances to the three sides is minimal.

(3°). Construction: In a given triangle, inscribe a triangle such that the squares of its sides, multiplied by given numbers, yield the smallest possible sum.

Solution I: As suggested in the problem statement, we assume that a triangle JKL exists, inscribed in a given triangle ABC, such that the sum $s = JK^2 + KL^2 + JL^2$ is minimal.

Lemma 1: If such a triangle exists, then its medians are perpendicular to the corresponding sides of ABC.

Proof: We first use **128** to relate the sum of the squares of the sides to the lengths of the medians. Suppose (*fig.* t366a) JJ_1 is a median of JKL. Then we have $JL^2 + JK^2 = \frac{1}{2}KL^2 + 2JJ_1^2$, so that $s = \frac{3}{2}KL^2 + 2JJ_1^2$. Now for fixed points K, L, we can minimize this sum by minimizing JJ_1; that is, by choosing J so that $JJ_1 \perp BC$. Since we can do this for any two vertices of JKL, s is not minimal unless all three medians are perpendicular to the sides of ABC on which the corresponding vertex is located. This proves our lemma.

Proof of (1°). Now let JKL be the inscribed triangle such that the sums of the squares of its sides is minimal (so that its medians are perpendicular to the sides of ABC), and let G be its centroid. We will use arguments similar to those of exercise 365, investigating the distances from G to the sides of ABC, to show that G coincides with point O' in that exercise. It then follows that JKL is the same triangle as PQR.

The result of exercise 295 shows that (using absolute value for area)

$$|GKL| = |GLJ| = |GJK|. \tag{1}$$

As in exercise 365, we note that quadrilateral $ALGK$ is cyclic (two opposite angles are right angles), so angles $\widehat{BAC}$, $\widehat{LGK}$ are supplementary. Then, from **256**, $|GKL| : |ABC| = (GK \cdot GL) : (AB \cdot AC)$. Likewise, $|GLJ| : |ABC| = (GL \cdot GJ) : (BC \cdot BA)$ and $|GJK| : |ABC| = (GJ \cdot GK) : (CA \cdot CB)$. But from (1) it now follows that $(GK \cdot GL) : (AB \cdot AC) = (GL \cdot GJ) : (BC \cdot BA) = (GJ \cdot GK) : (CA \cdot CB)$. A bit of algebra shows that this is equivalent to $GJ : GK : GL = a : b : c$.

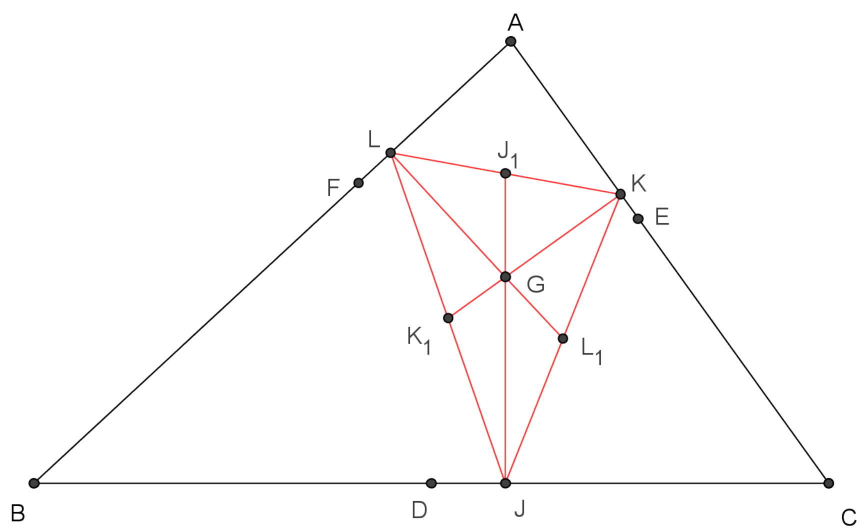

FIGURE t366a

This last relationship implies that G coincides with point O' in exercise 365. Indeed, construction 10 of **157** show that there are at most four points in the plane which satisfy $GJ : GK : GL = a : b : c$, and construction 9 of the same section shows that there is only one such point which is inside all three angles $\widehat{A}$, $\widehat{B}$, $\widehat{C}$. This in turn implies that triangle JKL, which is formed by the feet of the perpendiculars from G to the sides of ABC, is the same as triangle PQR in exercise 365. This concludes the proof of the first assertion in the problem.

(2°). We now show that G is the point which minimizes the sum of the squares of the distances to the sides of ABC. Let M be any point in the plane (*fig.* t366b), and let J', K', L' be the feet of the perpendiculars from M to BC, AC, AB respectively. Suppose G' is the centroid of triangle $J'K'L'$. The result of exercise 140 gives us

$$MJ'^2 + MK'^2 + ML'^2 = G'J'^2 + G'K'^2 + G'L'^2 + 3MG'^2 \tag{2}$$

.

We express this relationship using the sides of triangle $J'K'L'$. The result of **56** tells us that $G'J'$, $G'K'$, $G'L'$ are each $\frac{2}{3}$ the corresponding medians, and the result of exercise 137 tells us that the sum of the squares of the medians is $\frac{3}{4}$ the sum of the squares of triangle $J'K'L'$. Together, these results give:

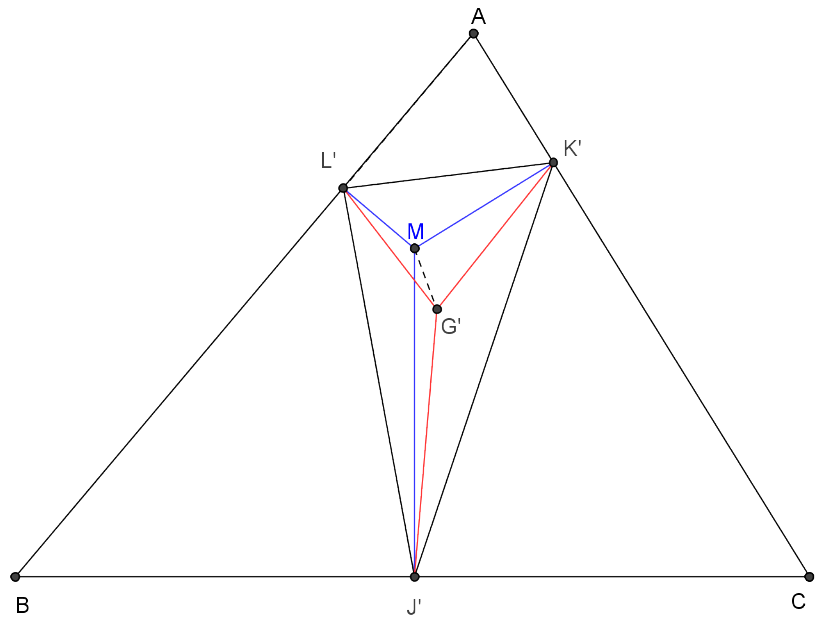

Figure t366b

$$G'J'^2 + G'K'^2 + G'L'^2 = \frac{1}{3}(J'K'^2 + K'L'^2 + L'J'^2). \tag{3}$$

From (2) and (3), we get $MJ'^2+MK'^2+ML'^2 = \frac{1}{3}(J'K'^2+K'L'^2+L'J'^2)+3MG'^2$.

This last relationship gives us everything we need. Indeed, we can minimize $J'K'^2 + K'L'^2 + L'J'^2$ by placing M at the point O' in exercise 365 (as proved in its solution), and at the same time $MG = MO' = 0$, which is also a minimum.

(3°). First we let ℓ, m, n be three *positive* numbers, postponing the discussion for three numbers with arbitrary signs. We will determine the triangle $J'K'L'$ for which $s = \ell \cdot J'K'^2 + m \cdot K'L'^2 + n \cdot L'J'^2$ is minimal by analogy with the argument in part 1° of the present exercise.

We choose point J_1 such that $K'J_1 : J_1L' = m : n$ (*fig.* t366c). By analogy with 1°, we will show that if $J'K'L'$ minimizes s, then $J_1J' \perp BC$. Indeed, we have $K'J_1 = \frac{m}{m+n}K'L'$; $J_1L' = \frac{n}{m+n}K'L'$ and we have, by **127**:

$$\frac{m}{m+n}K'L' \cdot J'L'^2 + \frac{n}{m+n}K'L' \cdot J'K'^2 = K'L' \cdot J_1J'^2 + \frac{mn}{(m+n)^2}K'L'^3,$$

or, dividing by $K'L'$ and multiplying by $m+n$:

$$m \cdot J'L'^2 + n \cdot J'K'^2 = (m+n) \cdot J_1J'^2 + \frac{mn}{m+n}K'L'^2; \tag{4}$$

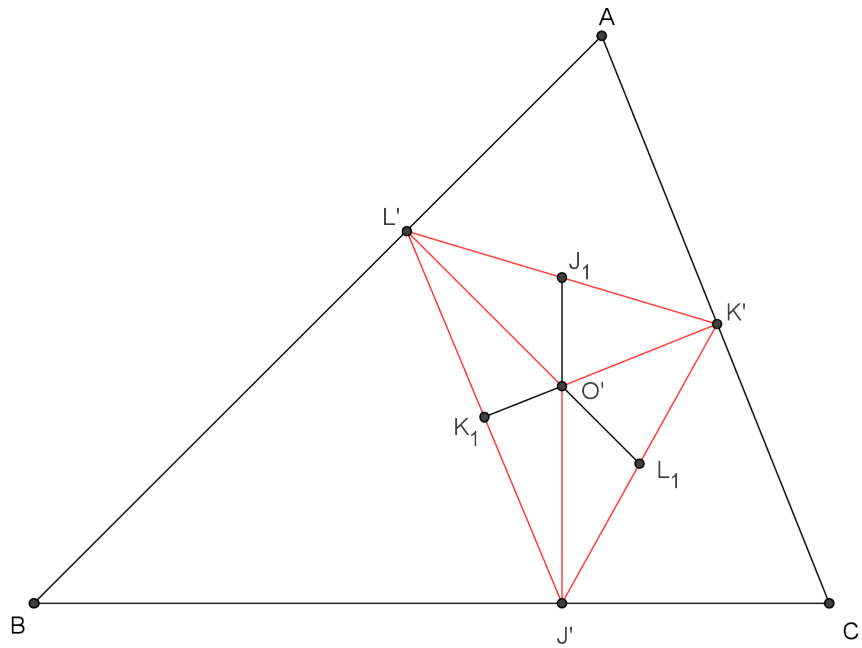

Figure t366c

$$s = \left(\ell + \frac{mn}{m+n}\right) \cdot K'L'^2 + (m+n) \cdot J_1J'^2. \tag{5}$$

If J_1J' were not perpendicular to BC, we could make the value of s smaller by leaving K', L' in place and moving J_1 to coincide with the foot of the perpendicular from J' to line BC. Thus, for s to be minimal, the line dividing segment $K'L'$ (internally) in the ratio $m : n$ must be perpendicular to BC. In the same way, we can show that the line $K'K_1$ which divides $L'J'$ in the ratio $n : \ell$ (internally) must be perpendicular to AC, and the line $L'L_1$ which divides $J'K'$ in the ratio $\ell : m$ (internally) must be perpendicular to AB. If we choose these positions for $J_1,\ K_1,\ L_1$, then we have $\frac{J_1K'}{J_1L'} \cdot \frac{K_1L'}{K_1L'} \cdot \frac{L_1J'}{L_1K'} = -1$, and **198** assures us that $J'J_1,\ K'K_1,\ L'L_1$ are concurrent at some point O'.

We now relate the ratios $\ell : m,\ m : n,\ n : \ell$ (into which we have divided the sides of $J'K'L'$) to the ratios of $|J'K'O'|,\ |K'O'L'|,\ |L'O'J'|$. For example, triangles $O'L'J',\ O'J'K'$ have a common side $O'J'$, so the ratio of their areas is the ratio of their altitudes from L', K'. But that ratio is just $J_1L' : K'J_1 = n : m$. (We can prove this as in the argument in the solution for exercise 295, by drawing similar right triangles which include these sides and also the altitudes in question.) That is, we have $|O'L'J'| : |O'J'K'| = J_1L' : K'J_1 = n : m = \frac{1}{m} : \frac{1}{n}$. Similarly, we have $|O'L'K'| : |O'K'J'| = \frac{1}{\ell} : \frac{1}{n}$; $|O'J'L'| : |O'L'K'| = \frac{1}{m} : \frac{1}{\ell}$. We can write this result as:

$$\ell \cdot |O'K'L'| = m \cdot |O'L'J'| = n \cdot |O'J'K'|.$$

On the other hand, we see that (as before) quadrilateral $O'L'AK'$ is cyclic, so $\widehat{L'O'K'}$, $\widehat{L'AK'}$ are supplementary, and so (**256**) $|OK'L'| : |ABC| = O'L' \cdot O'K' : AB \cdot AC$. Similarly, $|O'L'J'| : |ABC| = O'L' \cdot O'J' : BA \cdot BC$, and $|O'J'K'| : |ABC| = O'K' \cdot O'J' : CA \cdot CB$. It follows that:

$$\ell \cdot \frac{O'L}{AC} \cdot \frac{O'K'}{AB} = m \cdot \frac{O'L'}{BA} \cdot \frac{O'J'}{BC} = n \cdot \frac{O'J'}{BC} \cdot \frac{O'K'}{AC},$$

where we have divided each ratio by $|ABC|$. This can be written as:

$$\frac{O'J'}{\ell \cdot BC} = \frac{O'K'}{m \cdot AC} = \frac{O'L'}{n \cdot AB}.$$

That is, the distances from O' to the sides of triangle ABC are in proportion to the products $\ell \cdot BC$, $m \cdot AC$, $n \cdot AB$. As in the argument in the last paragraph of (1°) above, this determines the position of point O', and therefore the positions of J', K', L' on the sides of triangle ABC, so that inscribed triangle $J'K'L'$ minimizes the product of ℓ, m, n and the squares of the lengths of its sides.

Note 1. We briefly indicate what happens when ℓ, m, and n are not all positive. It turns out useful to consider the following cases:

(1) $m + n > 0$, $n + \ell > 0$, and $\ell + m > 0$;
(2) $m + n < 0$, $n + \ell < 0$, and $\ell + m < 0$;
(3) of the sums $m + n$, $n + \ell$, $\ell + m$, at least one is positive and at least one is negative, but none is zero;
(4) of the sums $m + n$, $n + \ell$, $\ell + m$, at least one is zero.

We will make use of directed line segments, and also signed areas (see the solutions to exercise 301 and 324). We assume that the required triangle is $J'K'L'$ (*fig.* t366c).

1°. As before, we find points J_1, K_1, L_1 which divide the sides of $J'K'L'$ in the ratios ℓ, m, n, taking into account the signs of these numbers. (That is, some of the points we take divide the corresponding side *externally* in the required ratio.) Then, in magnitude and sign, we have $K'J_1 : J_1L' = m : n$, $L'K_1 : K_1J' = n : \ell$, $J'L_1 : L_1K' = \ell : m$. We can then derive equation (5) above just as before. Now equation (5) represents s in terms of side $K'L'$ of triangle $J'K'L'$ and segment $J'J_1$. Analogous derivations, using other sides of $J'K'L'$ in place of $K'L'$, will give analogous expressions for s. That is:

$$\begin{aligned} s &= \left(\ell + \frac{mn}{m+n}\right) \cdot K'L'^2 + (m+n) \cdot J_1J'^2 \\ &= \left(m + \frac{n\ell}{n+\ell}\right) \cdot L'J'^2 + (n+\ell) \cdot K_1K'^2 \\ &= \left(n + \frac{\ell m}{\ell + m}\right) \cdot J'K'^2 + (\ell+m) \cdot L_1L'^2. \end{aligned} \tag{6}$$

Since the sums $\ell + m$, $m + n$, $n + \ell$ are all positive, the argument remains valid, as long as we apply the correct signs to the various segments and areas involved.

2°. If the sums $\ell+m,\ m+n,\ n+\ell$ are all negative, then the argument above shows that the expression s takes on a *maximal* value at the point O' we have constructed. Indeed, equation (6) shows that, for example, if we fix $K',\ L'$, a smaller length for $J'J_1$ results in a larger value for s.

3°. In this case, s has neither a maximum nor a minimum, for any triangle $J'K'L'$. Indeed, suppose, for example, $m+n>0,\ n+\ell<0$. Then an examination of equation (6) shows that if we fix $K',\ L'$, and move J' away from the foot of the perpendicular from J_1 (along line BC), then s keeps increasing. But if we fix $J',\ L'$, and move K_1 (along line AC), then s keeps decreasing.

4°. Suppose for example $m+n=0$. We will show again that s can take on neither a maximum nor a minimum for any triangle $J'K'L'$. Let H be the foot of the altitude from J' to $K'L'$ in triangle $J'K'L'$. Then we have $s=\ell\cdot K'L'^2+m\cdot(J'L'^2-J'K'^2)$. But $J'L'^2-J'K'^2=J'H^2+HL'^2-(J'H^2+K'H^2)=HL'^2-K'H^2$. Now $HL'=K'L'-K'H$ (both in magnitude and sign), so $HL'^2=K'L'^2-2K'H\cdot K'L'+K'H^2$, and we can write $s=\ell\cdot K'L'^2+m\cdot(K'L'^2-2K'H\cdot K'L')$. This expression shows that if we move point H (by moving point J' along line BC), we can arrange for s to take on any value at all.

Note 2. For the cases in which the sums indicated in the previous note all have the same sign, our reasoning has been based on the assumption that there exists a triangle $J'K'L'$ which minimizes (or maximizes) the expression s. But even if we make the assumption that s cannot take arbitrarily small values as we vary triangle $J'K'L'$, we still cannot be sure that there exists a triangle which actually minimizes s. We can still imagine a case in which, for example, the value of s remains larger than some fixed number, but can be made as close as we like to that number, without ever achieving a minimal value.

But in fact there is no such case. Our second solution, which is in some ways less natural than our first, does not require the assumption of the existence of a minimal triangle $J'K'L'$. We limit the discussion to the case where $\ell,\ m,\ n$ are all positive.

Solution II. We start with the following algebraic identity, which is not at all obvious, but is easily verified:

$$(7)\quad (a^2+b^2+c^2)\cdot(x^2+y^2+z^2)=(ax+by+cz)^2+(bz-cy)^2+(cx-az)^2+(ay-bx)^2.$$

This equation holds for any six real numbers. For our solution, we think of $a,\ b,\ c$ as the (fixed) sides of our given triangle, and $x,\ y,\ z$ as the (variable) distances from a point O' on the plane to its three sides. Then (see the solution to exercise 301) we have $ax+by+cz=2S$, where S is the area of the given triangle. (Note that in the most general case, we are using signed areas, as described in exercise 301).

Alternate proof of (1°): As in our first proof, let $J'K'L'$ be any triangle inscribed in the given triangle ABC, (*fig.* 366d) and let $J_2,\ K_2,\ L_2$ be the feet of the perpendiculars from its centroid O' to the sides of ABC. As in the derivation of equation (3) above, we can write:

$$(8)\qquad s=J'K'^2+K'L'^2+L'J'^2=3(O'J'^2+O'K'^2+O'L'^2).$$

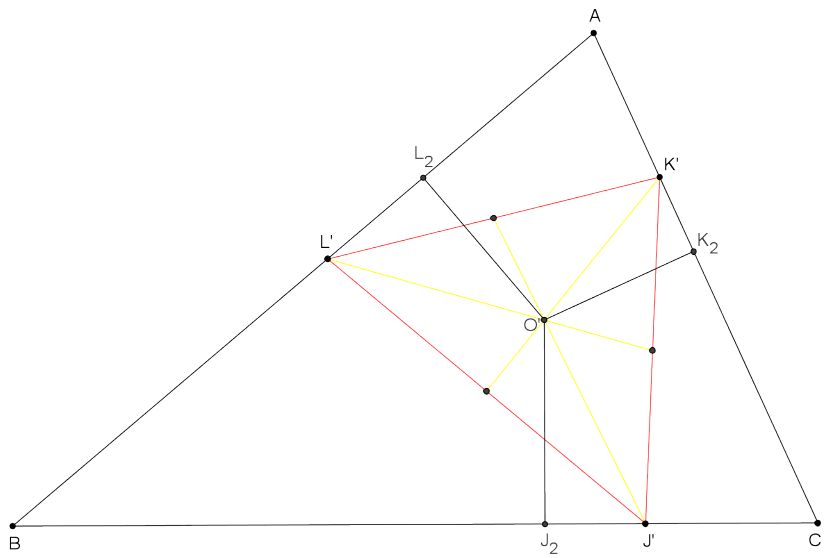

FIGURE t366d

We use the Pythagorean theorem to relate the segments on the right to $O'J_2$, $O'K_2$, $O'L_2$. We have $O'J'^2 = O'J_2^2 + J'J_2^2$, $O'K'^2 = O'K_2^2 + K'K_2^2$, $O'L'^2 = O'L_2^2 + L'L_2^2$, so that

$$s = 3(O'J_2 + O'K_2 + O'L_2) + 3(J'J_2^2 + K'K_2^2 + L'L_2^2),$$

and s is minimal when each sum on the right is minimal. The second sum is minimal when the pairs of points J_2, J'; K_2, K'; L_2, L' coincide, and this situation also minimizes the first sum. This happens when O' is the point we found in exercise 365, and triangle $J'K'L'$ coincides with triangle PQR in that exercise.

Alternate proof of (2°): We can write equation (7) in the form:

$$(x^2 + y^2 + z^2) = \frac{4S^2}{a^2 + b^2 + c^2} + \frac{(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2}{a^2 + b^2 + c^2}.$$

Now the expression on the right is clearly minimal when each of the expressions in parentheses is zero; that is, when $x : y : z = a : b : c$. Translating this to the geometric situation, it means that the sum of the squares of the distances from a point to the sides of a triangle is minimal when these distances are proportional to the sides of the triangle. As argued earlier, this means that this point must coincide with point O' of exercise 365, and the minimal triangle with triangle PQR. As in part $1\circ$ of our first solution, this means that $J' = J_2$, $K' = K_2$, and $L' = L_2$. hence the second sum is also minimized in this case.

Alternate proof of (3°): Let $s = \ell \cdot J'K'^2 + m \cdot K'L'^2 + n \cdot L'J'^2$, where ℓ, m, n are the three given positive numbers. We will use the following generalization of identity (7):

$$(\ell a^2 + mb^2 + nc^2) \cdot (\frac{x^2}{\ell} + \frac{y^2}{m} + \frac{z^2}{n}) =$$

$$(9) \qquad = (ax+by+cz)^2 + \frac{\ell(mbz-ncy)^2 + m(ncx-\ell az)^2 + n(\ell ay - mbx)^2}{\ell mn}.$$

This identity can be derived from (7) by replacing $a,\ b,\ c,\ x,\ y,\ z$ with $a\sqrt{\ell},\ b\sqrt{m},\ c\sqrt{n},\ \frac{x}{\sqrt{\ell}},\ \frac{y}{\sqrt{m}},\ \frac{z}{\sqrt{n}}$ respectively. Again, we can think of $a,\ b,\ c$ as the lengths of the sides of our given triangle, and $x,\ y,\ z$ as the distances from a (variable) point to its three sides.

As in 2° above, we note that $ax+by+cz$ is twice the area of the given triangle ABC, and so does not really depend on $x,\ y,\ z$. Hence the expression $\frac{x^2}{\ell}+\frac{y^2}{m}+\frac{z^2}{n}$ is minimal when the second expression on the right of (9) is zero; that is, when

$$(10) \qquad x : y : z = \ell a : mb : nc.$$

As in our first proof, we can find a unique point P inside triangle ABC such that equation (10) is satisfied for the distances $x,\ y,\ z$ from P to the sides of ABC. We will show that the triangle we seek is the one formed by the feet of the perpendiculars $PJ_3,\ PK_3,\ PL_3$ to the sides of ABC.

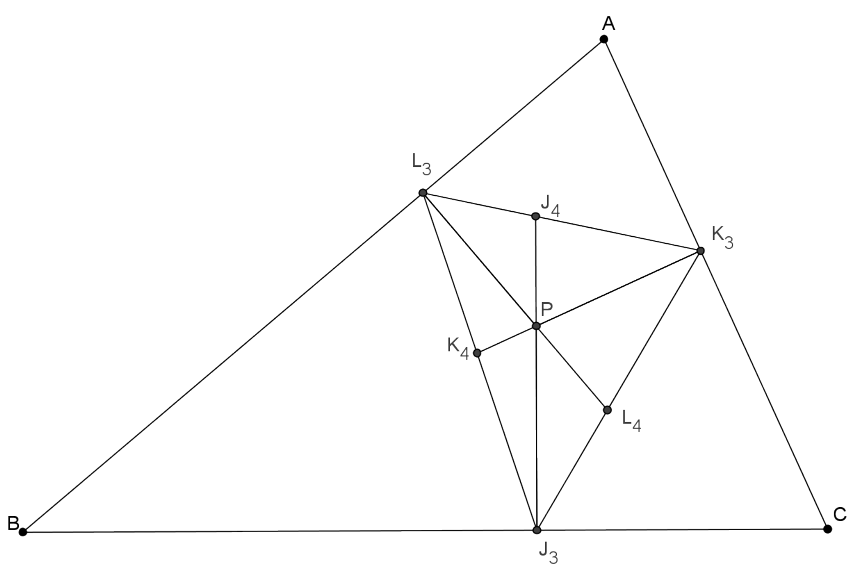

FIGURE t366e

Indeed, suppose $J_4,\ K_4,\ L_4$ are the intersections of these perpendiculars with sides $K_3L_3,\ L_3J_3,\ J_3K_3$ of $J_3K_3L_3$ (*fig.* t366e). As in the previous proofs, cyclic quadrilateral AL_3PK_3 and **256** give us $|PK_3L_3| : |ABC| = (PK_3 \cdot PL_3) : (AB \cdot AC) = yz : bc$. Similarly, $|PL_3J_3| : |ABC| = zx : ca$ and $|PJ_3K_3| : |ABC| = xy : ab$. Thus $|PK_3L_3| : |PL_3J_3| : |PJ_3K_3| = \frac{a}{x} : \frac{b}{y} : \frac{c}{z} = \frac{1}{\ell} : \frac{1}{m} : \frac{1}{n}$ (since $J_3K_3L_3$ satisfies (10)). Now it is not hard to see that $|PJ_4K_3| : |PL_3J_4| = K_3J_4 : J_4L_3$. (For example, we can draw perpendiculars to line J_3J_4 from $K_3,\ L_3$ and examine similar triangles to show that the altitudes to common side PJ_4 of these triangles have the ratio indicated.) Hence $K_3J_4 : J_4L_3 = m : n$. Likewise, $L_3K_4 : K_4J_3 = n : \ell$ and $J_3L_4 : L_4K_3 = \ell : m$.

So far, we have shown that if we choose point P to satisfy (10), and construct a new triangle whose vertices are the feet of its perpendiculars to the sides of ABC, then these perpendiculars divide the sides of the new triangle in the ratios $m : n$, $n : \ell$, $\ell : m$. We must show that the value of $s = \ell \cdot J'K'^2 + m \cdot K'L'^2 + n \cdot L'J'^2$ is minimal for this triangle.

To this end, we take any triangle JKL at all inscribed in ABC (*fig.* t366f), and divide its sides at J_0, K_0, L_0 in the ratios $m : n$, $n : \ell$, $\ell : m$. By direct computation we have (in magnitude and sign) $\frac{J_0K}{J_0L} \cdot \frac{K_0L}{K_0J} \cdot \frac{L_0J}{L_0K} = -1$, so that (**198**) lines JJ_0, KK_0, LL_0 are concurrent at some point O_0. Also by direct computation, we have

$$\text{(11)} \qquad \frac{JK_0}{K_0L} = \frac{J_0K}{J_0L} \cdot \frac{L_0J}{L_0K}.$$

Now we apply Menelaus' theorem (**192**) to triangle KJJ_0 with transversal LL_0, to get $\frac{O_0J}{O_0J_0} \cdot \frac{LJ_0}{LK} \cdot \frac{L_0K}{L_0J} = 1$, which we can write as

$$\frac{JO_0}{O_0J_0} = \frac{JL_0}{L_0K} \cdot \frac{LK}{LJ_0} = \frac{JL_0}{L_0K} \cdot \frac{LJ_0 + J_0K}{LJ_0} = \frac{JL_0}{L_0K} \cdot \left(1 + \frac{J_0K}{LJ_0}\right)$$
$$= \frac{JL_0}{L_0K} + \frac{JL_0}{L_0K} \cdot \frac{J_0K}{LJ_0} = \frac{JL_0}{L_0K} + \frac{J_0K}{J_0L} \cdot \frac{L_0J}{L_0K}.$$

Combining this equation with (11), we have:

$$\text{(12)} \qquad \frac{JO_0}{O_0J_0} = \frac{JL_0}{L_0K} + \frac{JK_0}{K_0L} = \frac{\ell}{m} + \frac{\ell}{n} = \frac{\ell(m+n)}{mn}$$

(See Lemma 2, at the end of this solution, for another way to arrive at equation (12)). Thus we have:

$$\text{(12}a\text{)} \qquad O_0J_0 = \frac{mn}{\ell(m+n)} \cdot JO_0;$$

$$\text{(12}b\text{)} \qquad JJ_0 = JO_0 + O_0J_0 = JO_0 \cdot \left(1 + \frac{mn}{\ell(m+n)}\right) = JO_0 \cdot \left(\frac{mn + n\ell + \ell m}{\ell(m+n)}\right)$$

As in our first solution, we now apply Stewart's theorem (**127**) to triangle JKL and segment JJ_0. This time we get:

$$\text{(13)} \qquad (m+n) \cdot J_0J^2 = m \cdot JL^2 + n \cdot JK^2 - \frac{mn}{m+n} \cdot KL^2.$$

And applying Stewart's theorem to triangle O_0KL and segment OJ_0, we find:

$$\text{(14)} \qquad (m+n) \cdot O_0J_0^2 = m \cdot O_0L^2 + n \cdot O_0K^2 - \frac{mn}{m+n} \cdot KL^2.$$

We substitute the values found from (12a) and (12b) into (13) and (14) to get:

$$\text{(15)} \qquad \frac{(mn + n\ell + \ell m)^2}{\ell^2(m+n)} \cdot O_0J^2 = m \cdot JL^2 + n \cdot JK^2 - \frac{mn}{m+n} \cdot KL^2;$$

$$(16) \qquad \frac{m^2n^2}{\ell^2(m+n)} \cdot O_0J^2 = m \cdot O_0L^2 + n \cdot O_0K^2 - \frac{mn}{m+n} \cdot KL^2.$$

We multiply equation (16) by $\frac{mn+n\ell+\ell m}{mn}$, then subtract equation (15) from the result, to get (after algebraic simplification):

$$(17) \quad \ell \cdot KL^2 + m \cdot LJ^2 + n \cdot JK^2 = (mn + n\ell + \ell m) \cdot \left(\frac{O_0J^2}{\ell} + \frac{O_0K^2}{m} + \frac{O_0L^2}{n} \right).$$

Note that either side of (17) is in fact the value of s (the quantity we are required to minimize), and that if we let $\ell = m = n = 1$, then equation (17) becomes equation (8). In this sense, (17) generalizes (8).

Equation (17) is very close to what we need. Indeed, if we let X, Y, Z be the feet of the perpendiculars from O_0 to BC, CA, AB respectively, we have $O_0J^2 = O_0X^2 + XJ^2$, $O_0K^2 = O_0Y^2 + YK^2, O_0L^2 = O_0Z^2 + ZL^2$. Rewriting (17) using these values, we have:

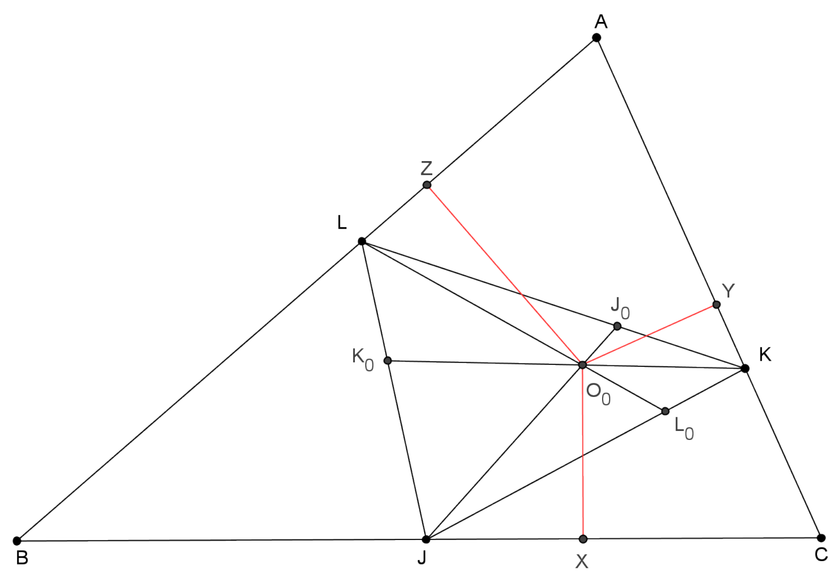

FIGURE t366f

$$s = \ell \cdot KL^2 + n \cdot LJ^2 + n \cdot JK^2 =$$

$$= (mn + n\ell + \ell m) \cdot \left[\left(\frac{O_0X^2}{\ell} + \frac{O_0Y^2}{m} + \frac{O_0Z^2}{n} \right) + \left(\frac{XJ^2}{\ell} + \frac{YK^2}{m} + \frac{ZL^2}{n} \right) \right].$$

An examination of this last equation gives us the result: the value of s is minimal when point O_0 is chosen to satisfy the condition of (10). Indeed, we have shown that in that case $J,\ K,\ L$ will coincide with $X,\ Y,\ Z$, so $\left(\frac{O_0X^2}{\ell}+\frac{O_0Y^2}{m}+\frac{O_0Z^2}{n}\right)$ is minimal, and $XJ = YK = ZL = 0$.

Note. The argument that led to equation (12) actually contains a proof of a statement interesting in its own right. Sometimes called Van Aubel's Theorem, we state it below, with an alternative proof, as a lemma.

Lemma 2: (Van Aubel's Theorem). If segments $AL,\ BM,\ CN$ are concurrent at point J (with points $L,\ M,\ N$ on the sides of triangle ABC, as in figure t366g, then

$$\frac{AJ}{JL} = \frac{AN}{NB} + \frac{AM}{MC}.$$

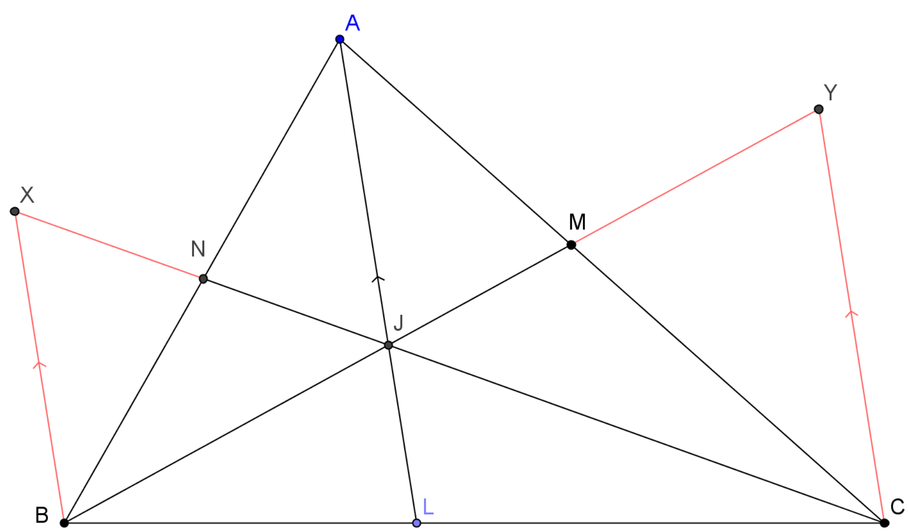

FIGURE t366g

Proof of Lemma 2: We draw $BX,\ CY$, both parallel to AL (*fig.* t366g). We first relate LJ to $BX,\ CY$. From similar triangles $JLC,\ XBC$ we have $JL : XB = LC : BC$. In the same way, from triangles $JLB,\ YCB$, we have $JL : YC = BL : BC$. Adding, we have $JL : XB + JL : YC = (BL + LC) : BC = 1$, or

$$\frac{1}{JL} = \frac{1}{XB} + \frac{1}{YC}. \tag{17}$$

Multiplying this by AJ we find:

$$\frac{AJ}{JL} = \frac{AJ}{XB} + \frac{AJ}{YC} \tag{18}$$

.

We now relate these ratios to those in the problem statement. From similar triangle ANJ, BNX we have $\frac{AJ}{XB} = \frac{AN}{NB}$. From similar triangle AMJ, CMY we have $\frac{AJ}{YC} = \frac{AM}{MC}$. Substituting these values into (18) gives us our result.

Note. The reasoning leading to equation (17) is similar to the argument in the solution of Exercise 129. In fact, if we draw XY, we reproduce the diagram of that exercise: a trapezoid and its two diagonals. Equation (17) gives the relationship between the bases of a trapezoid and (half) the line through its diagonals, parallel to the bases.

Problem 367. In a given circle, inscribe a triangle such that the sum of the squares of its sides, multiplied by three given numbers, is as large as possible.

Solution I. For this solution, we assume that there exists a triangle ABC inscribed in the given circle, such that $s = \ell \cdot BC^2 + m \cdot CA^2 + n \cdot AB^2$ is as large as possible. We also assume, for now, that ℓ, m, n are positive. If necessary, we re-label the vertices of triangle ABC so that

$$\ell \leq m \leq n. \tag{1}$$

We choose point D on segment BC so that $BD : DC = m : n$, and apply Stewart's theorem (as we did in the derivation of equation (4) in Exercise 366), to get

$$m \cdot AC^2 + n \cdot AB^2 = (m+n) \cdot AD^2 + \frac{mn}{m+n} \cdot BC^2.$$

so that

$$s = \left(\ell + \frac{mn}{m+n}\right) \cdot BC^2 + (m+n) \cdot AD^2. \tag{2}$$

Now if we fix B and C, then the position of D is also fixed, and the value of s given in (2) depends only on A, and occurs when AD is as large as possible. By **64**, this will be when AD is a diameter. Hence we see that if ABC is the triangle which maximizes s, then the diameter through A divides BC internally in the ratio $m : n$. In the same way, we can show that the diameter through B divides AC in the ratio $n : \ell$, and the diameter through C divides AB in the ratio $\ell : m$.

Suppose (*fig.* t367a) that these three diameters are AD, BE, CF. Next we apply Lemma 2 from Exercise 366 to triangle ABC and these three diameters, to get:

$$\frac{AO}{OD} = \frac{AF}{FB} + \frac{AE}{EC} = \frac{\ell}{m} + \frac{\ell}{n} = \frac{\ell(m+n)}{mn}. \tag{3}$$

Hence if R is the radius of the given circle, we have:

$$AD = AO + OD = \frac{mn + \ell m + \ell n}{\ell(m+n)} \cdot AO = \frac{mn + \ell m + \ell n}{\ell(m+n)} \cdot R. \tag{4}$$

Note that equation (3) or equation (4) determine the position of point D along the diameter through A, independently of points B or C. Thus we have the following

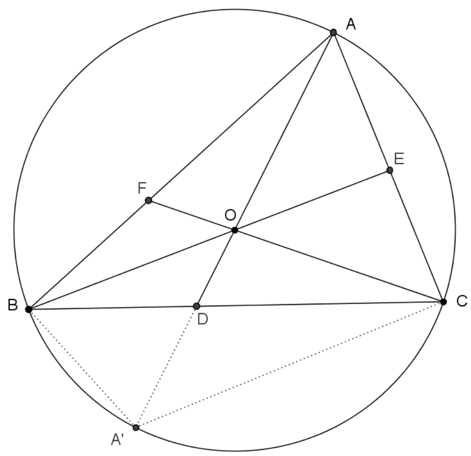

FIGURE t367a

construction. We choose a point A arbitrarily, and locate D along diameter AA' as defined by equation (3) or (4). We must then draw chord BD through D so that $BD : DC = m : n$. This task can be completed by dilating the given circle by the ratio $-m : n$ (around point D as center), and finding the intersection of the original circle with its image. Either can be a position of B, and point C is located at the intersection of BD and the original circle. (This argument is analogous to the one in the last paragraph of the solution to exercise 165. Here the two given circles in that exercise coincide.)

When does this construction work? It is necessary and sufficient that (a) point D, defined by equations (3) or (4) lie inside the given circle, and (b) that there be a chord through D which that point divides in the ratio $m : n$. Condition (a) is equivalent (**64**) to $\frac{\ell}{m} + \frac{\ell}{n} > 1$ (so that $OD < OA' = R$), or to

$$\frac{1}{\ell} < \frac{1}{m} + \frac{1}{n}. \tag{5}$$

To explore when condition (b) holds, we note that (**64**) A is the furthest point from D on the given circle, and A' is the closest. Thus condition (b) is equivalent to the condition that $\frac{A'D}{AD} < \frac{m}{n} < \frac{AD}{DA'}$, or $\frac{2R-AD}{AD} < \frac{m}{n} < \frac{AD}{2R-AD}$.

We replace AD with its equivalent as given in equation (4), to get

$$\frac{-mn + \ell m + \ell n}{mn + \ell m + \ell n} < \frac{m}{n} < \frac{mn + \ell m + \ell n}{-mn + \ell m + \ell n}. \tag{6}$$

However, some algebra will show that this is not really a second condition. Indeed, if we take the first inequality above, clear denominators and cancel terms, we get $n^2\ell < m^2 n + mn^2 + \ell m^2$, or $\ell(n^2 - m^2) < mn(m + n)$, or $\ell(n - m) < mn$, or $\ell n < \ell m + mn$, or $\frac{1}{m} < \frac{1}{n} + \frac{1}{\ell}$. Similarly, the second inequality leads to $\frac{1}{n} < \frac{1}{m} + \frac{1}{\ell}$. Both of these are simple consequences of equation (1). That is, if we choose positive numbers $\ell < m < n$, then these two conditions are automatically satisfied, and the only necessary and sufficient condition for the triangle to exist is (5).

Note. We can interpret this condition geometrically. The numbers $\ell,\ m,\ n$ must be chosen so that $\frac{1}{\ell},\ \frac{1}{m},\ \frac{1}{n}$ can be the lengths of the sides of a triangle. In other words, the altitudes of some triangle must be proportional to $\ell,\ m,\ n$.

This first solution has the same failing as the first solution we gave to exercise 366: we are assuming the existence of a triangle which maximizes s. (See the note to that solution.) Our second solution does not have this failing.

Solution II. As in the first solution, we assume that

(7) $$0 < \ell \le m \le n.$$

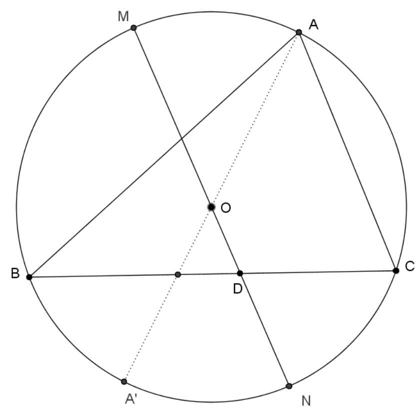

FIGURE t367b

Suppose ABC is any triangle at all inscribed in the given circle (*fig.* t367b). We find point D which divides BC in the ratio $m : n$. As in the first solution, the value of s can be expressed as in equation (2) above. We draw diameter MN through D, and assume that it is labeled so that $MD > DN$. We now express s in terms of MD and AD. We have (**131**) $BD \cdot DC = MD \cdot (2R - MD)$. But $BD = \frac{m}{m+n} \cdot BC$ and $DC = \frac{n}{m+n} \cdot BC$, so $BD \cdot DC = \frac{mn}{(m+n)^2} \cdot BC^2$, and $BC^2 = \frac{(m+n)^2}{mn} \cdot MD \cdot (2R - MD)$. We substitute this into (2) to obtain:

$$s = \left(\frac{\ell(m+n)^2}{mn} + m + n\right) \cdot MD \cdot (2R - MD) + (m+n) \cdot AD^2$$

$$= \left(\frac{\ell(m+n)^2}{mn} + m + n\right) \cdot 2R \cdot MD - \frac{\ell(m+n)^2}{mn} \cdot MD^2 - (m+n) \cdot MD^2 + (m+n) \cdot AD^2.$$

But, by direct computation, $\frac{\ell(m+n)^2}{mn} + m + n = \frac{(m+n)(\ell m + mn + n\ell)}{mn}$, so we have

$$s = \frac{m+n}{mn} \cdot \big(2(mn + n\ell + \ell m)R \cdot MD - \ell(m+n) \cdot MD^2\big) + (m+n)(AD^2 - MD^2).$$

The first term in this express depends only on the position of D, while the second also depends on the position of A. We can eliminate this second dependence. Indeed, M is the most distant point on the circle from D, so $AD^2 - MD^2$ is negative except if A and M coincide. Hence, for any position of D (which determines a position of M), the value of s cannot be maximal unless A and M coincide. That is, a necessary condition for s to be maximal is that A coincide with M.

But this condition is not sufficient. We must also maximize the value of

$$\frac{m+n}{mn} \cdot \left(2(mn + n\ell + \ell m)R \cdot MD - \ell(m+n) \cdot MD^2\right).$$

We can study this expression using techniques from algebra. First note that the value of MD that maximizes the expression is not determined by the fraction $\frac{m+n}{mn}$: this fraction will only affect the maximal value of the function. So we can ignore this part of the expression.

Letting $x = MD$, we can write the expression as a quadratic function of this variable:

$$y = -\ell(m+n) \cdot x^2 + 2(mn + n\ell + \ell m)R \cdot x, \tag{8}$$

(where R, the radius of the given circle, is constant). This function has the form $y = Ax^2 + Bx$, where $A < 0$, and such a function has its maximum where

$$MD = x = \frac{-B}{2A} = \frac{(mn + n\ell + \ell m)R}{\ell(m+n)}. \tag{9}$$

A look at equation (4) shows that this expression determines the same position for D as in our first solution.

But even this condition may not quite be sufficient: we must know that point D is inside the given circle; that is, that

$$\frac{(mn + n\ell + \ell m)R}{\ell(m+n)} < 2R, \tag{10}$$

This condition implies that $mn + n\ell + \ell m < 2\ell m + 2\ell n$, or $mn < \ell m + \ell n$, which gives us the condition that $\frac{1}{\ell} < \frac{1}{m} + \frac{1}{n}$, which is the same condition we were led to in the first solution.

We must also be sure that there exists a point D such that $BD : DC = m : n$. It is not hard to see (**64**) that this condition is equivalent to $DN : MD < m : n < MD : DN$. Substituting the value of MD given by (8), we arrive at the inequalities in (6), and the discussion in the first solution about the existence of a maximal value for s now applies.

Note. If point D, as found above, lies on or outside the given circle, then the largest value of s occurs when triangle ABC then degenerates into two copies of the diameter of the given circle.

In the discussion above we have assumed that ℓ, m, n are all positive. However, the situation when at least one of them is not is resolved by the discussion as well.

Indeed, suppose two of the coefficients, say m and n, are negative, and let A' be the point diametrically opposite A (as in figure t367a). Then we have $s = \ell \cdot BC^2 + m \cdot AC^2 + n \cdot AB^2 = \ell \cdot BC^2 + m(4R^2 - A'C^2) + n(4R^2 - A'B^2) = \ell \cdot BC^2 - m \cdot A'C^2 - n \cdot A'B^2 + 4(m+n)R^2$. The term $4(m+n)R^2$ is constant, and we are led to the problem of finding triangle $A'BC$ which maximizes $s = \ell \cdot BC^2 - m \cdot A'C^2 - n \cdot A'B^2$, whose coefficients are all positive. This problem is resolved by the discussion above.

If the number of negative values among the coefficients $\ell,\ m,\ n$ is odd, we can seek the maximum of the function $s' = -s = -\ell \cdot BC^2 - m \cdot AC^2 - n \cdot AB^2$, in which the number of negative coefficients is even. We have solved this problem in the discussions above, and its solution will give us a *minimum* value for s.

Problem 368. A necessary and sufficient condition for the existence of a solution to Exercise 127 (a point whose distances to the three vertices of a triangle ABC are proportional to three given numbers $m,\ n,\ p$) is that there exist a triangle with sides $m \cdot BC,\ n \cdot CA,\ p \cdot AB$.

Solution. First we show that the indicated condition is necessary. Suppose there exists a point D such that $AD : BD : CD = m : n : p$, for given numbers $m,\ n,\ p$. We invert these three points in a circle around D, with any radius r. If the image of points $A,\ B,\ C$ are $A',\ B',\ C'$ respectively, then these three image points certainly form a triangle. We have (**218**) $A'B' = AB \cdot \frac{r^2}{AD \cdot BD}$, $B'C' = BC \cdot \frac{r^2}{BD \cdot CD}, A'C' = AC \cdot \frac{r^2}{AD \cdot CD}$. So $\frac{A'B'}{B'C'} = \frac{AB \cdot r^2}{AD \cdot BD} \cdot \frac{BD \cdot CD}{r^2 \cdot BC} = \frac{AB \cdot CD}{AD \cdot BC}$, with analogous results for the ratios $B'C' : C'A'$ and $C'A' : A'B'$. Direct computation of these ratios will show that $A'B' : B'C' : C'A' = (CD \cdot AB) : (AD \cdot BC) : (BD \cdot CA)$, and since $AD : BD : CD = m : n : p$, we can write $A'B' : B'C' : C'A' = (p \cdot AB) : (m \cdot BC) : (n \cdot CA)$. These numbers are the sides of triangle $A'B'C'$, so the indicated condition is necessary for the existence of point D.

To show that the given condition is also sufficient, we suppose that there exists a triangle T with sides $p \cdot AB,\ m \cdot BC,\ n \cdot CA$. The result of exercise 270b assures us that there is an inversion with some center D (and some power of inversion whose value will not concern us) which takes points $A,\ B,\ C$ onto points $A',\ B',\ C'$ which form a triangle congruent to T. It is not hard to show that point D satisfies the requirements of the problem. Indeed, we have, as in the previous paragraph, $A'B' : B'C' : C'A' = (CD \cdot AB) : (AD \cdot BC) : (BD \cdot CA)$. But we've assumed that $A'B' : B'C' : C'A' = (p \cdot AB) : (m \cdot BC) : (n \cdot CA)$. These two continued proportions imply that $AD : BD : CD = m : n : p$, so D is the point we seek, and the given condition is sufficient as well as necessary.

Problem 369. We join the vertices of a triangle ABC to points $D,\ E,\ F$ on the opposite sides of the triangle so that the segments $AD,\ BE,\ CF$ are equal. Through an interior point O of the triangle we draw segments $OD',\ OE',\ OF'$, parallel to these, with $D',\ E',\ F'$ on the corresponding sides. Show that the sum of these segments is constant, no matter what point is chosen for O.

Solution. Let the distances from O to $BC,\ CA,\ AB$ be $x,\ y,\ z$ respectively. By the result of exercise 301, we have $\frac{x}{h} + \frac{y}{k} + \frac{z}{\ell} = 1$, where $h,\ k,\ \ell$ are the altitudes from $A,\ B,\ C$ respectively in triangle ABC. In figure t369, right triangles $AA_1D,\ OO_1D'$ are clearly similar, so we have $OD' : AD = OO_1 : AA_1 = x : h$. In the same way, we can show that $OE' : BE = y : k$ and $OF' : CF = z : \ell$. Thus we have $\frac{OD'}{AD} + \frac{OE'}{BE} + \frac{OF'}{CF} = 1$. But $AD = BE = CF$, so multiplying by their common value gives $OD' + OE' + OF' = AD$, a constant value.

Note. This exercise generalizes the result of exercise 42.

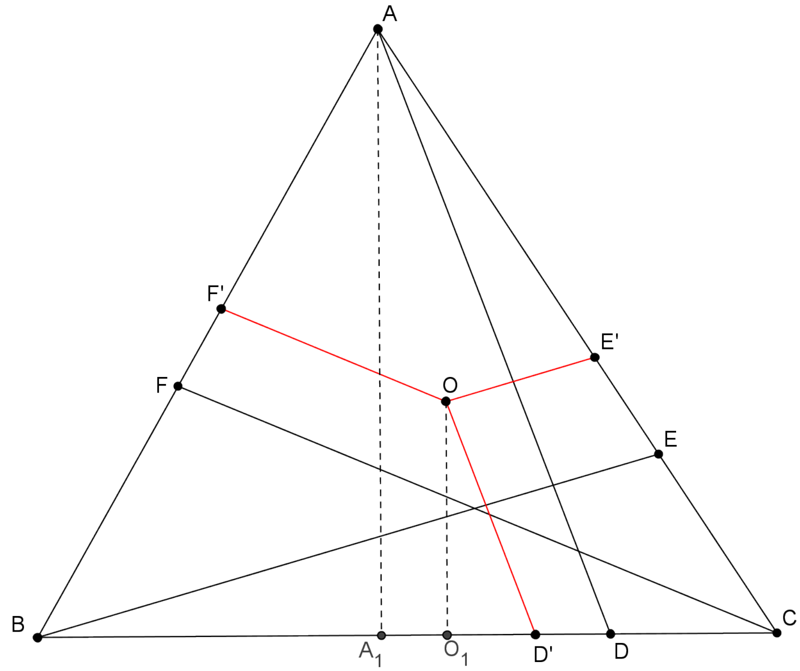

Figure t369

Problem 370. When three lines are concurrent, there always exist numbers such that the distance from an arbitrary point in the plane to one of them is equal to the sum or the difference of its distances to the other two, multiplied by these numbers. Formulate the result in a manner entirely independent of the position of the point by an appropriate convention for the signs of the segments.

Conversely, the sum or the difference of the distances from an arbitrary point M in the plane to two fixed lines, multiplied by given numbers, is proportional to the distance from M to a certain fixed line, passing through the intersection of the first two.

Solution. We first treat a simple case, assuming a specific position for the 'arbitrary point', then consider how to generalize for any position of the point in the plane.

In this problem, and the next, we will be using signed areas, so we will not use absolute value to denote area. Instead, we will use the notation $Area(ABC)$, for example, to denote the (signed) area of triangle ABC.

Figure t370 shows lines a, b, c and point M. The distances from M to a, b, c respectively are x, y, and z. We draw line c' through M parallel to c. We have $Area(OAB) = \frac{1}{2} \cdot AB \cdot z = \frac{1}{2}z \cdot AM + \frac{1}{2}z \cdot MB = Area(OAM) + Area(OMB) = \frac{1}{2}x \cdot OA + \frac{1}{2}y \cdot OB$. It follows (algebraically) that $z = \frac{OA}{AB} \cdot x + \frac{OB}{AB} \cdot y$. Now if we move the position of M the shape of triangle OAB does not change: the new triangle remains similar to the original. It follows that the coefficients $\frac{OA}{AB}$, $\frac{OB}{AB}$ in this expression for z do not depend on the position of M. This proves the original assertion.

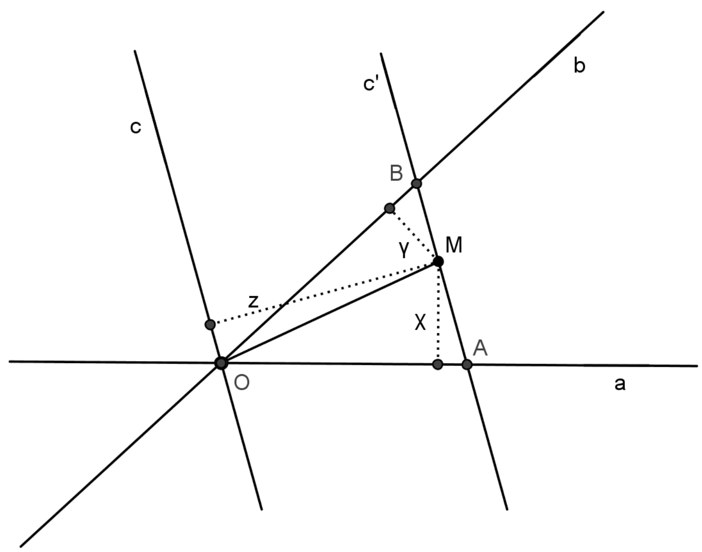

Figure t370

But we have made significant assumptions about the position of M. For some positions, we will have $Area(OAB) = Area(OAM) - Area(OMB)$ and so forth. A more general assertion, following the argument above, is that $z = \pm\frac{OA}{AB} \cdot x \pm \frac{OB}{AB} \cdot y$. (The original assertion allows for these coefficients to be positive or negative for different positions of M.)

To formulate the result still more generally, we will consider the area of any triangle as either positive or negative depending on the orientation of its perimeter (as in the solution to problem 324). We will also need some further conventions about signs. We want to say, for example, that the area of triangle MPQ (whose base PQ lines on line a) is always $\frac{1}{2}PQ \cdot x$ (where x is the length of the perpendicular from M to a). For this, we must give x a sign as well. We will call x 'positive' if, for an observer at point M, the positive direction of line a is from right to left. We will call x 'negative' in the opposite situation. We make analogous conventions for the distances y and z from M to lines b and c respectively.

Using these conventions, we have both in absolute value and in sign, we have $area(OAB) = \frac{1}{2}AB \cdot z = \frac{1}{2}(AM + MB) \cdot z = \frac{1}{2}AM \cdot z + \frac{1}{2}MB \cdot z = area(OAM) + area(OMB) = \frac{1}{2}OA \cdot x\frac{1}{2}OB \cdot y$. For different positions of M in the plane, the ratio $AB : OA : OB$ retains its magnitude and sign. So in general, we have $z = \frac{OA}{AB}x - \frac{OB}{AB}y$, and the coefficients of x and y are constant.

Conversely, suppose the (signed) distances to two (oriented) lines a and b from a variable point M are x and y respectively, and consider the expression $mx + ny$, for (positive or negative) constants $m,\ n$. If O is again the intersection of a and b, we can lay off segments $OA,\ OB$ along lines $a,\ b$ so that $OA : (-OB) = m : n$ (in magnitude and sign). We then draw a line c through O parallel to AB and let z be the (signed) distance from M to c. We have, by our first (direct) statement, $z = \frac{OA}{AB} \cdot x - \frac{OB}{AB} \cdot y = \frac{OB\cdot(-m)}{AB\cdot n} \cdot x - \frac{OB}{AB} \cdot y$, or $\left(\frac{-AB}{OB} \cdot n\right) z = mx + ny$, and the distance z from M to the fixed line c (which passes through the intersection of $a,\ b$) is indeed proportional to $mx + ny$.

Note. Dynamic geometry software can be useful in exploring this sort of problem. Students can move M about the plane and observe that triangle OAB retains its shape. They can also observe that sometimes areas must be added, and sometimes subtracted, and that in between, the areas vanish. The truth of the various assertions using conventions about signed areas and segments can thus be conveniently examined.

Problem 371. Find the locus of points such that the sum of their distances to n given lines, taken with appropriate signs and multiplied by given numbers, is constant; in other words, the locus of points such that the areas of the triangles with a vertex at the point, and with n given segments as bases, have a constant algebraic sum. (The preceding exercise provides a solution of the problem for n lines, provided that we know how to solve it for $n - 1$ lines.) Deduce that the midpoints of the three diagonals of a complete quadrilateral are collinear.

Solution. We use the conventions for directed line segments given in the solution to exercise 370. Let $a_1,\ a_2, \dots a_n$ be the n lines, each with a positive direction indicated along it. Let $x_1,\ x_2, \dots\ x_n$ be the distances to these lines from some point X, considered positive or negative as described in the previous solution. Let $k_1,\ k_2, \dots, k_n$ be the given (positive or negative) numbers.

We solve the problem first for $n = 2$. Suppose the two given lines intersect, and we seek the locus of points X such that

$$k_1 x_1 + k_2 x_2 = k_0, \qquad (1)$$

where k_0 is some constant. From the converse proposition in exercise 370, we know that there exists a line c such that $k_1 x_1 + k_2 x_2 = k_c x_c$, where x_c is the distance from X to line c and k_c is some constant. So in this case, equation (1) can be written as $k_c x_c = k_0$, or $x_c = \frac{k_0}{k_c}$, which is another constant. That is, the distance from X to line c is constant, and the locus of X is a line parallel to c.

Now suppose the two given lines are parallel, and oriented in the same direction. If X is a point satisfying (1), and x_p is the (perpendicular) distance between the two lines, then we can write $x_2 = x_1 + x_p$ for any position of X (in magnitude and sign). Substituting in (1), we can rewrite that equation as $(k_1 + k_2)x_1 = k_0 - k_2 x_p$; that is, $x_1 = \frac{k_0 - k_2 x_p}{k_1 + k_2}$, which is again constant (as long as $k_1 + k_2 \neq 0$). Again, the locus of X (if there are any such points at all) is a line, parallel to the two given lines. If $k_1 + k_2 = 0$ there are no such points, unless $k_0 - k_2 x_p = 0$ as well, in which case X can be any point on the plane.

Finally, we suppose the two given lines are parallel, and oriented in opposite directions. This case reduces to the previous, if we reverse the orientation of one of the lines (say a_2) and at the same time reverse the sign of the associated constant (k_2). The sign of x_2 then reverses as well, and the conclusions above remain valid. This concludes the analysis for the case $n = 2$.

We now proceed by induction. We assume that the locus of X for any $n-1$ lines is another line, and investigate the case for n lines. That is, we seek the locus of points X such that

$$k_1x_1 + k_2x_2 + \cdots + k_{n-1}x_{n-1} + k_nx_n = k_0. \tag{2}$$

We can assume that no k_i is zero. (If some $k_i = 0$, then line a_i contributes no constraint to the position of X.) Of any set of more than two non-zero numbers, we can certainly choose two whose sum is not zero. So we can assume that the lines have been numbered so that $k_{n-1} + k_n \neq 0$.

We next apply the statement for $n = 2$: there exists a line c such that the distance x_c from any point to c is proportional to $k_{n-1}x_{n-1} + k_nx_n$. This allows us to replace the two lines $a_{n-1},\ a_n$ with the single line c and reduce the number of lines to $n-1$. That is, condition (2) is equivalent to

$$k_1x_1 + k_2x_2 + \cdots + +k_{n-2}x_{n-2} + k_cx_c = k_0, \tag{2a}$$

for some constant k_c. By our induction hypothesis, the locus satisfying this condition is a line.

As noted in the problem statement, we can interpret this result geometrically, in terms of (signed) areas of triangles. As in the previous exercise, we use $Area(ABC)$ to denote the area of triangle ABC, rather than absolute value.

Suppose we have line segments $A_1B_1,\ A_2B_2, \ldots A_nB_n$, and suppose k_0 is some constant. If we set $k_i = \frac{A_iB_i}{2r}$ for some constant r, then equation (2) can be written as

$$\frac{A_1B_1}{2r}x_1 + \frac{A_2B_2}{2r}x_2 + \cdots + \frac{A_nB_n}{2r}x_n = k_0,$$

or

$$\frac{1}{2}A_1B_1x_1 + \frac{1}{2}A_2B_2x_2 + \cdots + \frac{1}{2}A_nB_nx_n = k_0r,$$

or

$$Area(A_1B_1X) + Area(A_2B_2X) + \cdots + Area(A_nB_nX) = k_0r. \tag{3}$$

Finally, we use this result to prove that the midpoints of the three diagonals of a complete quadrilateral are collinear. Let the complete quadrilateral be $ABCDEF$, and let $L,\ M,\ N$ be the midpoints of its diagonals (*fig.* t371). We have, in magnitude and sign, $Area(ACL)+Area(BCL) = 0$; $Area(ADL)+Area(BDL) = 0$ (note the orientation of these triangles). It follows that

$$Area(ACL) + Area(BCL) + Area(ADL) + Area(BDL) = 0. \tag{4}$$

Likewise, we have $Area(ACM)+Area(ADM) = 0$; $Area(BCM)+Area(BDM) = 0$, so

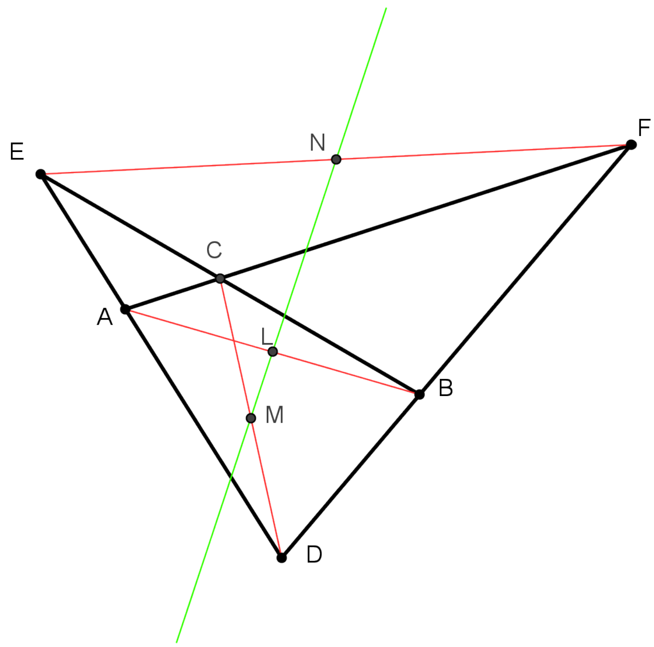

FIGURE t371

$$(5) \qquad Area(ACM) + Area(BCM) + Area(ADM) + Area(BDM) = 0.$$

That is, both L and M belong to the locus of points X such that $Area(ACX) + Area(BCX) + Area(ADX) + Area(BDX) = 0$. We prove the assertion by showing that point N belongs to this locus as well.

This takes a bit more work than (4) and (5) did. We have $Area(AEN) + Area(AFN) = 0$; $Area(BEN) + Area(BFN) = 0$; $Area(CEN) + Area(CFN) = 0$; $Area(DEN) + Area(DFN) = 0$. We add the first two of these equations, and subtract from them the second two, to get:

$$\{Area(AEN) - Area(DEN)\} + \{Area(AFN) - Area(CFN)\}$$

$$(6) \qquad + \{Area(BEN) - Area(CEN)\} + \{Area(BFN) - Area(DFN)\} = 0.$$

Now we note that, for example, $Area(AFN) - Area(CFN) = Area(AFN) + Area(FCN) = Area(ACN)$, with similar substitutions possible for other terms in (6), which then becomes:

$$Area(ACN) + Area(BCN) + Area(ADN) + Area(BDN) = 0.$$

Equations (4), (5), (7) show that $L,\ M,\ N$ all belong to the locus of points X such that

$$(7) \qquad Area(ACX) + Area(BCX) + Area(ADX) + Area(BDX) = 0.$$

Since this locus is a line, the assertion is proved.

Note. A subtle point completing this argument is to show that equation (7) does not hold for every point on the plane. (If it did, we could not assert the collinearity of L, M, N.) In fact, we can easily show that it does not hold for point A or point C. For these points, some of the 'triangles' in equation (7) degenerate into segments, so their area is 0, and we have $Area(BCA) + Area(BDA) = Area(ADC) + Area(BDC) = 0$. It is not hard to see that these conditions imply that quadrilateral $ADBC$ is a parallelogram, and we cannot even make the assertion in the problem statement. Hence (7) in fact does determine a line.

Problem 371b. The three circles whose diameters are the diagonals of a complete quadrilateral have the same radical axis. This axis passes through the intersection of the altitudes of each of the four triangles formed by three sides of the quadrilateral.

Solution. Suppose (*fig.* t371b) that AB, CD, EF are the diagonals of complete quadrilateral $ABCDEF$. Triangle BCF is formed by three of the sides of the quadrilateral, and we let H be the intersection of its altitudes BK, CL, FM. Quadrilateral $BCKL$ is cyclic (two opposite angles are right angles), so points B, K, C, L lie on the same circle (not shown) and (**131**):

$$(1) \qquad HB \cdot HK = HL \cdot HC.$$

Now $AK \perp BK$, so K lies on the circle with diameter AB. Similarly L lies on the circle with diameter CD. We can read equation (1) as saying that H has the same power with respect to both these circles. Analogously, we have (from cyclic quadrilateral $BKFM$):

$$(2) \qquad HB \cdot HK = HM \cdot HF.$$

Again, $HM \perp EM$, so M lies on the circle with diameter EF, and (2) says that H has the same power with respect to this circle as it has with respect to the circle on diameter AB. Thus H has the same power with respect to all three circles on the diameters of the three diagonals of $ABCDEF$, and lies on the radical axis of any pair of these circles.

In just the same way, we can show that the orthocenters of triangles BDE, ACE, ADF have the same powers with respect to these three circles. So each of the four orthocenters lie on the intersection of the radical axes of pairs of these circles. Since these orthocenters don't coincide, this must mean that the radical axes of pairs of these three circles are all the same line (and the orthocenters all lie along this line).

Note. The simplicity of this argument belies the complex nature of the result, which combines in an unexpected ways the definitions of orthocenter and radical axis. If the circles intersect, we get the equally unexpected result that they all pass through the same pair of points.

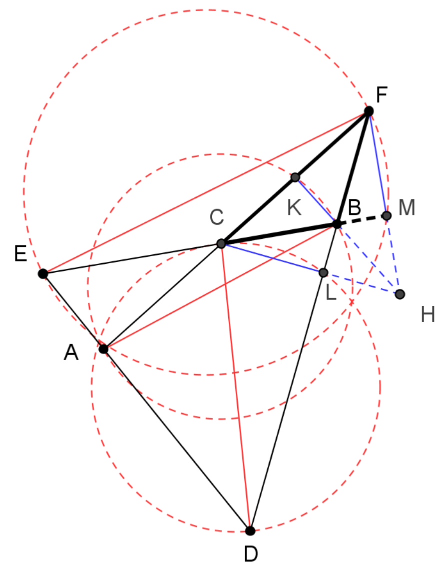

Figure t371b

Problem 372. The opposite sides of a complete quadrilateral, and its diagonals, form a set of three angles such that the polars of any point O in the plane relative to these three angles are concurrent. (Transform by reciprocal polars, and take O as the center of the directing circle.)

These same lines intercept three segments on an arbitrary transversal such that the segment which divides two of them harmonically also divides the third harmonically.

Three segments with this property are said to be *in involution.*

Solution. Let $PQRS$ be four vertices of the given quadrilateral. (For clarity, figure t372a shows only part of the complete quadrilateral.) Let opposite sides PQ, RS intersect at K. Let line m be the polar of point O with respect to angle $\widehat{PKS}$ (**203**); that is, m contains the harmonic conjugate of point O with respect to the segment cut off by angle $\widehat{PKS}$ along any line through O.

We take the polar of each point in figure t372a, with respect to an arbitrary circle (not shown) centered at O. The result is given in figure t372b. The complete quadrilateral determined by $PQRS$ is transformed into another complete quadrilateral, determined by lines p, q, r, s. Point K is transformed into line k. By **205**, any line through point K is transformed into a point on k. In particular, lines PQ, RS, OK, m are transformed into points along k, which we denote (unconventionally, but with a certain clarity) as (pq), (rs), (ok) and M. However, since line OK passes through the center of the directing circle, its pole (ok) lies at infinity along line k (**204**).

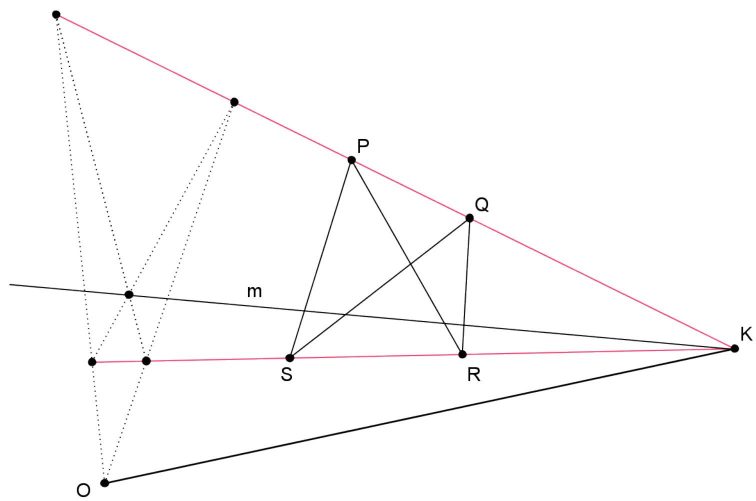

Figure t372a

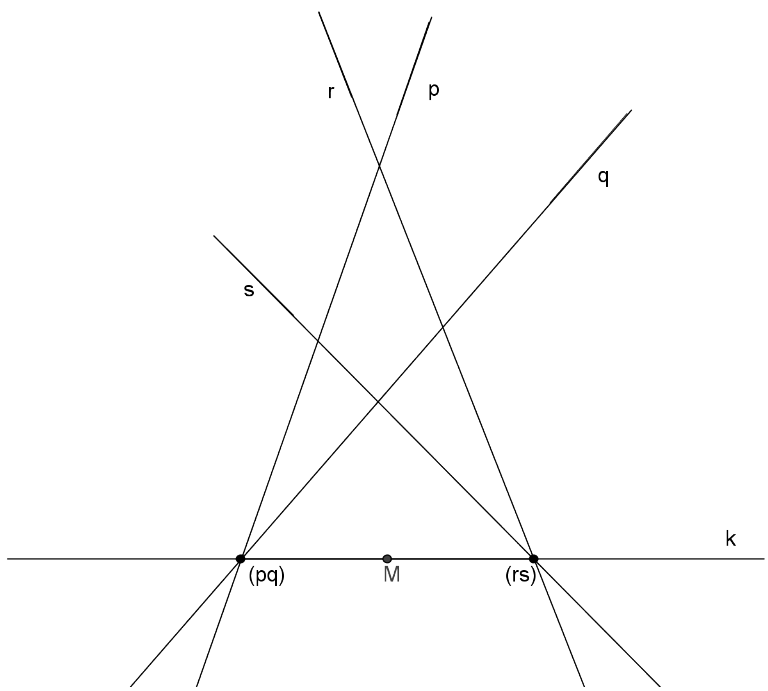

Figure t372b

By definition of the polar with respect to an angle, lines PK, m, SK, OK form a harmonic pencil. Thus points (pq), M, (rs), (ok) form a harmonic range along line k. Since the harmonic conjugate of a point at infinity, with respect to a given segment, is the midpoint of the segment, point M is equidistant from (pq) and (rs). That is, M is the midpoint of one of the diagonals of complete quadrilateral $pqrs$.

We can repeat this argument for the angle determined by PS and QR (the other two opposite sides of quadrilateral $PQRS$). Under the transformation by reciprocal polars, these lines correspond to points (ps), (qr), two more vertices of the transformed complete quadrilateral, and by an analogous argument, the polar of O with respect to the angle determined by PS, QR is transformed into the midpoint of another diagonal of $pqrs$.

A third iteration of this argument will show that the polar of O with respect to the angle determined by diagonals PR, QS of the original complete quadrilateral is transformed into the midpoint of the third diagonal of $pqrs$. But these three midpoints are collinear (**194**, or exercise 371). Therefore the polars of these three lines, which are the polars of O with respect to the angles described in the problem, are concurrent (at the pole of the line containing the midpoints of the diagonals of $pqrs$). This proves the first assertion of the problem.

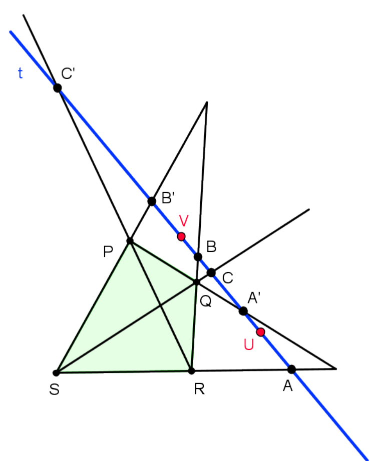

Figure t372c

To prove the second assertion of the problem, suppose line t intersects the opposite sides and the diagonals of complete quadrilateral $PQRS$ (*fig.* t372c). Suppose the segments these lines determine along T are AA', BB', CC'. Then, by the result of exercise 219b, there exists a segment UV which divides both AA' and BB' harmonically. The definition of the polar of a point with respect to an angle (**203**) tells us that the polar of point U with respect to the pair of lines PQ, RS passes through point V, and the same thing is true for the polar of point U with respect to lines PS, QR. It follows from the first assertion of this problem that the polar of U with respect to the third pair PR, QS of lines also passes through V. Therefore segment UV divides segment CC' harmonically.

Notes. The solution to problem 219b shows that points U and V exist if and only if the circles on diameters AA', BB' do not intersect. This is a condition on the position of line t.

Problem 373. The Simson line (Exercise 72) which joins the feet of the perpendiculars from a point P on the circumscribed circle of a triangle to the three sides, divides the segment joining P with the intersection H of the altitudes of the triangle into equal parts. (Prove, using Exercise 70, that the points symmetric to P relative to the three sides, are on a line passing through H.)

Deduce from this, and from exercise 106 that the points of concurrence of the altitudes of four triangles formed by four lines, taken three at a time, are collinear.

Solution. There are three assertions here:

(1) The points symmetric to P in the triangle's three sides are collinear;
(2) The Simson line bisects PH;
(3) The four orthocenters determined by four intersecting lines are concurrent.

Solutions. (1°) If the given triangle (*fig.* t373a) is ABC, let X, Y, Z be the feet of the perpendiculars from P to its three sides. By the result of exercise 72, these three points are collinear. Let H be the orthocenter of the triangle, and let Q be the reflection of P in line BC. We will first show that Q is on a line through H parallel to XY.

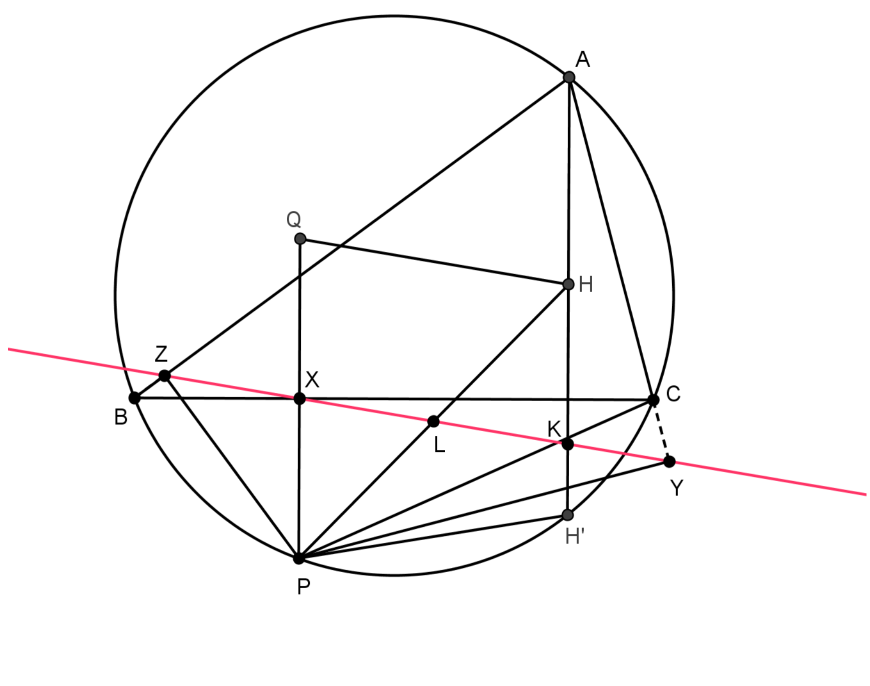

FIGURE t373a

To show this, let H' be the reflection of point H in BC. By the result of exercise 70, H' lies on the circumcircle of ABC. Let XY intersect HH' at K and

PH at L. We will use angles to show that $HQ \parallel XY$. First note that since $H,\ Q$ are symmetric respectively to $H',\ P$ in line BC, that

$$(1) \qquad \widehat{PH'K} = \widehat{QHK},$$

and also that $\widehat{PH'K} = \widehat{PCA}$ (they both intercept arc $\widehat{PBA}$), so $\widehat{PH'K}$ supplements $\widehat{PCY}$. But we can tell more about $\widehat{PCY}$. Points $X,\ Y$ lie on the circle with diameter PC (because $\widehat{PXC},\ \widehat{PYC}$ are right angles). Thus $\widehat{PCY} = \widehat{PXY}$, and $\widehat{PH'K}$ supplements $\widehat{PXY} = \widehat{XKH}$ (this last equality because $PQ \parallel AH'$). By (1), this means that $\widehat{QHK}$ supplements $\widehat{XKH}$, which shows that $QH \parallel XY$.

In just the same way, we can show that the reflections of P in $AB,\ AC$ also lie on a line through H parallel to XY. Since there is only one such line, these three points are collinear.

Note. For some positions of P, details of this argument must be changed. Some angles which are here said to be equal become supplementary if P is closer to C than to B. However, the argument still holds in these cases.

(2°). It is now easy to show that L is the midpoint of PH. In triangle PQH, X is the midpoint of side PQ, and XL is parallel to side QH. It follows from **114** that L is the midpoint of PH.

(3°) Suppose $a,\ b,\ c,\ d$ are any four lines which intersect in pairs. From exercise 106, we know that if we draw the circumcircles of triangles $abc,\ abd,\ acd,\ bcd$ (where abc denotes the triangle formed by lines $a,\ b,\ c$, etc.), these four circles all pass through some point P.

Since P is on the circumcircle of triangle abc, the feet of the perpendiculars from P to $a,\ b,\ c$ all lie along some line k. Since P is also on the circumcircle of triangle bcd, the foot of the perpendicular from P to d is also on line k, which is the Simson line for P relative to all four triangles.

It follows, from the first part of this exercise, the four orthocenters in question all lie on a line parallel to k, and twice as far from P as k is.

Problem 374. We fix points $A,\ B,\ P,\ P'$ on a circle S, and let C be a variable point on the same circle. Show that the intersection M of the Simson lines for P and P', with respect to the triangle ABC, describes a circle S'.

Find the locus of the center of S' when $A,\ B$ remain fixed, and $P,\ P'$ move along S so that the distance PP' remains constant.

Also find the locus of the point M when the points $A,\ B,\ C$ are fixed, and $P,\ P'$ are variable and diametrically opposite.

Solution. We first prove an initial lemma, then treat the three assertions of the exercise separately.

Lemma 1. If two points are chosen on the circumcircle of a triangle, the acute angle between their Simson lines is equal to half the minor arc between the two points.

Proof: In figure t374a, we have chosen two points on the circumcircle of triangle ABC. We have point P and the feet $K,\ L$ of its perpendiculars to $AC,\ AB$ respectively. We also have point P' and the feet $K',\ L'$ of its perpendiculars to AC, AB. Simson lines $KL,\ K'L'$ of points $P,\ P'$ intersect at T. We will prove

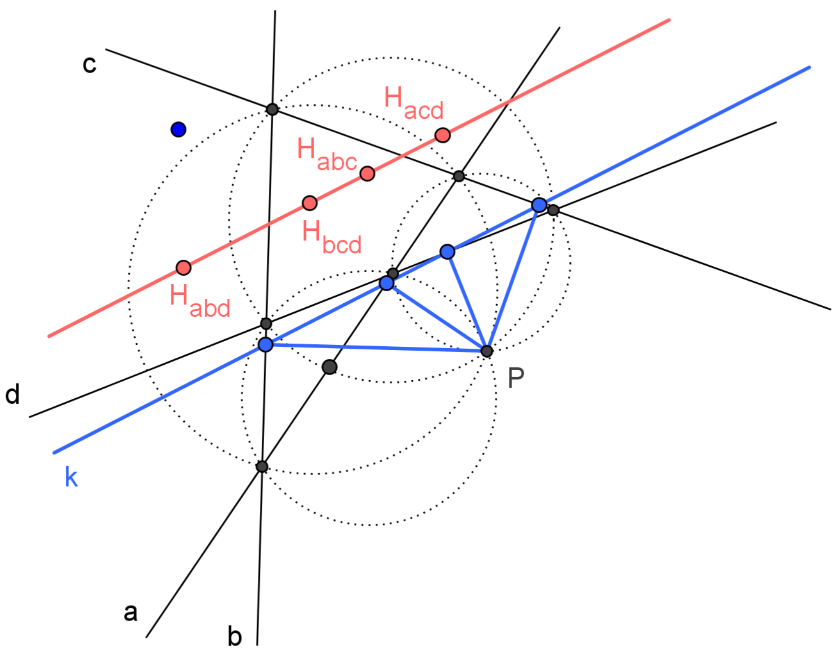

Figure t373b

the lemma by dividing acute angle $\widehat{KTK'}$ into two smaller angles, each of which is equal to an angle inscribed in the circumcircle.

We create the two smaller angles by drawing $TU \perp AC$. Note that since $PK \perp AC$, $P'K' \perp AC$, we have $TU \parallel PK \parallel P'K'$.

We first look at $\widehat{UTK}$, which is equal to $\widehat{LKP}$ (since $TU \parallel PK$). Now quadrilateral $PKAL$ is cyclic, having right angles at K and L, so $\widehat{PKL} = \widehat{PAL} = \frac{1}{2}\,\widehat{PB}$. Thus $\widehat{UTK} = \frac{1}{2}\,\widehat{PB}$.

Similarly, using parallel lines TU, $P'K'$ and cyclic quadrilateral $P'L'K'C$ (in which $P'C$ is a diameter of the circumcircle), we have $\widehat{K'TU} = \widehat{TK'P'} = \widehat{L'CP'} = \frac{1}{2}\,\widehat{BP'}$.

Adding these two angles, we find that $\widehat{KTK'} = \widehat{KTU} + \widehat{UTK'} = \frac{1}{2}(\widehat{PB} + \widehat{BP'}) = \frac{1}{2}\,\widehat{PP'}$. This concludes the proof.

Solutions. (1°). This situation is now easy to deal with. If P, P' are two points on the circumcircle of triangle ABC (*fig.* t374b), then the angle between

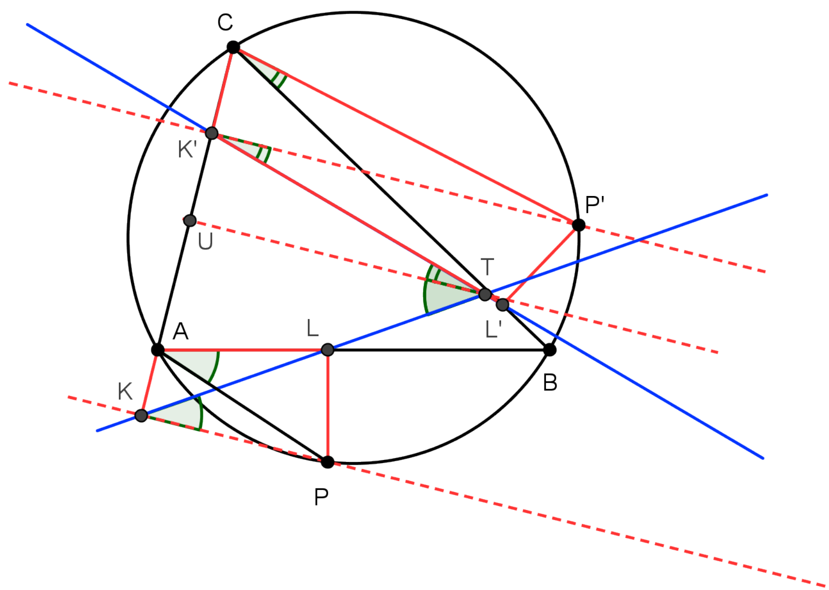

Figure t374a

their Simson lines depends only on arc $\widehat{PP'}$, and not on the position of point C along the circle. Points $Z,\ Z'$ (the feet of the perpendiculars from $P,\ P'$ to AB) also do not depend on point C. So angle $\widehat{ZMZ'}$ is constant, and M (the intersection of the Simson lines) describe an arc of a circle.

In figure 374b, the angle at M is acute, but is still constant as C moves. For other positions of C, the angle at M will be the supplement of the angle shown, and M will describe a full circle S'. Note that S' passes through $Z,\ Z'$, which are the feet of the perpendiculars from $P,\ P'$ to AB.

(2°). Now suppose (*fig.* t374c) that the distance PP' is fixed. We have a fixed circle (S), two fixed points ($A,\ B$), two variable points ($P,\ C$), and point P' whose position depends on P (and S). However, it turns out that the locus of centers of S' does not depend on C, but only on the length PP'.

Let $O,\ O'$ be the centers of circles $S,\ S'$ respectively, and let C' be the midpoint of segment AB. We will show that the length $C'O'$ is constant, and is equal to the distance of PP' from O. The argument requires some auxiliary construction.

Let K be the midpoint of ZZ'. Then $PZ \parallel O'K \parallel P'Z'$, so (**113**) the intersection I of PP' and $O'K$ is the midpoint of PP' We will show that $OIO'C'$ is a parallelogram. We know that $C'O \perp AB$, and $IO' \perp AB$ as well, so $C'O \parallel O'I$.

We now will show, by means of a rather complicated argument that $C'O = O'I$. We know, from lemma 1, that $\widehat{ZMZ'} = \frac{1}{2}\ \widehat{PP'}$, so $\widehat{ZMZ'} = \frac{1}{2}\widehat{POP'}$. But, in circle

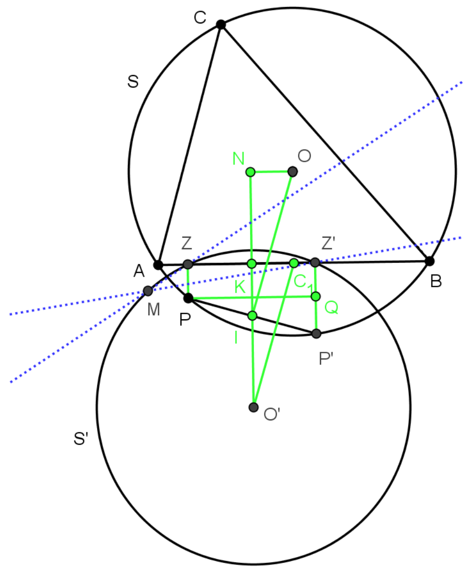

FIGURE t374b

S', $\widehat{ZMZ'}$ is an inscribed angle, and $\widehat{ZO'Z'}$ is a central angle intercepting the same arc, so $\widehat{ZO'Z'} = 2(\widehat{ZMZ'}) = \widehat{POP'}$. This means that isosceles triangles POP', $ZO'Z'$ are similar, and (since corresponding altitudes are in the same ratio as corresponding sides) $O'K : OI = ZZ' : PP'$.

On the other hand, suppose N is the foot of the perpendicular from O to $O'I$, and Q is the foot of the perpendicular from P to $P'Z'$. Then $PP' \perp OI$ and $PQ \parallel ZZ'$, so $PQ \perp IN$. Thus angles $\widehat{P'PQ}$, $\widehat{OIN}$ have pairs of sides which are perpendicular, so (**43**) these angles are either equal or supplementary. An examination of their senses of rotation will show that in fact they are always equal. Thus right triangle QPP', NIO are similar, and $IN : OI = ZZ' : PP'$.

Comparing the proportions at the end of the last two paragraphs, we conclude that $O'K = IN$. Subtracting segment IK from each of these, we find that $O'I = KN = C'O$.

We have now shown that two sides of quadrilateral $O'IOC'$ are both equal and parallel, so (**46**, converse 2°) this figure is a parallelogram, and its other pair of sides is equal. That is, $C'O' = OI$. Now the length of OI does not depend on the position of point P, but only on the radius of circle S (which is constant) and the length PP' (which is also constant). Hence O' always lies on a the circle centered at C' with radius equal to OI. (Recall that C' does not depend on C, P, or P'.)

We can show that any point O' on this circle is part of the required locus by following the proof in reverse. We draw $O'I \perp AB$, locating point I as the fourth vertex of parallelogram $O'C'OI$. Then we can rotate segment IO about I by 90° to get a line which will intersect circle S at points P, P'. Then O' is the center of

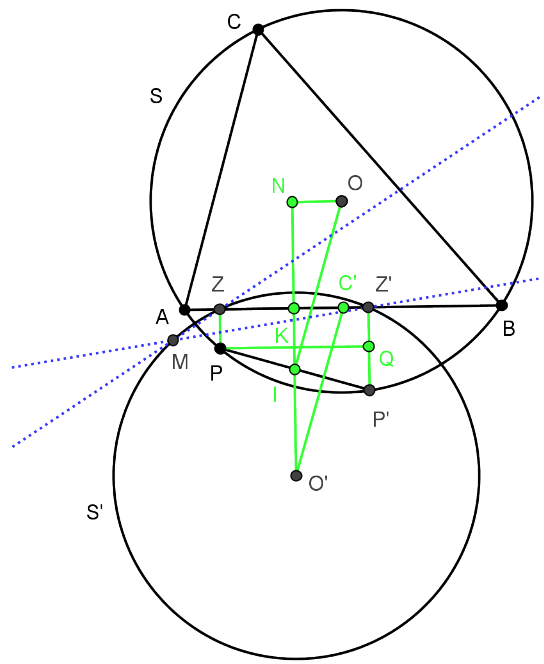

FIGURE t374c

the circle along which the intersection of the Simson lines corresponding to P, P' will lie.

(3°). We will show that this locus is none other than the nine-point circle (exercise 101) of the original triangle ABC. We will draw freely on the results of exercises 101 and 373.

Let O, H be the circumcenter and orthocenter of triangle ABC, and let K, L be the midpoints respectively of segments HP, HP' (*fig.* t374d). By the result of exercise 373, K and L are on the Simson lines of points P, P' (which are shown in blue in figure t374d).

We examine segment KL. Since it joins the midpoints of two sides of triangle HPP', it is half the length of PP' (**55**), which is the length of the diameter of the nine-point circle (exercise 101). Since O is on line PP' (and is the midpoint of segment PP'), segment KL passes through the midpoint O_1 of HO (**114**). (Note that O_1 stays fixed as P, P' move around the circle.) Hence (exercise 101) KL is always a diameter of the nine-point circle of triangle ABC. And since (from lemma 1) $\widehat{LMK} = 90°$, point M lies on this nine-point circle.

We now investigate the converse: We are given a point M on the nine-point circle of triangle ABC (*fig.* t374f), and we must find points P, P' diametrically opposite, and such that the Simson lines of these two points intersect at M.

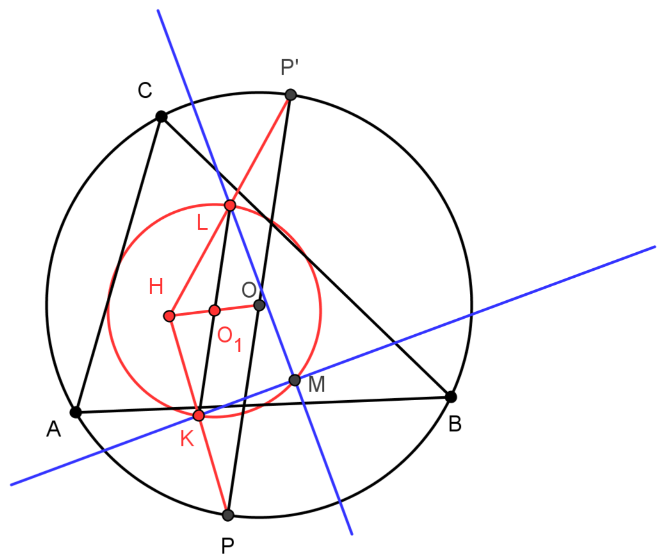

FIGURE t374d

Lemma 2. Let P be a point on the circumcircle of triangle ABC (*fig.* t374e), let T be the foot of the perpendicular from P to side BC, and let P_1 be the intersection of PT with the circumcircle. If p is the Simson line of point P, then $AP_1 \parallel p$.

Proof of Lemma 2. Note that T is, by definition, on the Simson line p of point P. If the intersection of p and AB is point Y, then $PY \perp AB$, again by the definition of the Simson line p. Hence quadrilateral $PYTB$ is cyclic, and $\widehat{YTP} = \widehat{ABP}$. But $\widehat{ABP} = \frac{1}{2}\ \widehat{AP} = \widehat{AP_1B}$. So $\widehat{AP_1P} = \widehat{YTP}$, and $AP_1 \parallel p$. This proves the result of the lemma.

Proof of Converse Statement. We first move from the nine-point circle to the circumcircle (*fig.* t374f). We know that these two circles are homothetic with center H (the orthocenter of ABC). Thus if we extend HM to intersect the circumcircle at M' we have $HM = MM'$.

Let us suppose P, P' are the required points. Then, from exercise 373, HP is bisected by p (the Simson line of P) at their intersection K. Since KM joins the midpoints of two sides of triangle HPM', we have $PM' \parallel p$. That is, we seek a point P for which this is true. Let $\widehat{AM'P} = \alpha$. We will locate P from the known segment HM' by computing α.

Let P_1 be the intersection of perpendicular PT from P to BC. By lemma 2, $AP_1 \parallel p$. Hence $\widehat{MPM'} = \alpha$ as well. If PT and $M'A$ intersect at U, then $\widehat{TUM'} = \widehat{UPM'} + \widehat{UM'P} = 2\alpha$.

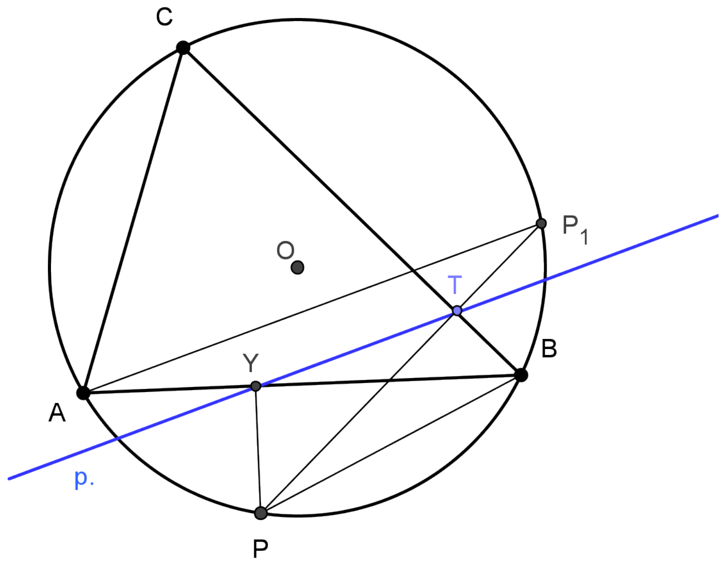

FIGURE t374e

We next draw $B'M' \parallel BC$ through M', and note that $B'M' \perp PP_1$. If the intersection of these lines is V, then triangle UVM' is a right triangle, so $2\alpha = \widehat{VUM'} = 90° - \widehat{AM'B'}$. This allows us to compute α.

Thus we have the following construction: we extend HM to M' on the circumcircle (so that $HM = HM'$), and draw $B'M' \parallel BC$. We then construct a line making an angle equal to $\frac{1}{2}(90° - \widehat{AM'B'})$ with AM'. The intersection of this line with the circumcircle will give us point P. U Indeed, for this position of P we can drop a perpendicular to BC and locate the corresponding points T, U, V, P_1 and also line AP_1. The angle relationships we've derived assure us that $PM' \parallel AP_1$, and lemma 2 guarantees that both these lines are parallel to p, the Simson line for point P. In triangle HPM', we know (from exercise 373) that line p passes through K, the midpoint of HP, so (**113**) it must pass through M, which is the midpoint of HM'. The result of lemma 1 tells us that the Simson line for P' (diametrically opposite P) also passes through M.

The proof of the converse statement is due to Behzad Mehrdad.

Problem 375. Find the locus of the midpoints of a triangle inscribed in a fixed circle, whose altitudes pass through a fixed point.

Solution. Let O be the center of the given circle, and let H be the fixed orthocenter of the variable inscribed triangle ABC. The centroid G of the triangle

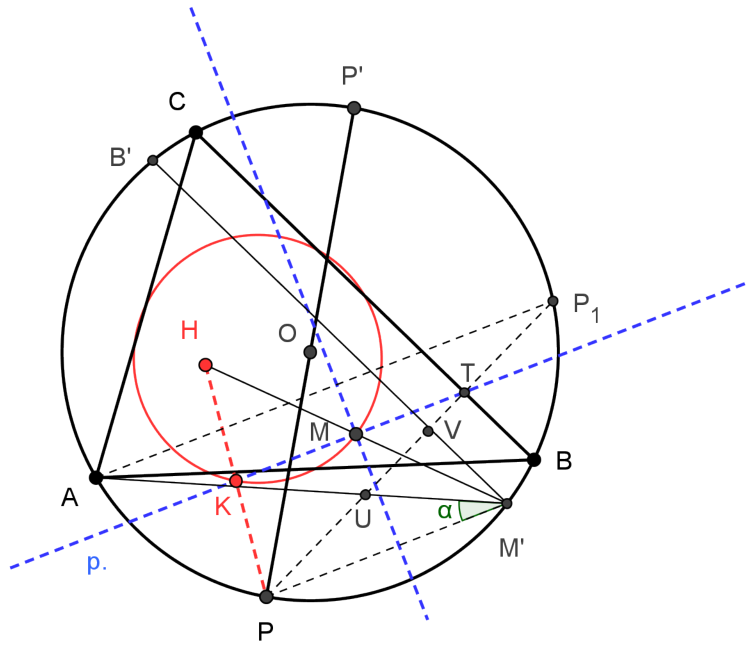

Figure t374f

is on segment OH, and $OG : GH = 1 : 2$. (see exercise 158). It follows that any triangle ABC such as described has the fixed point G as its centroid.

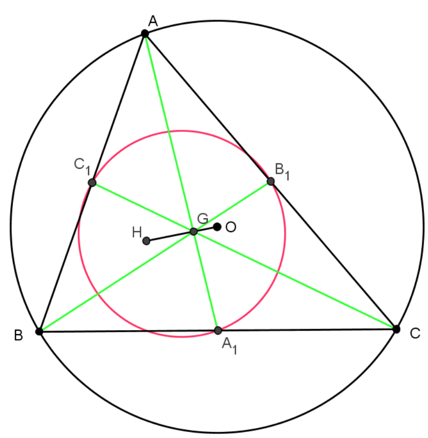

Figure t375

Now medians $AA_1,\ BB_1,\ CC_1$ pass through point G, and $AG : GA_1 = BG : GB_1 = CG : GC_1 = 2 : 1$ (**56**), so points $A_1,\ B_1,\ C_1$ lie on a circle homothetic to circle O with the center of homothecy at G and ratio of homothecy $-\frac{1}{2}$. Note that this circle is the nine-point circle (exercise 101) of triangle ABC.

Conversely, it is not hard to see that any point on the nine-point circle can serve as the midpoint of a side of a triangle such as required. We can simply follow the construction of the solution 'backwards'. That is, suppose we are given points $O,\ G,\ H$, circle O, and circle G, centered at G and homothetic to O with ratio $-\frac{1}{2}$. If we choose any point B_1 on circle G, we can draw B_1G, and take its intersection with circle O as vertex B of the required triangle. Then we can draw OB_1. A perpendicular to OB_1 will intersect circle O at two points which can be taken as vertices $A,\ C$.

Problem 376. We transform the nine-point circle (Exercise 101) of a triangle by an inversion whose pole is the midpoint of a side, and with a power equal to the power of the pole relative to the inscribed circle or, equivalently (Exercise 90b), relative to the escribed circle corresponding to that side. Show that the line which is the transform of the circle is precisely the common tangent of these two circles, other than the sides of the triangle. It follows that the nine-point circle is tangent to the inscribed circle and to the escribed circles.

Lemma 1. The tangent to the nine-point circle of a triangle at a midpoint of a side is parallel to the tangent to the circumcircle at the opposite vertex.

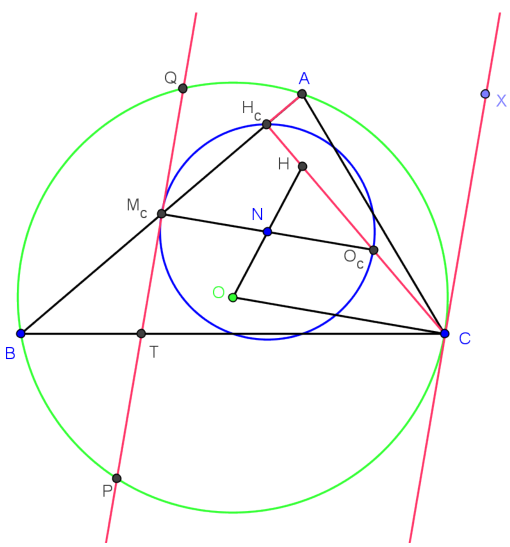

FIGURE t376a

Proof. In figure t376a, O is the circumcenter of triangle ABC, H is its orthocenter, N its nine-point center, and O_c is the midpoint of segment HC. Point M_c is the midpoint of AB, and H_c is the foot of the altitude from C. (Note that O_c, M_c and H_c are all on the nine-point circle.) In exercise 101 we showed that N is the midpoint of OH. Thus O_cN connects the midpoints of two sides of triangle

HOC, and is parallel to OC. But since $\widehat{M_cH_cO_c}$ is a right angle, segment M_cO_c is a diameter of the nine-point circle. Since $M_cO_c \parallel OC$, it follows that the tangents to the two circles at M_c and C, which are perpendicular to these radii, are themselves parallel.

Note. The conclusion also follows from the more general observation that the nine-point circle is homothetic to the circumcircle, with factor $-\frac{1}{2}$ and the centroid as the center of homothecy.

Lemma 2. Let M_c be the midpoint of side AB of triangle ABC. The tangent to the nine-point circle of a triangle at M_c forms an angle equal to angle $\widehat{A}$ with side BC.

Proof. In figure t376a, we need to show that $\widehat{QTC} = \widehat{BAC}$. This is not hard, if we examine arcs along circle O. Since $PQ \parallel CX$ (lemma 1), we know that $\widehat{QC}$=$\widehat{PC}$, and we have $\widehat{BAC} = \frac{1}{2}\ \widehat{BC}$= $\frac{1}{2}(\widehat{BP} + \widehat{PC}) = \frac{1}{2}\ \widehat{BP} + \widehat{QC}$= $\widehat{QTC}$.

Note. Analogously, but using subtraction of arcs instead of addition, we can show that line PQ makes an angle equal to $\widehat{B}$ with side AC. (This angle is not shown in figure t376a.)

Solution (to exercise t376). In figure t376b, the original triangle is ABC, its incenter is I, its excenter opposite vertex C is I_c. Point M_c is the midpoints of side AB, F and F_c are the points of contact of circles I and I_c with side AB, and M is the intersection of the common internal tangents of circles I, I_c. Finally, H_c is the foot of the altitude from C to side AB.

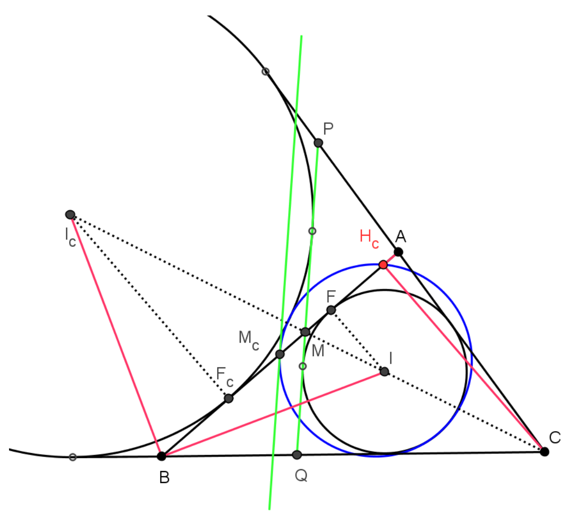

Figure t376b

The result of exercise 90a tells us that $M_cF = M_cF_c$. Since the power of a point with respect to a circle is the square of its tangent to the circle, this means that the powers of M_c with respect to circles I, I_c are equal, as asserted in the problem statement. We invert around M_c by this power, and examine the images of various elements under this inversion.

First we note that by **221**, 1°, circles I, I_c are their own images under this inversion.

Next we prove that point H_c inverts into point M, using a complicated but interesting argument. We know that BI, BI_c are the internal and external angle bisectors of triangle ABC at vertex B, so these lines divide segment CM harmonically (**115**, Remark, or **201**, corollary 2). In our diagram, we already have the parallel projection of the harmonic range $(CIMI_c)$ onto line BA (in a direction perpendicular to BC): it is the four points (H_cFMF_c). By **121** (or otherwise), these points also form a harmonic range, so FF_c divides MH_c harmonically. Then, from **189**, we have $M_cF^2 = M_cF_c^2 = M_cM \cdot M_cH_c$, and this equation says that point H_c inverts (by definition) onto point M under the inversion we are considering.

Next we look at the image of the nine-point circle of triangle ABC. The pole of inversion is M_c, which is on this circle, so the image is a line (**220**). And point H_c is on the nine-point circle, so its image, which is M, is on the line. In fact, by **220** (corollary), the image of the nine-point circle is a line through M parallel to its tangent at M_c.

This line is labeled PQ in figure t376b, and we will show that in fact it is the other common internal tangent (the one that is not side AB) to circles I, I_c. By lemma 2, and the fact that PQ is parallel to the tangent to the nine-point circle at M_c, we have $\widehat{PQC} = \widehat{BAC}$. It follows that in fact PQ is the reflection in II_c of AB. Indeed, II_c bisects $\widehat{ACB}$, so $\widehat{ACM} = \widehat{MCQ}$, and triangles MAC, MQC are congruent by AAS.

So the nine-point circle inverts into the common tangent to circles I and I_c. We have already noted that circles I and I_c are their own images under our inversion. It follows that the image of the nine-point circle is tangent to the images of I and I_c, and therefore the nine-point circle is itself tangent to I and I_c (**219**, corollary).

In the same way, we can show that the nine-point circle is tangent to the other two escribed circles of triangle ABC.

Notes. This fascinating result is sometimes called *Feuerbach's Theorem*. It can in fact be proven without using inversion. See, for example, Altshiller-Court, College Geometry, New York: Barnes and Noble, 1952; repr. Dover Publications, 2007.

Problem 377. If R and r (respectively) are the radii of the circumscribed and inscribed circles in a triangle, and if d is the distance between their centers, show that $d^2 = R^2 - 2Rr$ (Use Exercise 103 and **126**).[2]

Conversely, if the radii of two circles and the distance between their centers satisfy this relation, we can inscribe infinitely many triangles in one circle that are also circumscribed about the other.

Obtain analogous results replacing the inscribed circle by an escribed circle.

Solution. We first derive the result $d^2 = R^2 - 2Rr$. Let O and I be the circumcenter and incenter respectively of triangle ABC (*fig.* t377). We need to relate R to r, so it is reasonable to draw at least one perpendicular bisector and one angle bisector. Taking our cue from the result of exercise 103, we let M be

[2]For another solution to this problem, see Exercise 411.

the midpoint of BC, and let K be the intersection of the perpendicular bisector of side BC with the bisector of angle $\widehat{A}$. By exercise 103, K lies on the circumcircle of ABC.

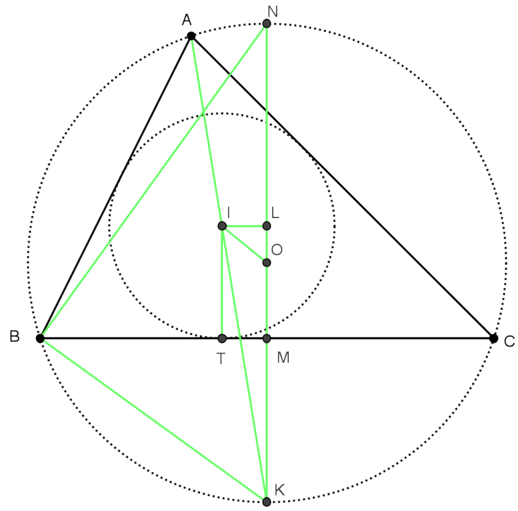

FIGURE t377
m

In triangle OIK, we have $OI = d$, $OK = R$. We need to relate R^2 and d^2, which suggests that we apply the result labeled 1° in **126** to this triangle. Letting L be the foot of the perpendicular from I to line OK, we find that $d^2 = OI^2 = OK^2 + IK^2 - 2OK \cdot KL = R^2 + IK^2 - 2R \cdot LK$.

We need to replace IK and LK with expressions involving R and r. From exercise 103, we know $IK = BK$, and we can work with BK if we draw triangle BKN, where N is the point on the circumcircle diametrically opposite K. Then BKN is a right triangle, and from **123** we have $IK^2 = BK^2 = NK \cdot MK = 2R \cdot MK$. It follows that $d^2 = R^2 - 2R \cdot (LK - MK) = R^2 - 2R \cdot LM$. But LM is just r, so we can write $d^2 = R^2 - 2Rr$. This gives us the first result.

(To see that $LM = r$, we can draw inradius $IT \perp BC$, and note that $ILMT$ is a rectangle.)

We now look at the converse. Suppose we are given circles O and I, satisfying the condition $d^2 = R^2 - 2Rr$. We will first show that this condition implies that circle I lies inside circle O. Indeed, d^2 is certainly not less than than 0, so $R^2 \geq 2Rr$, and since $R > 0$, this implies that $R \geq 2r$. Then $(R - r)^2 = R^2 - 2Rr + r^2 > R^2 - 2Rr = d^2$. That is, $d < R - r$. This inequality tells us that circle I lies inside circle O.

Now we can construct figure t377, knowing the positions of the two circles with their centers. We take any point A on circle O and draw line AI. Let K be its second point of intersection with circle O. We construct tangent BC to circle I, perpendicular to OK and passing between points I and K. (There is another tangent to circle I perpendicular to OK, which we ignore.)

Let M be the intersection of this tangent with OK, and let L be the foot of the perpendicular from I to OK. From triangle OIK and **126** we have $d^2 =$

$R^2 + IK^2 - 2R \cdot KL$. But the given condition is that $d^2 = R^2 - 2Rr$. From these two equations we find $IK^2 = 2R(KL - r) = 2R \cdot (KL - LM) = 2R \cdot KM = KB^2$. (We have used the fact that $LM = IT = r$ and that $2R \cdot KM = KN \cdot KM = KB^2$, which we derive just as we did in the proof of the direct statement.) It follows that $IK = KB$.

We turn our attention to triangle ABC. Circle O is of course its circumcircle. We now show that circle I is its incenter. In circle O, $\widehat{BK} = \widehat{KC}$, so line AK is an angle bisector in triangle ABC. From exercise 103, the incenter of ABC is the point on AK whose distance from K is equal to KB. But we have shown that $KB = IK$, so this incenter is I. And since the given circle I is tangent to BC, this circle is in fact the incircle of ABC. So any point A on circle O can be the vertex of a triangle inscribed in O and circumscribed about I.

Finally, we note that if we replace circle I with escribed circle I_a (with radius r_a), analogous reasoning will show that $OI_a^2 = R^2 + 2Rr_a$.

Note. A related result, giving conditions for the existence of a quadrilateral circumscribed about one circle and inscribed in another, is given in exercise 282.

Problem 378. In any triangle ABC:

1^o. The line joining the projection of B onto the bisector of $\widehat{C}$ with the projection of C onto the bisector of $\widehat{B}$ is precisely the chord joining the points of contact $E,\ F$ (*fig.* 94, Exercise 90b) of the inscribed circle with sides $AC,\ AB$;

2^o. The line joining the projection of B to the bisector of $\widehat{C}$ with the projection of C onto the bisector of the exterior angle at B is the chord of contact E_3F_3 of the escribed circle for angle $\widehat{C}$ with these same sides;

3^o. The line joining the projection of B onto the bisector of the exterior angle at C with the projection of C onto the bisector of the exterior angle at B is the chord of contact E_1F_1 of the escribed circle for angle $\widehat{A}$ with these same sides;

4^o. The projections of A onto the bisectors of the interior and exterior angles at B and C are on the same line parallel to BC, and their consecutive distances are equal to $p - c,\ p - a,\ p - b$;

5^o. The six points obtained by projecting each of the vertices $A,\ B,\ C$ onto the exterior angle bisectors at the other two vertices belong to the same circle (this reduces to Exercise 102). This circle is orthogonal to the escribed circles. Its center is the same as the center of the inscribed circle of $A'B'C'$, the triangle whose vertices are the midpoints of the sides of ABC. Its radius is equal to the hypotenuse of the right triangle whose legs are the radius of the inscribed circle and the semiperimeter of triangle $A'B'C'$. There are three analogous circles, each of which passes through two projections onto exterior angle bisectors, and four projections onto interior angle bisectors.

Solutions. The first three parts of this exercise are closely related. In each case, collinearity can be proven by examining angles. We show that a pair of sides, one from each angle, are collinear, by showing that the angles are either equal or supplementary. Also in each case, we identify cyclic quadrilaterals which contribute to the solution. Each of the quadrilaterals can be shown to be cyclic by identifying certain right angles. Right angles, in turn, can be identified because certain lines are given to be perpendicular, or because interior and exterior angles bisectors at the same vertex are perpendicular, or because a radius is perpendicular to a chord at the point of contact. Another element the various cases have in common is the identification of an isosceles triangle, formed by two tangents to a circle from the same point.

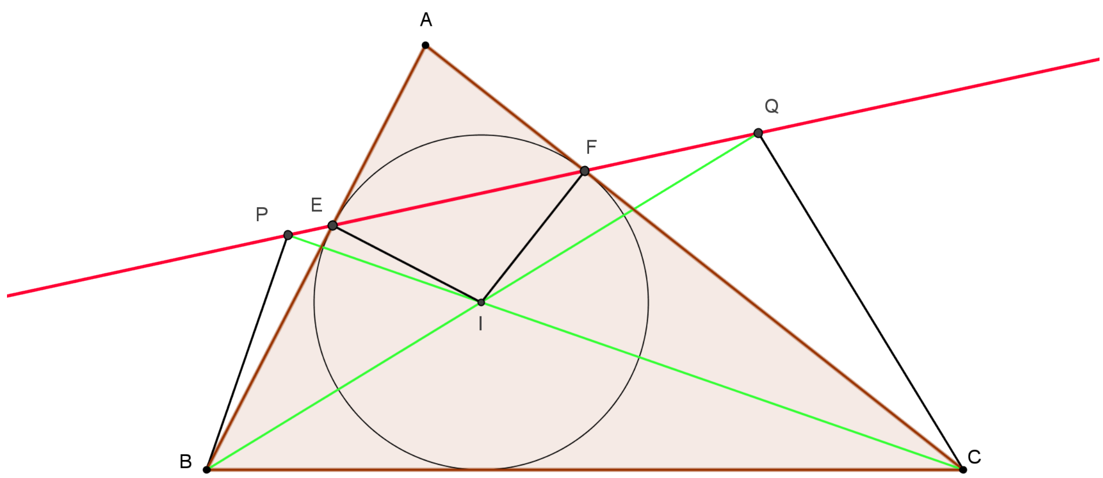

FIGURE t378a

The strategy is to identify angles that will give us the required collinearity, then relate the measures of these angles to the angles of the original triangle. We denote the angles of the original triangle simply as $\widehat{A},\ \widehat{B}, \widehat{C}$.

At each step of the proof, it is important to observe which points we know to be collinear and which we must prove collinear. Otherwise, an error in logic can easily creep into the argument.

(1°). Let P be the foot of the perpendicular from B to the (internal) angle bisector through C (*fig.* t378a), let Q be the foot of the perpendicular from C to the internal angle bisector through B, and let I be the incenter of triangle ABC. We will show that angles $\widehat{AFE},\ \widehat{CFQ}$ are equal. Indeed, quadrilateral $IFQC$ is cyclic, because $\widehat{IFC},\ \widehat{CQI}$ are right angles. Then $\widehat{CFQ} = \widehat{CIQ}$, which is an exterior angle for triangle BIC, so $\widehat{CFQ} = \widehat{CIQ} = \frac{1}{2}(\widehat{B} + \widehat{C})$.

On the other hand, $\widehat{AFE}$ is a base angle of isosceles triangle AEF, so $\widehat{AFE} = \frac{1}{2}(180° - \widehat{A}) = \frac{1}{2}(\widehat{B} + \widehat{C})$.

Thus $\widehat{CFQ} = \widehat{AFE}$, and (by exercise 4) $E,\ F,\ Q$ are collinear.

We can show the collinearity of $E,\ F,\ P$ similarly, using cyclic quadrilateral $IEPB$ and isosceles triangle AEF. Thus $E,\ F,\ P,\ Q$ are collinear.

(2°). Let P_3 be the foot of the perpendicular from B to the internal angle bisector through C (*fig.* t378b), let Q_3 be the foot of the perpendicular from C to the external angle bisector through B, and let I_c be the excenter of triangle ABC

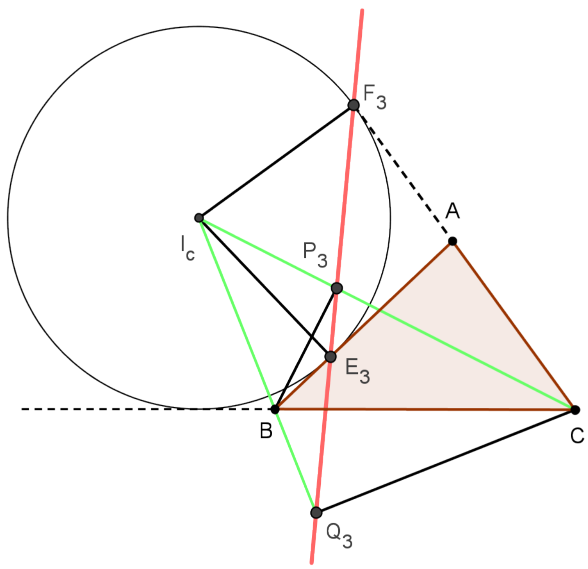

FIGURE t378b

opposite vertex C. We will show that angles $\widehat{CF_3Q_3}$, $\widehat{CF_3E_3}$ are equal. Indeed, quadrilateral $I_cF_3CQ_3$ is cyclic, because $\widehat{I_cF_3C}$, $\widehat{I_cQ_3C}$ are right angles. Then $\widehat{CF_3Q_3} = \widehat{CI_cQ_3} = \widehat{CI_cB}$. Now in triangle BI_cC, $\widehat{CI_cB} = 180° - \frac{1}{2}\widehat{C} - \widehat{CBI_c}$, and $\widehat{CBI_c} = \widehat{B} + \frac{1}{2}(180° - \widehat{B})$, and a quick calculation shows that $\widehat{CI_cB} = 90° - \frac{1}{2}\widehat{B} - \frac{1}{2}\widehat{C} = \frac{1}{2}\widehat{A}$.

On the other hand, in isosceles triangle AF_3E_3, exterior angle $\widehat{A} = \widehat{AF_3E_3} + \widehat{AE_3F_3} = 2\widehat{AF_3E_3}$, so $\widehat{CF_3E_3} = \widehat{AF_3E_3} = \frac{1}{2}\widehat{A} = \widehat{CF_3Q_3}$. Thus points E_3, F_3, Q_3 are collinear.

We can show the collinearity of E_3, F_3, P_3 similarly. From cyclic quadrilateral $BE_3P_3I_c$, we find that $\widehat{BE_3P_3} = 180° - \widehat{CI_cB} = 180° - \widehat{CI_cQ_3}$, and we have already seen that this last angle is $\frac{1}{2}\widehat{A}$. So $\widehat{BE_3P_3} = 180° - \frac{1}{2}\widehat{A}$.

On the other hand, $\widehat{BE_3F_3} = 180° - \widehat{AE_3F_3}$, and we have already seen that this last angle is equal to $\frac{1}{2}\widehat{A}$. Thus $\widehat{BE_3F_3} = \widehat{BE_3P_3}$, and points E_3, P_3, F_3 are collinear.

(3°). Let P_1 be the foot of the perpendicular from B to the external angle bisector through C (*fig.* t378c), let Q_1 be the foot of the perpendicular from C to the external angle bisector through B, and let I_a be the excenter of triangle ABC opposite vertex A. We will show that angles $\widehat{CF_1Q_1}$, $\widehat{CF_1E_1}$ are equal. Indeed, quadrilateral $I_aF_1CQ_1$ is cyclic, because $\widehat{I_aF_1C}$, $\widehat{I_aQ_1C}$ are right angles. Then $\widehat{CF_1Q_1} = \widehat{CI_aQ_1} = \widehat{CI_aB}$. Now in triangle BI_aC, $\widehat{CI_aB} = 180° - \frac{1}{2}(180° - \widehat{B}) - \frac{1}{2}(180° - \widehat{C}) = \frac{1}{2}\widehat{B} + \frac{1}{2}\widehat{C}$.

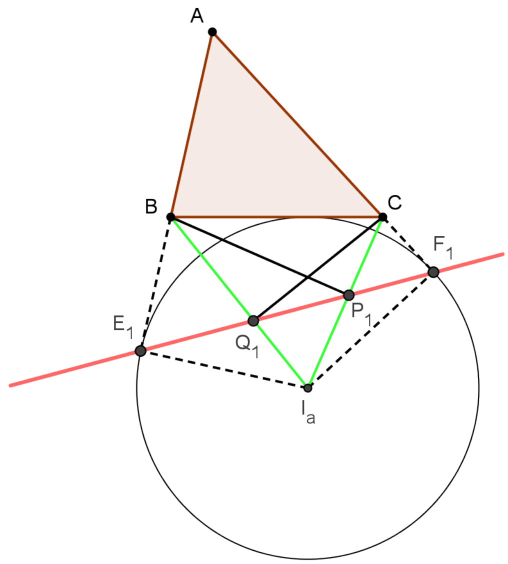

Figure t378c

On the other hand, in isosceles triangle AF_1E_1, $\widehat{AF_1E_1} = \widehat{AE_1F_1} = \frac{1}{2}(180° - \widehat{A}) = \frac{1}{2}(\widehat{B} + \widehat{C}) = \widehat{CF_1Q_1}$. Hence points $E_1,\ F_1,\ Q_1$ are collinear.

We can show the collinearity of $E_1,\ F_1,\ P_1$ similarly,, using cyclic quadrilateral $BE_1I_aP_1$. We find that $\widehat{BE_1P_1} = \widehat{BI_aP_1} = \widehat{CI_aB}$, and we have already seen that this last angle is $\frac{1}{2}(\widehat{B} + \widehat{C})$. So $\widehat{BE_1P_1} = \frac{1}{2}(\widehat{B} + \widehat{C})$.

We have seen that $\widehat{BE_1F_1}$ has the same measure. Thus $\widehat{BE_1F_1} = \widehat{BE_1P_1}$, and points $E_1,\ P_1,\ F_1$ are collinear.

(4°.) Note that p here denotes the *semiperimeter* of triangle ABC. As usual, we let $AB = c,\ AC = b,\ BC = a$. Let $P_4,\ P_5$ be the feet of the perpendiculars from A onto the interior and exterior angle bisectors at vertex B, and let $Q_4,\ Q_5$ be the corresponding points for the bisectors at vertex C. Let $M_b,\ M_c$ be the midpoints of $AC,\ AB$ respectively.

The central observation of this solution is that quadrilaterals $AP_4BP_5,\ AQ_4CQ_5$ are both rectangles (by construction, and because interior and exterior angle bisectors at the same vertex are perpendicular).

We first prove that $P_4,\ P_5,\ Q_4,\ Q_5$ are collinear. Indeed, in rectangle AP_4BP_5, diagonals $AB,\ P_4P_5$ bisect each other. So line P_4P_5 passes through the midpoint of side AB. And (from isosceles triangle BM_cP_5) $\widehat{CBP_5} = \widehat{P_5BA} = \widehat{BP_5P_4}$, so $BC \parallel P_4P_5$, and P_4P_5 lies along the line connecting the midpoints of sides $AB,\ AC$ of triangle ABC. In just the same way, we can show that $Q_4,\ Q_5$ lies along the same line, which is the 'parallel' referred to in the problem statement.

We now proceed to the metric results, considering P_4Q_5 first. We have $P_4Q_5 = P_4M_c + M_cM_b + M_bQ_5$. But $M_cM_b = \frac{1}{2}BC$, and $P_4M_c = AM_c = \frac{1}{2}AB$, and

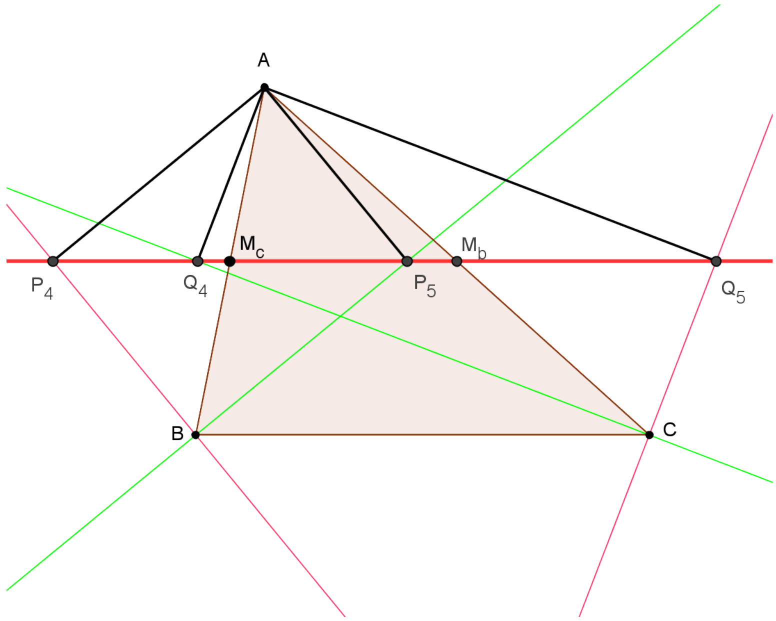

FIGURE t378d

$M_bQ_5 = AM_b = \frac{1}{2}AC$. Thus $P_4Q_5 = \frac{1}{2}(a+b+c) = p$ (in the notation of the exercise).

Then $P_4Q_4 = P_4Q_5 - Q_4Q_5 = P_4Q_5 - AC$ (from rectangle AQ_4CQ_5), and this last expression is equal to $p-b$. In the same way, $P_5Q_5 = P_4Q_5 - P_4P_5 = P_4Q_5 - AB = p - c$.

Finally, $Q_4P_5 = P_4Q_5 - P_4Q_4 - P_5Q_5 = p - (p-b) - (p-c) = b + c - p = \frac{b}{2} + \frac{c}{2} - \frac{a}{2} = p - a$.

Note. After the rectangles are identified, the solution is made simple by looking at P_4Q_5 before considering the lengths of the required segments.

(5°) We break this exercise into the following statements:

a. The projections of the three vertices onto the three angle bisectors are concyclic;

b. Their circle is orthogonal to the escribed circles;

c. Their circle is concentric with the incircle of $A'B'C'$;

d. The radius of this circle is as described in the exercise;

e. There are three analogous circles.

Figure t378e shows the six feet of the perpendiculars referred to (P_a, Q_a, etc.) and the three excenters (I_a, I_b, I_c) of triangle ABC.

(5a) An interior angle bisector is perpendicular to an exterior angle bisector at the same vertex of a triangle. It follows that AI_a, BI_b, CI_c are the altitudes of triangle $I_aI_bI_c$, and statement of the problem reduces to the statement of exercise 102.

We will call this circle S.

(5b). We will prove that circles I_a, S are orthogonal by showing that the power of point I_a with respect to S is equal to the square of the radius of circle I_a (**135**).

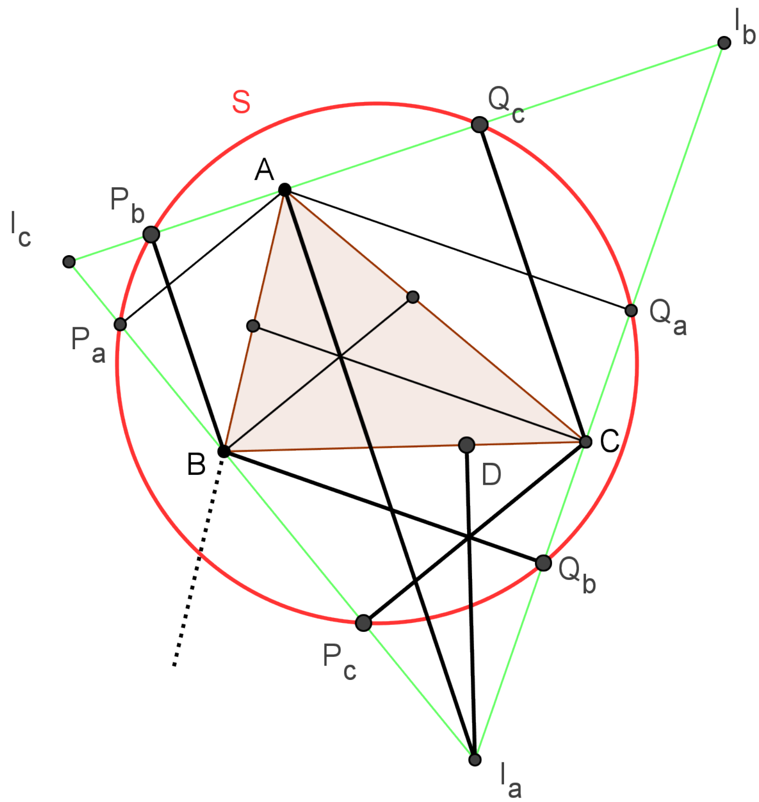

FIGURE t378e

Since we need a radius of circle I_a, we draw $I_aD \perp BC$, with D on BC. Now $I_aQ_a \cdot I_aQ_b$ is the power of I_a with respect to circle S. We will show that power is equal to I_aD^2, by relating both quantities to distances between the excenters and the vertices of the triangle.

To do this, we look first at I_aD, and find similar triangles involved the 'target' segments: triangles I_aDB, I_aAI_b. To prove that they are similar, we note that they are right triangles, and that we can find one more pair of equal angles. Indeed, $\widehat{I_aBD}$ is half of an exterior angle at B of the original triangle, so it is equal to $\frac{1}{2}(\widehat{A} + \widehat{C})$. For the same reason, $\widehat{I_bAC} = \frac{1}{2}(\widehat{B} + \widehat{C})$ and $\widehat{I_bCA} = \frac{1}{2}(\widehat{A} + \widehat{B})$. So $\widehat{AI_bC} = 180° - \frac{1}{2}(\widehat{A} + \widehat{B} + \widehat{C}) - \frac{1}{2}\widehat{B} = 90° - \frac{1}{2}\widehat{B} = \frac{1}{2}(\widehat{A} + \widehat{C})$.

Thus triangles I_aDB, I_aAI_b are similar, and $I_aD : I_aB = I_aA : I_aI_b$. That is, $I_aD = \frac{I_aA \cdot I_aB}{I_aI_b}$. This expresses the radius of an escribed circle in terms of the distances between vertices and excenters, so it accomplishes half our goal.

We now examine the product $I_aQ_a \cdot I_aQ_b$, by applying **123** to right triangles I_aI_bA and I_aI_bB. We find that $I_aA^2 = I_aI_b \cdot I_aQ_a$ and $I_aB^2 = I_aI_b \cdot I_aQ_b$. So $I_aQ_a \cdot I_aQ_b = \frac{I_aA^2 \cdot I_aB^2}{I_aI_b^2}$, which is the square of the expression we obtained for I_aD.

Thus circles I_a, S are orthogonal by **135**.

(5c). In figure t378f, A', B', C' are the midpoints of the sides of triangle ABC. We will show that the incenter of $A'B'C'$ lies on a certain diameter of circle S, then that there are other diameters that this incenter lies on. This will show that the incenter of $A'B'C'$ is also the center of S. The diameter of circle S that we need is the one which bisects chord P_bQ_c. We will show that it lies along the bisector of angle $\widehat{B'A'C'}$.

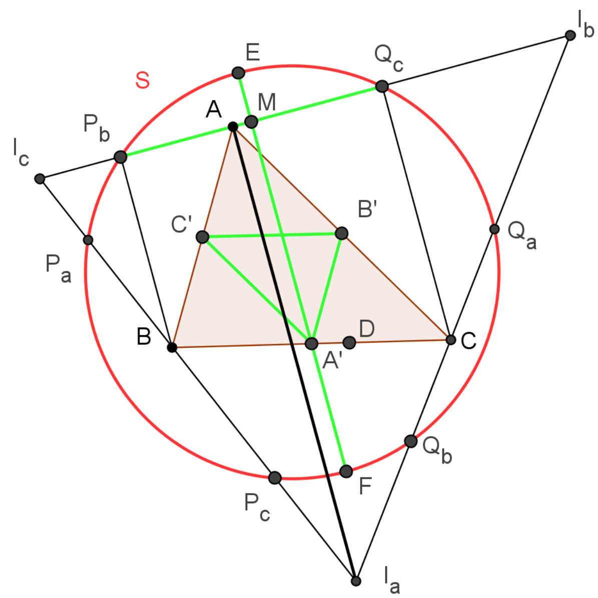

FIGURE t378f

Let M be the midpoint of P_bQ_c, and EF the diameter of S through M. Then $EF \perp P_bQ_c$. And since $P_bB \perp P_bQ_c$, $Q_cC \perp P_bQ_c$ by construction, segment MA' must pass through the midpoint of BC, which is A' (**113**).

Now $I_aA \parallel EF$ (they are both perpendicular to P_bQ_c), so (**43**) angles $\widehat{I_aAC}$, $\widehat{MA'C'}$ are equal. Since $\widehat{BAC} = \widehat{B'A'C'}$, this means that EF bisects $\widehat{B'A'C'}$, and the incenter of $A'B'C'$ lies on this diameter of S.

We can repeat this argument for chord Q_aQ_b and the diameter of S which bisects it, so the the incenter of $A'B'C'$ lies at the intersection of two diagonals of S; that is, at its center. This completes the proof.

Note. Knowing $I_aA \parallel EF$, the easiest way to see that EF bisects $\widehat{A'B'C'}$ is to recall that $A'B'C'$ is homothetic to ABC, with center of homothecy at the centroid of ABC and coefficient $-\frac{1}{2}$.

(5d). We let O be the center of circle S, and we draw segment P_aQ_a (*fig.* t378g). We have shown, in 4°, that this segment passes through B' and C'. We draw $ON \perp P_aQ_a$ and radius OP_a of circle S. We can get the length of OP_a from right triangle ONP_a. Indeed, we saw in 4°, that $P_aQ_a = p$ (where p is the semiperimeter of ABC), so $P_aN = \frac{p}{2}$. This is exactly the semiperimeter of $A'B'C'$. And ON is the distance from the incenter of $A'B'C'$ (from 5°c) to a side of that triangle, so it is equal to the inradius of $A'B'C'$. These observations complete the proof.

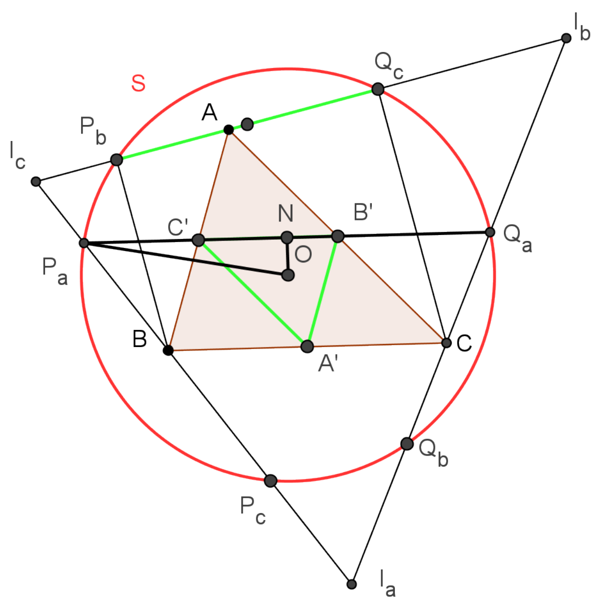

FIGURE t378g

(5e). The existence of circle S was given us by the result of exercise 102. We could apply this result because the interior angle bisectors of triangle ABC are the altitudes of triangle $I_aI_bI_c$.

If, as a vertex of this triangle, we replace one of the excenters with the incenter, we still get a triangle whose sides lie on (interior or exterior) angle bisectors of ABC, and whose altitudes are the remaining angle bisectors. Thus the result of exercise 102 still applies, and we get three analogous circles.

For example, if we consider triangle II_bI_c (*fig.* t378h), the feet of its altitudes are (again) points A, B, C. If we drop perpendiculars from these points onto the sides of II_bI_c, we get the six points of exercise 102, which therefore lie on the same circle.

Problem 379. Each escribed circle of triangle ABC is tangent to the extensions of two of its sides. We draw the lines through pairs of these points of contact.

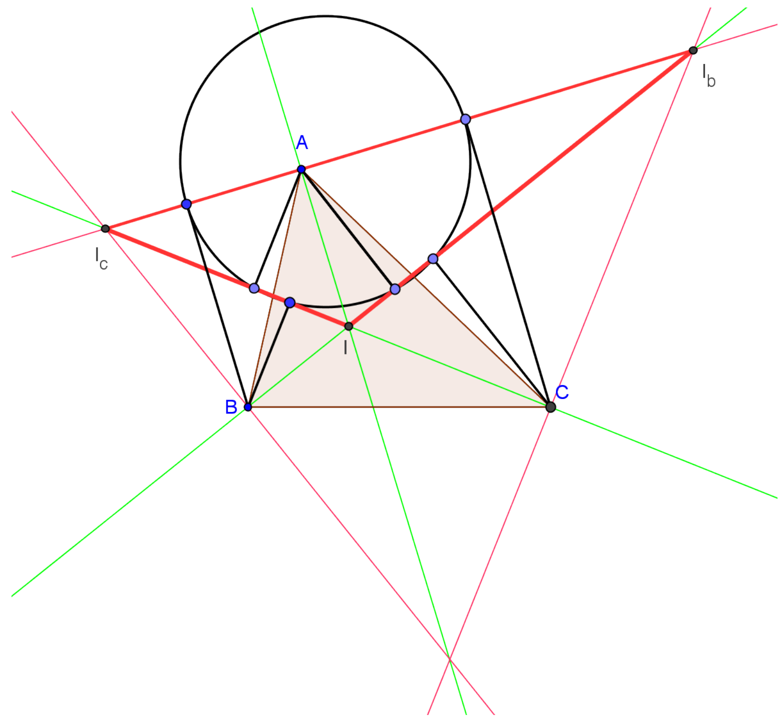

FIGURE t378h

Show that these three lines form a triangle whose vertices are on the altitudes of the original triangle, and the intersection of these altitudes is the center of the circumscribed circle for the new triangle.

Solution. In figure t379, the triangle constructed in the exercise is $A'B'C'$, and the points of contact of the escribed circles are as labeled. Segment AK is the altitude from A in triangle ABC, and H is its orthocenter. We will show that lines B_1B_2, C_1C_2 intersect line AK at the same point, by showing that the point of intersection of each line with the altitude divides segment AK (externally) in the same ratio. As usual, we let p denotes the semiperimeter of triangle ABC, and a, b, c the lengths of its sides. Our chief tool will be Menelaus' theorem (**192**).

First suppose that line B_1B_2 intersects AK at a point A'. We will show that the ratio $AA' : A'K$ is equal to $a : (b+c)$. To do this, we apply Menelaus' Theorem to triangle ABK with transversal B_2B_1A':

$$\frac{B_1A}{B_1B} \cdot \frac{B_2B}{B_2K} \cdot \frac{A'K}{A'A} = 1, \tag{1}$$

which gives us information about the ratio we seek. We need to express the other four segments in (1) in terms of a, b, and c. The results of exercise 90b give us $BB_2 = BB_1 = p$; $AB_1 = p - c$. To compute B_2K, we write $B_2K = B_2B - KB$. Then $B_2B = p = \frac{a+b+c}{2}$, and the second theorem of **126** allows us to express KB as required: $KB = \frac{a^2+c^2-b^2}{2a}$. Hence $B_2K = \frac{a+b+c}{2} - \frac{a^2+c^2-b^2}{2a} = \frac{ab+ac+b^2-c^2}{2a} =$

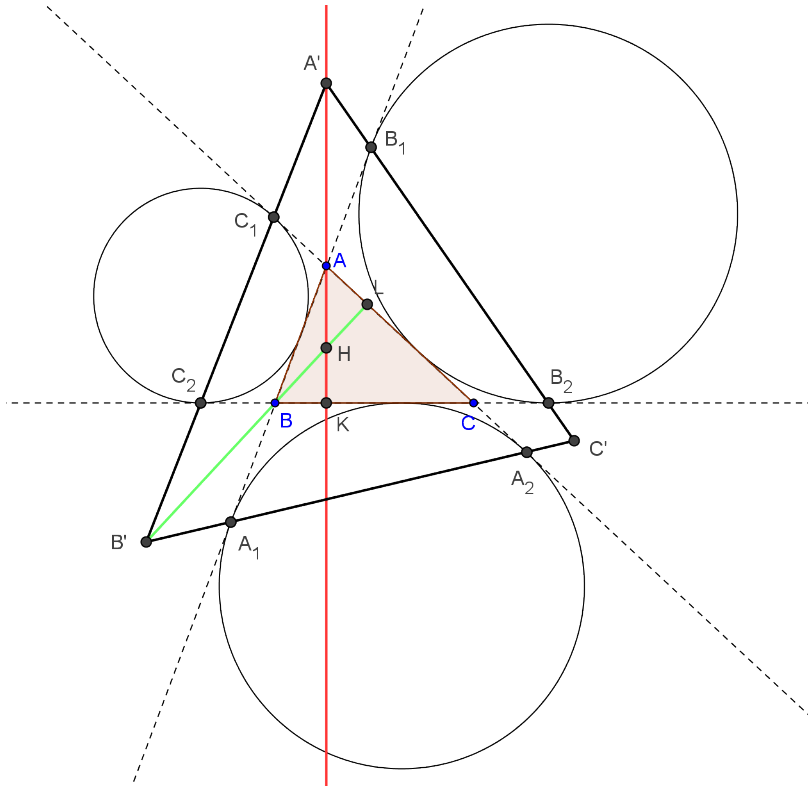

FIGURE t379

$\frac{(b+c)(a+b-c)}{2a}$. A quick computation shows that $\frac{a+b-c}{2} = p - c$, so we can write $B_2K = \frac{(b+c)(p-c)}{a}$.

We substitute all these values into (1):

$$\frac{p-c}{p} \cdot \frac{pa}{(b+c)(p-c)} \cdot \frac{KA'}{A'A} = 1,$$

to find that $\frac{A'A}{KA'} = \frac{a}{b+c}$.

We can now perform this same computation, using triangle ACK and transversal C_2C_1. We find that this line intersects AK at a point dividing AK externally into just the same ratio. But in fact this is expected: the expression we obtained for this ratio is symmetric in b and c, so interchanging the roles of these two sides (i.e. computing with C_1C_2 in place of B_1B_2) should not change the result of the computation.

Thus the intersection of B_1B_2 and C_1C_2 lies on AK, the altitude from A in triangle ABC, and must be the point we have labeled A'. In the same way, we can show that points B', C' lie on the other two altitudes of the triangle.

The second assertion of the exercise is easier to prove. Let H be the orthocenter of ABC, and let L be the foot of the altitude to AC. (Note that the first assertion of this exercise implies that $B'L \perp AC$.) Then we can show that H is equidistant from vertices A', B' by showing that triangle $HA'B'$ is isosceles. Indeed, from right triangle $A'C_2K$, $\widehat{HA'B'} = \widehat{KA'C_2} = 90° - \widehat{A'C_2K}$. From isosceles triangle CC_1C_2, we find that $\widehat{A'C_2K} = \widehat{C_1C_2C} = \widehat{C_2C_1C}$, and from right triangle $B'LC_1$, we have $\widehat{C_2C_1C} = \widehat{B'C_1L} = 90° - \widehat{C_1B'L} = 90° - \widehat{A'B'H}$. This implies that $\widehat{HA'B'} = 90° - (90° - \widehat{A'B'H}) = \widehat{A'B'H}$, so triangle $A'B'H$ is isosceles, and H is equidistant from vertices A', B' of triangle $A'B'C'$.

In the same way, we can show that H is equidistant from vertices B', C', so that H is the circumcenter of triangle $A'B'C'$.

Problem 380. Suppose we know (a) the point O corresponding to itself (**150b**) in two similar figures F, F' with the same orientation, and (b) a triangle T similar to the triangle formed by this point with two other corresponding points. Suppose we also know the point O' corresponding to itself (**150b**) in similar figures F', F'' with the same orientation, and a triangle T' similar to the triangle formed by this point with two other corresponding points. Construct the point O_1 corresponding to itself (**150b**) in the similar figures F, F'', and a triangle T_1 similar to the triangle formed by this point with two other corresponding points.

(Place triangles T, T' so that they have a common side $\omega\alpha'$, and denote by α, α'' the vertices opposite this side in the two triangles. Then T_1 is the triangle $\omega\alpha\alpha''$. Take the inverses A, A', A'' of α, α', α'' with respect to the pole ω: then triangle O_1OO' must be similar to $A_1A''A$.)

Solution. We construct triangle T_1 first. Let A be any point of figure F. If we construct triangle OAA' similar to triangle T, we obtain the point A' of figure F' which corresponds to point A of figure F. And if we construct triangle $O'A'A''$ similar to triangle T', we will obtain the point A'' of figure F'' which corresponds to point A of figure F.

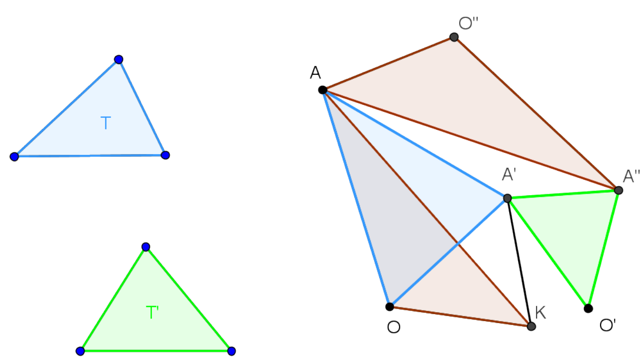

FIGURE t380

Next we construct triangle $OA'K$, similar to triangle $O'A'A''$ and having the same sense of rotation. Now $\widehat{AOA'}$ is the angle between figures F and F' (**50**), while

$\widehat{A'OK}$, which is equal by construction to $\widehat{A'O'A''}$, is the angle between figures F' and F''. Hence $\widehat{AOK}$ is the angle between figures F and F''.

We next show that $\frac{OA}{OK}$ is the ratio of similitude of figures $F,\ F''$. This ratio is the product of the ratios of similitude for $F,\ F'$ and $F',\ F''$. We know that $\frac{OA}{OA'}$ is the ratio of similitude of figures F and F', while $\frac{O'A'}{O'A''}$ is the ratio of similitude of figures F' and F''. But we can write $\frac{OA}{OK} = \frac{OA}{OA'} \cdot \frac{OA'}{OK} = \frac{OA}{OA'} \cdot \frac{O'A'}{O'A''}$, which means that $\frac{OA}{OK}$ is the ratio of similitude of $F,\ F''$.

So triangle AOK contains the angle between figures F and F'', and also two sides in the ratio of these figures. Thus it is similar to the triangle formed by the required point O'' and a pair of corresponding points from those figures. This solves the first part of the exercise.

To find the actual point O'', we can 'apply' the shape of triangle OAK to two corresponding points of F and F''. That is, we construct triangle $O''AA''$ similar to triangle OAK (with the same orientation). Since $\widehat{AO''A''}$ is the angle between figures F and F'', and the ratio $O''A'' : O''A$ is their ratio of similarity, and since $A,\ A''$ are corresponding points, it follows that O'' is the point of figure F which coincides with its corresponding point in F''.

Our construction implies that the problem has, in general, a single solution.

Notes. The construction requires small adjustments in some special cases. If points $O,\ A,\ K$ turn out to be collinear, and A and K are distinct, then figures $F,\ F''$ are homothetic. To find point O'', we merely divide segment AA'' in the ratio $O''A : O''A'' = OA : OK$ (both in magnitude and sign).

If points A and K coincide, then the corresponding segments of F and F'' are equal, parallel, and have the same orientation. Figures F and F'' can be obtained from one another by a translation whose magnitide and direction is give by segment AA'', and there is no point O''.

Finally, if points $A,\ A''$ coincide, then the two figures $F,\ F''$ coincide completely, and we can choose any point in the plane for O''.

Problem 381. Construct a polygon knowing the vertices of the triangles whose bases are its various sides, and similar to given triangles.

(The preceding exercise allows a reduction of one in the number of sides of the required polygon. One can continue until there are only two vertices to determine.)

When is the problem impossible or under-determined?

Solution. Suppose the required polygon is $A_1A_2\cdots A_{n-1}A_n$, (*fig.* t381) and suppose the given vertices are $V_1,\ V_2, \ldots,\ V_{n-1},\ V_n$, so that $A_1V_1A_2$ is similar to the given triangle T_1, $A_2V_2A_3$ is similar to triangle T_2, and so on.

We reduce this problem to exercise 380, starting at the 'end' of the given polygon. We think of V_{n-1} as the fixed point of two figures $F_{n-1},\ F_n$ which are similar and similarly oriented. We think of A_{n-1} and A_n as corresponding points in $F_{n-1},\ F_n$. In the same way, we think of V_n as the fixed point of two figures $F_n,\ F_1$ which are similar and similarly oriented, and of A_n and A_1 as corresponding points in $F_n,\ F_1$.

Using the result of exercise 380, we can construct V'_{n-1}, the fixed point of figures $F_{n-1},\ F_1$, as well as a triangle T'_{n-1} similar to the triangle formed by V'_{n-1} and two corresponding points in figures $F_{n-1},\ F_n$. This reduces the problem to

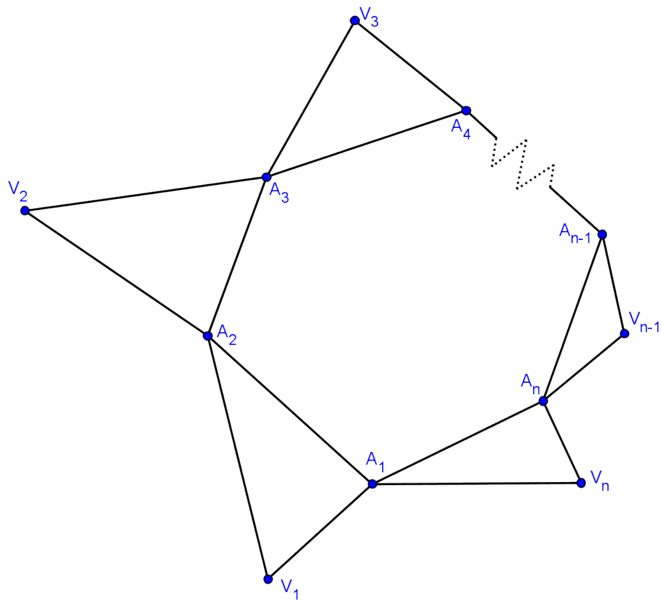

FIGURE t381

that of finding polygon $A_1A_2\cdots A_{n-1}$, given the points $V_1,\ V_2,\ldots,\ V_{n-2},\ V'_{n-1}$ and triangles $T_1,\ T_2,\ldots T_{n-2},\ T'_{n-1}$.

If we repeat this process of reducing the number of vertices in the polgyon we seek, we eventually come to the following problem: construct segment A_1A_2, given two points $V_1,\ V'_2$ which are vertices of triangles $V_1A_1A_2,\ V'_2A_2A_1$, similar respectively to the given triangles $T_1,\ T'_2$. We can now think of V_1 as the point corresponding to itself in two similar figures $F_1,\ F_2$, which include $A_1,\ A_2$ as corresponding points. Likewise, we can think of V'_2 as the point corresponding to itself in two similar figures $F_1,\ F_3$, which include $A_2,\ A_1$ as corresponding points.

Now we again apply exercise 380, to construct A_1 as the point corresponding to itself in figures $F_1,\ F_3$, using points $V_1,\ V'_2$ and triangles $T_1,\ T'_2$ as in that exercise. After locating point A_1, we can find point A_2 as the third vertex of triangle $V_1A_1A_2$, similar to triangle T_1. Then we can construct points $A_3,\ A_4\ldots A_n$ analogously.

The construction is valid so long as we do not encounter one of the conditions described in exercise 380 as preventing that construction.

Note. A more advanced discussion would rest on the notion of a transformation (a 'scaling rotation') which is a composition of a rotation and a dilation with the same center. Here, these centers are V_i, we are composing scaling rotations with centers at the points V_i, and we seek a fixed point of this composition.

In composing these scaling rotations, it may happen (as noted in the solution to problem 380) that two of them result in a translation, which has no fixed point. This may not cause a problem, since further composition can yield a scaling rotation after all. There is no solution if the sum of the angles through which we are rotating (the sum of the angles at the vertices V_i) is 0 or an integer multiple of $360°$.

Problem 382. Let ABC be a triangle, and let $O,\ a,\ b,\ c$ be four arbitrary points. Construct (a) a triangle BCA' similar to triangle bcO and with the same orientation ($B,\ C$ being the points corresponding to $b,\ c$); (b) a triangle CAB' similar to caO with base CA, and (c) a triangle ABC' similar to abO with base AB. Show that triangle $A'B'C'$ is similar to, but with opposite orientation from, the triangle whose vertices are the inverses of the points $a,\ b,\ c$ with pole O.

Solution. In light of exercises 380-381, we introduce three figures $F_a,\ F_b,\ F_c$, each pair of which is similar and similarly oriented. Figure F_a includes point A; figure F_b includes point B as the point corresponding to A in F_a, and figure F_c includes point C as the point corresponding to B in F_b. We think of point A' as the fixed point of the similarity between figures $F_b,\ F_c$ and B' as the fixed point of the similarity between figures $F_c,\ F_a$. We can then show that C' is the fixed point of the similarity between figures $F_a,\ F_b$.

We first examine the angles between the three figures. The angle between figures F_b and F_c is equal to $\widehat{BA'C} = \widehat{bOc}$, and has the same orientation. Likewise, the angle between figures F_c and F_a will be $\widehat{CB'A} = \widehat{cOa}$. It follows that the angle between figures F_c and F_a is equal to $\widehat{aOb} = \widehat{AC'B}$.

Next we examine the ratios of corresponding sides in the three figures. This ratio, for figures $F_b,\ F_c$ is $A'B : A'C = Ob : Oc$. For figures $F_c,\ F_a$ it is $B'C : B'A = Oc : Oa$. It follows that the ratio of corresponding sides for figures F_a and F_b is equal to $Oa : Ob = C'A : C'B$. Therefore C' is the fixed point of the similarity taking F_a onto F_b.

Let $a',\ b',\ c'$ be the points inverse to $A,\ B,\ C$ respectively, with respect to some circle centered at O. We are now in a position to prove that triangles $A'B'C',\ a'b'c'$ are similar but oppositely oriented. We do this by showing that they have equal but oppositely oriented angles. We introduce an auxiliary point A'', corresponding to A' in figure F_a.

Triangle $C'A''A'$ is similar to triangle $C'AB$, and has the same orientation, so that $\widehat{C'A'A''} = \widehat{C'BA} = \widehat{Oba}$ (both in magnitude and orientation). In the same way, triangles $B'A'A''$, $B'CA$ are similar, with the same orientation, so that $\widehat{B'A'A''} = \widehat{B'CA} = \widehat{Oca}$.

It follows that $\widehat{B'A'C'} = \widehat{B'A'A''} + \widehat{A''A'C'} = \widehat{Oca} + \widehat{abO} = \widehat{c'a'O} + \widehat{Oa'b'} = \widehat{c'a'b'}$, where $a',\ b',\ c'$ are the points inverse to $a,\ b,\ c$ respectively in a circle centered at O (**217**).

Thus we have, both in magnitude and sense of rotation, $\widehat{B'A'C'} = \widehat{c'a'b'}$. A similar argument gives the same result for the other two angles of triangle $A'B'C'$. Triangles $A'B'C'$, $a'b'c'$ are similar, but with opposite senses of rotation.

Problem 383. On two given segments as chords, construct circular arcs subtending the same arbitrary angle $\widehat{V}$. Show that, as $\widehat{V}$ varies, the radical axis of the two circles turns around a fixed point. (This point can be considered to be determined by the fact that the triangles with this vertex and with the two given segments as bases are equivalent, and they have the same angle at the common vertex.)

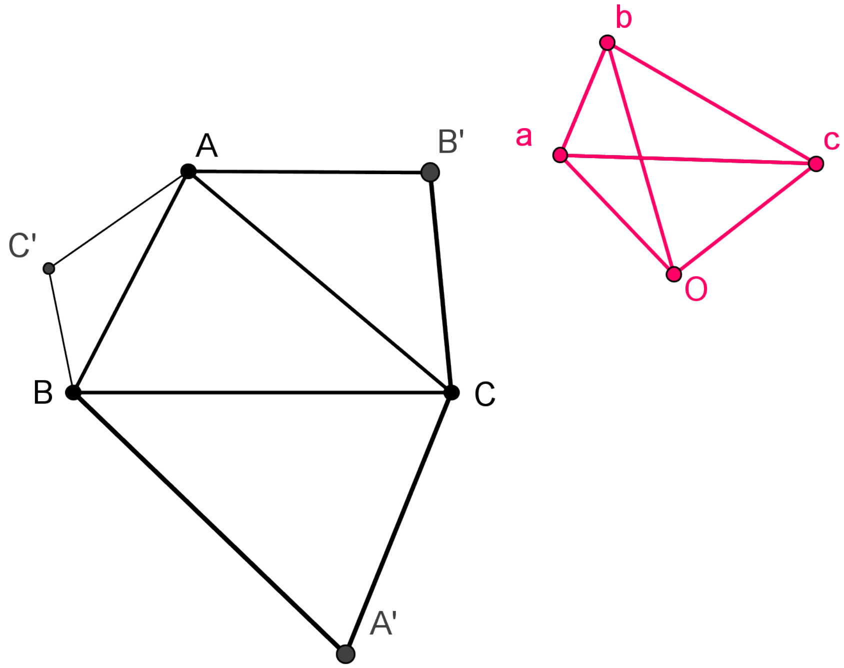

Figure t382

Solution. Suppose (*fig.* t383) the given segments are AB and CD, and P is the intersection of the lines they belong to. We assume, for now, that the angles we construct on AB and CD have the same orientation.

Following the hint in the problem statement, we will prove the following lemma:

Lemma. There exists exactly one point O with the following properties:

- (a) triangles OAB, OCD have the same area;
- (b) angles $\widehat{AOB}$, $\widehat{COD}$ have the same orientation;
- (c) angles $\widehat{AOB}$, $\widehat{COD}$ are equal.

We prove this lemma by constructing the point in question.

For condition (a) to be true, the distances from O to lines AB, CD must be in the ratio $CD : AB$ (note the reversal of the order AB, CD). By **157**, the locus of these points consists of two lines, both passing through P. It is not hard to see, from figure t383, that for any point X on one of these lines, angles $\widehat{AXB}$, $\widehat{CXD}$ have the same orientation, while for any point Y on the other line, angles $\widehat{AXB}$, $\widehat{CXD}$ have the opposite orientation. Thus point O satisfies conditions (a) and (b) if and only if it lines on line PX.

The set of points satisfying condition (c), in addition to (a) and (b), is a bit more difficult to describe. We will show that condition (c) requires point O to be on a second line, which is the locus of points such that $OM^2 - ON^2$ is constant, where

M, N are the midpoints of segments AB, CD respectively. (See **128b**, corollary.) We do this by examining relationships within triangles AOB, COD.

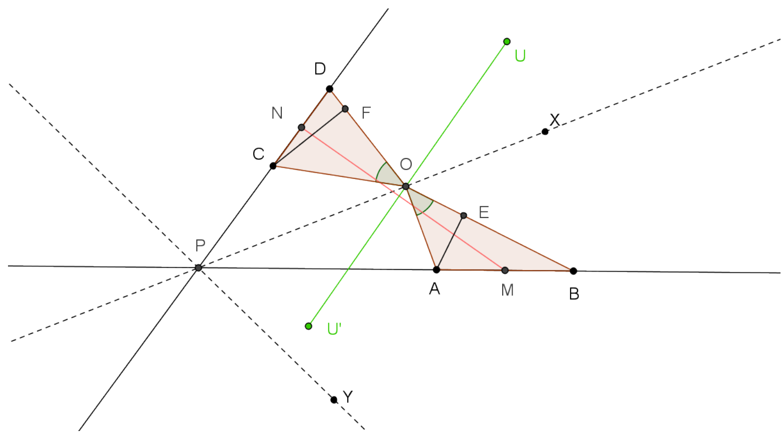

FIGURE t383

Let E be the foot of the perpendicular frm A to OB, and let F be the f of the perpendiculars from C to OD. From **126** we have:

$$AB^2 = OA^2 + OB^2 \pm 2OB \cdot OE, \tag{1}$$

$$CD^2 = OC^2 + OD^2 \pm 2OD \cdot OF. \tag{2}$$

Because $\widehat{AOB} = \widehat{COD}$, the ambiguous signs on the right are either both positive or both negative. And in fact we will show that these rightmost products in the equations above are equal. Indeed, from the equality of the same two angles we know that right triangles OAE, OCF are similar, so that

$$\frac{OA}{OE} = \frac{OC}{OF} \tag{3}$$

.

Now we use condition (a). Because triangles AOB, COD have the same area, and $\widehat{AOB} = \widehat{COD}$, we have (**256**):

$$OA \cdot OB = OC \cdot OD. \tag{4}$$

We now divide equation (4) by equation (3), to get

$$OB \cdot OE = OD \cdot OF. \tag{5}$$

This equation shows that the rightmost products in equations (1) and (2) are equal, and we have already noted that the signs are the same. This observation, together with the fact that we want to look at the difference of squares of certain segments, suggests that we subtract these first two equations. Doing so, we have:

$$AB^2 - CD^2 = OA^2 + OB^2 - OC^2 - OD^2. \tag{6}$$

We now use the result of **128** to write

$$OA^2 + OB^2 = 2OM^2 + \frac{1}{2}AB^2,$$

$$OC^2 + OD^2 = 2ON^2 + \frac{1}{2}CD^2.$$

Substituting these values into (6), we find:

$$OM^2 - ON^2 = \frac{1}{4}\left(AB^2 - CD^2\right). \tag{7}$$

By the result of **128b**, the locus of points satisfying (7) is some line UU', and point O must lie on this line. This argument shows that there is exactly one point O satisfying conditions (a), (b), and (c).

Note. For condition (c) to hold, it is necessary, but not sufficient, that point O be on line UU'. This is because we used condition (a) to define line UU'. It would have been more straightforward simply to find the locus of points satisfying condition (c) independent of the other conditions, but this problem is in general not an elementary one.

We now turn to the solution of the exercise itself. Most of the work has already been done, in proving our lemma. Suppose arcs AV_1B, CV_2D are the loci of points at which segments AB, CD both subtend angle V. Let B', D' be the second points of intersection of these arcs with lines OB, OD respectively (where O is the point located in our lemma). We have $\widehat{AOB} = \widehat{AB'B} \pm \widehat{OAB'}$, with the ambiguous sign depending on whether O lies inside or outside circle AV_1B. Analogously, $\widehat{COD} = \widehat{CD'D} \pm \widehat{OCD'}$. Since $\widehat{AB'B} = \widehat{CD'D} = V$ and $\widehat{AOB} = \widehat{COD}$ by construction, it follows that $\widehat{OAB'} = \widehat{OCD'}$, and therefore triangles OAB', OCD' are similar. Thus we have

$$OA : OB' = OC : OD'. \tag{8}$$

Dividing equation (4) by equation (8), we find that $OB \cdot OB' = OD \cdot OD'$, so O has the same power with respect to circles AV_1B and CV_2D. Thus O is always on the radical axis of these two circles. Since the position of O does not depend on the particular angle V (or equivalently, on the particular circles AV_1B, CV_2D, this proves the required statement.

Notes. We have assumed that angles $\widehat{AOB}$, $\widehat{COD}$ have the same orientation as well as being equal, and that the same is true for angles $\widehat{AV_1B}$, $\widehat{CV_2D}$. If these angles have opposite orientations, the same argument holds, but with the roles of points C and D reversed. Point O will be located at the intersection of line UU' with line PY (rather than PX).

Problem 384. A quadrilateral $ABCD$ (a *kite* or *rhomboid*) is such that the adjacent sides AD, AB are equal, and the other two sides are equal as well. Show that this quadrilateral is circumscribed about two circles. Find the locus of the centers of these circles if the quadrilateral is articulated, one of its sides remaining fixed.

Solution. Suppose the interior angle bisector at B in triangle ABC intersects AC at point O. Then (**115**) $AB : BC = AO : OC$. But from the problem statement, this is also the ratio $AD : DC$, which means that O is on the interior angle bisector at D of triangle ADC as well. It follows that O is equidistant from lines AB, BC, AD, DC, and so is the center of a circle tangent to lines AB, BC, CD, DA.

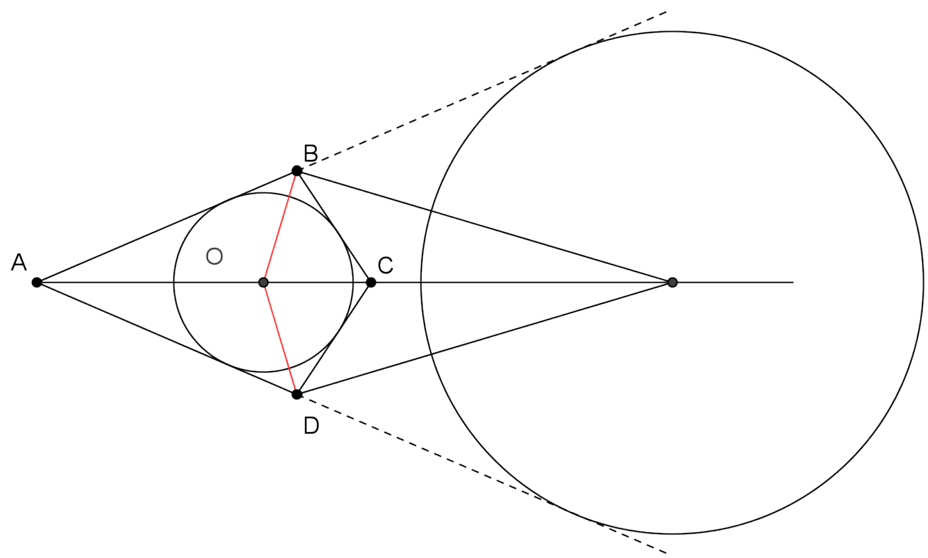

FIGURE t384

Similarly, the bisector of the exterior angle at B in triangle ABC intersects AC (extended) at point O' such that $AO' : O'C = AB : BC = AD : DC$, so O' is on the exterior angle bisector at D of triangle ADC, and so is the center of another circle tangent to lines AB, BC, CD, DA.

Now suppose segment AB remains fixed as the quadrilateral is articulated. Then point C describes a circle centered at B with radius BC. But the ratio $AO : OC = AB : BC$ remains fixed, as does point A, so O describes a circle homothetic to the one described by C, with center of homothecy at A and ratio $AB : (AB+BC)$ (**142**). Similarly, O' describes a circle homothetic to that described by C, with center of homothecy at A and ratio $(AB - BC) : AB$.

Problem 385. More generally, if a quadrilateral $ABCD$ has an inscribed circle, and is articulated while the side AB remains fixed, then it has an inscribed circle in all its positions (Exercise 87). Find the locus of the center O of the inscribed circle.

(To make the situation definite, assume the inscribed circle is inside the polygon, and lay off lengths $AE = AD$ (in the direction of AB) and $BF = BC$ (in the direction of BA), both on side AB. Using Exercise 87, reduce the question to Exercise 257.)

Show that the ratio of the distances from O to two opposite vertices remains constant.

Solution. Quadrilateral $ABCD$ is circumscribed about a circle (*fig.* t385) if and only if $AD + BC = AC + BD$. As the quadrilateral is articulated, the lengths of its sides do not change, so this relationship either continues to hold or never holds. That is, the articulated quadrilateral always has an inscribed circle, if the original quadrilateral does, or never has an inscribed circle, if the original does not.

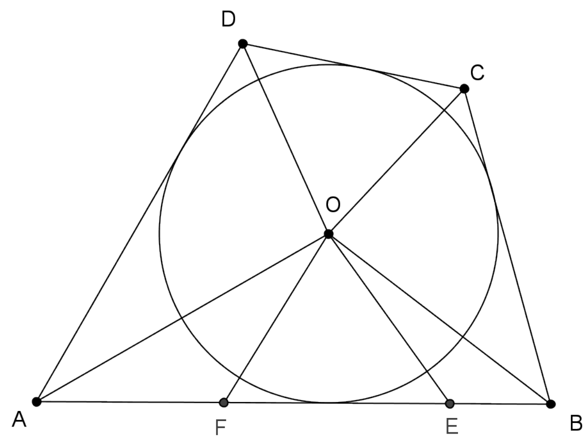

FIGURE t385

Next we find the locus of point O, the incenter, as the quadrilateral is articulated. To find this locus, we find E on line AB so that AE is in the same direction as AB and $AE = AD$. Similarly, we find F on line AB so that BF is in the same direction as BA and $BF = BC$. Note that E and F could both be inside segment AB, or one or both of them may lie outside AB. But in any case, $AE + BF = AD + BC > AB$, so segments AE and BF must overlap.

We will reduce the situation to that of exercise 257 by showing that segments AB, EF subtend supplementary angles at O.

We know that O is the intersection of the angle bisectors of $ABCD$, so triangles ODA, OEA are symmetric with respect to line AO and are therefore congruent. Hence $OD = OE$. Similarly, triangles OCB, OFB are symmetric with respect to line BO, so $OC = OF$.

We next note that $CD = EF$. Indeed, $AB + CD = AD + BC$, so $CD = AD + BC - AB = AE + BF - AB = EF$ (since AE and BF overlap). It follows that triangles OCD, OFE are congruent (**24,** case 3), so $\widehat{COD} = \widehat{EOF}$. But adding up the six angles in triangles AOB, COD we have:

$$\widehat{AOB} + \frac{1}{2}(\widehat{A} + \widehat{B}) + \widehat{COD} + \frac{1}{2}(\widehat{C} + \widehat{D})$$

$$= \widehat{AOB} + +\widehat{COD} + \frac{1}{2}(\widehat{A} + \widehat{B} + \widehat{C} + \widehat{D})$$

$$= \widehat{AOB} + \widehat{COD} + 180^\circ = 360^\circ.$$

So $\widehat{AOB} + \widehat{COD} = 180^\circ$, and

$$(1) \qquad \widehat{AOB} + \widehat{EOF} = 180^\circ$$

as well. Therefore if O is a position of the incenter of $ABCD$, it lies on the locus of points at which AB, EF subtend supplementary angles. By the result of exercise 257, this locus is a circle whose center lies on line AB.

Conversely, any point on this locus is the incenter of some position of quadrilateral $ABCD$. Indeed, let point O satisfy condition (1), where E and F are constructed as above from some original position of articulated quadrilateral $ABCD$. We can construct an articulated version of this quadrilateral such that O is its incenter, by finding triangles AOD, BOC congruent respectively to AOE, BOF. Then $\widehat{AOD} + \widehat{BOC} = \widehat{AOE} + \widehat{BOF} = \widehat{AOB} + \widehat{EOF} = 180^\circ$. Therefore $\widehat{AOB} + \widehat{COD} = 360^\circ - (\widehat{AOD} + \widehat{BOC} = 180^\circ$, so $\widehat{COD} = \widehat{EOF}$.

We know that O is the intersection of the angle bisectors at A and B of quadrilateral $ABCD$. So if $ABCD$ is circumscribed, then O must be its incenter. We now show that $ABCD$ is in fact circumscribed.

Now $OC = OF$ and $OD = OE$ by construction, so triangles COD, FOE are congrent (**24,** case 2), and $CD = EF$. Thus $AD + BC = AE + BF = AB + EF = AB + CD$, so that quadrilateral $ABCD$ is indeed circumscribed, and the locus of its incenter O coincides with the locus of points at which AB, EF subtend supplementary angles.

Finally, we show that the ratio $OA : OC$ is constant. Let S be the circle which is the locus of O. It is not obvious, but was shown in the solution to exercise 257, that when the endpoints of AB are inverted in S, their images are the endpoints of EF. In this case, A inverts onto F and B onto E. The result of exercise 242 tells us that the ratio of the distances of any point on circle S to two inverse points is constant. So, for example, the ratio $OA : OF$ is constant, and since $OF = OC$, the ratio $OA : OC$ is also constant. In the same way we can show that $OB : OD$ is also constant.

Notes. This generalization of exercise 384 is far from obvious, but it can be broken down into smaller sections which are not so hard, once the subgoal of each section is given. It is a bit of a challenge to construct a working model of the articulated quadrilateral $ABCD$ using dynamic geometry software.

Problem 386. Given four fixed points A, B, C, D on a circle, take an arbitrary point P in the plane, and denote by Q the second intersection point of the circles PAB and PCD. Find the locus of Q as P moves on a line or on a circle. Find the locus of points P such that Q coincides with P.

Solution. We observe first that lines AB, PQ, CD are the radical axes respectively of circles PAB, CAB, of circles PAB, PCD, and of circles ACD, PCD. Hence these three lines pass through the radical center O of these three circles (**139**). Since O is the intersection of lines AB, CD, it remains fixed as point P

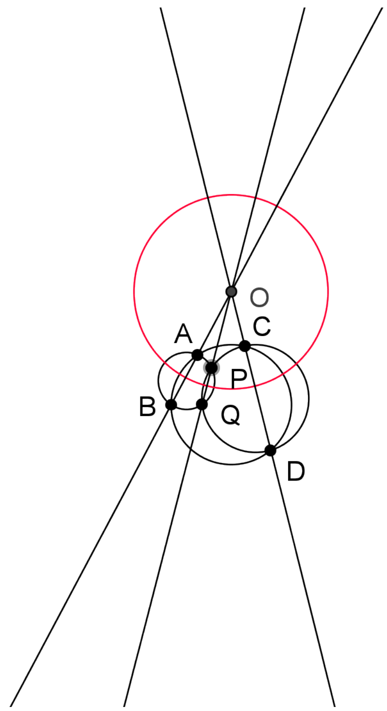

Figure t386

moves about the plane. This means that the product $OP \cdot OQ$ remains fixed: it is equal to the products $OA \cdot OB$ and $OC \cdot OD$, so its value is the power of O with respect to the original circle. This implies that P and Q are images of each other in an inversion about pole O with power equal to the power of O with respect to circle $ABCD$. This observation essentially solves the problem.

For instance, if point P describes a line not passing through O, then point Q describes a circle through O, and conversely (**220, 221**). If P describes a circle not passing through O, then Q describes a circle which is the inversion of that circle. If P describes a line or circle through O, Q likewise describes a line through O. Points P and Q coincide if and only if P lies on the circle of inversion.

Notes. The fact that point O remains fixed as points P varies is surprising in itself. Students might be shown this phenomenon with a dynamic sketch, and asked to explain it.

Problem 387. We join the vertices of a square $ABCD$ with an arbitrary point P in the plane. Let A', B', C', D' be the second points of intersection of these four lines with the circle circumscribed about $ABCD$. Show that $A'B' \times C'D' = A'D' \times B'C'$.

Conversely, let $A'B'C'D'$ be a cyclic quadrilateral such that $A'B' \times C'D' = A'D' \times B'C'$.

Find a point P such that the lines PA', PB', PC', PD' intersect the circumscribed circle in the vertices of a square.

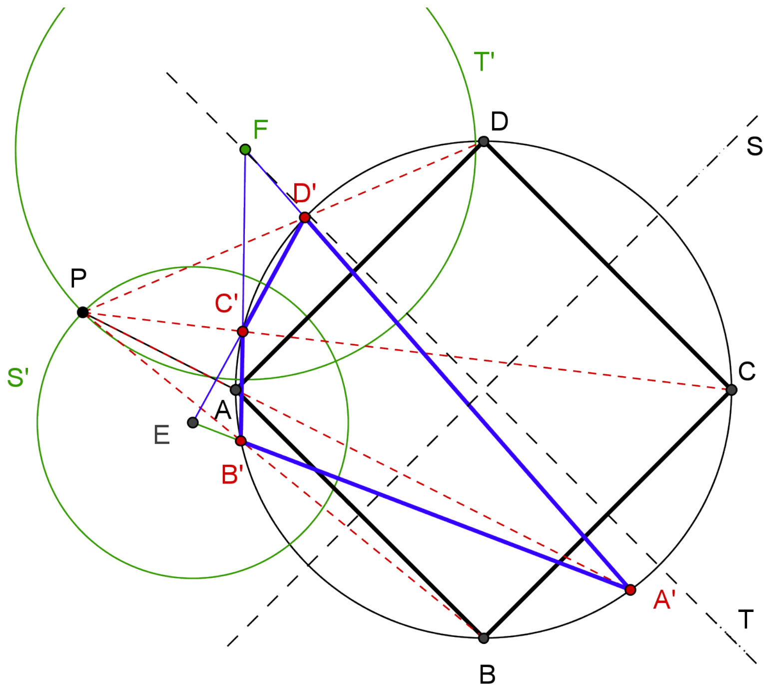

FIGURE t387

Solution. In figure t367, we have (**131**) $PA \cdot PA' = PB \cdot PB' = PC \cdot PC' = PD \cdot PD' = p$, where p is the power of point P with respect to circle $ABCD$. That is, $A',\ B,',C',D'$ are the images of $A,\ B,\ C,\ D$ under an inversion about pole P with power p. By **218**, then, we have:

$$A'B' = \frac{p \cdot BA}{PA \cdot PB}, \tag{1}$$

with analogous expressions for $B'C',\ C'D',\ D'A'$. By direct computation, taking into account that $AB = BC = CD = DA$, we have $A'B' \cdot C'D' = A'D' \cdot B'C' = \frac{p^2 c\dot{A}B^2}{PA \cdot PB \cdot PC \cdot PD}$.

We next consider the converse statement. Suppose cyclic quadrilateral $A'B'C'D'$ is such that $A'B' \cdot C'D' = A'D' \cdot B'C'$. (We assume that $A'B'C'D'$ is convex; that is, not self-intersecting.) Let us suppose that we know where point P (as described in the problem statement) lies. We consider the inversion around P as pole, with a power p equal to the power of P with respect to circle $A'B'C'D'$. That is, we assume that this inversion takes $A',\ B',\ C',\ D'$ into $A,\ B,\ C,\ D$ respectively, and these last points lie on circle $A'B'C'D'$. Note that this circle is its own image under this inversion.

Let us consider line S, the perpendicular bisector of AB. It is also the perpendicular bisector of CD (since we are assuming that $ABCD$ is a square), and we invert it, around pole P with power p. Line S is perpendicular to line AB, but also to any circle through A and B. Indeed, any such circle has its center on line S, so a tangent to S at their point of intersection will be perpendicular to the radius at that point; that is, perpendicular to S. The image of S will thus be a circle S' which is orthogonal to any circle through A' and B' (**219**), and in particular will be orthogonal to line $A'B'$. Likewise, S' will be orthogonal to every circle through C' and D', in particular to line $C'D'$. It follows that lines $A'B'$, $C'D'$ pass through the center E' of S'; that is, E' is the intersection of lines $A'B'$, $C'D'$. Finally, line S is certainly orthogonal to circle $ABCD$, so the images of these two objects are also orthogonal That is, circle S' is orthogonal to circle $A'B'C'D'$.

But does such a circle S' exist? Well, we have chosen quadrilateral $A'B'C'D'$ to be convex, so point E', the intersection of $A'B'$ and $C'D'$, lies outside the quadrilateral. Therefore S' is just the circle centered at E' with radius equal to the length of the tangent from E' to circle $A'B'C'D'$. We can now construct circle S'.

Likewise, line T, the perpendicular bisector of AD and BC, passes through point O, inverts into a circle T' whose center is at F', the intersection of lines $A'D'$, $B'C'$, and is orthogonal to circle $A'B'C'D'$. So we can construct circle T'.

Now under the inversion we seek, S', T' invert into lines S and T, so the pole that effects this inversion can only be one of their intersections P or Q of these two circles. These two circles must intersect, because one of them intersects arcs $\overparen{A'B'}$, $\overparen{C'D'}$, while the other intersects arcs $\overparen{A'D'}$, $\overparen{B'C'}$, of circle $A'B'C'D'$.

Finally, we find a pole P of inversion for which PA', PB, PC' PD' form a square. Let us look at an inversion with pole P and power equal to the power of P with respect to circle $A'B'C'D'$. We will show that in fact under this inversion, the image $ABCD$ of quadrilateral $A'B'C'D'$ is a square. Indeed, we know that any circle through A' and B' is orthogonal to S'. We look in particular at circle $PA'B'$, which inverts into line AB. Since circle $PA'B'$ is orthogonal to S', line AB is perpendicular to line S, the image of S'. For the same reason, line CD must be perpendicular to line S. Similar reasoning starting with circle T' shows that lines AD, BC must be perpendicular to line T. This reasoning shows that $ABCD$ is a parallelogram. And since A', B', C', D' line on a circle, their images A, B, C, D must lie on a circle (the image of the circle through $A'B'C'D'$). Thus parallelogram $ABCD$ must be a rectangle. Finally, the algebraic reasoning associated with equation (1) leads us from the relation $A'B' \cdot C'D' = A'D' \cdot B'C'$ to the relation $AB \cdot CD = AD \cdot BC$, and if we apply this formula to a rectangle (whose opposite sides must be equal), we quickly find that the rectangle is in fact a square.

The same reasoning applies to an inversion around point Q. These two points, constructed as indicated, give solutions to the problem: lines connecting them to A', B', C', D' intersect circle $A'B'C'D'$ again in the four vertices of a square.

Note. We have indicated in passing how to construct a circle with a given center and perpendicular to a given circle. Students can be asked to do this construction as an exercise, before undertaking this problem.

(This is a particular case of Exercise 270b, 5^o. However, the problem here admits of two solutions, while there is only one in the general case. What is the reason for this difference?)

Solution. The problem has two solutions because quadrilateral $A'B'C'D'$ is assumed to be cyclic. See, for example the solution to exercise 270, $5°$, note 2.

Problem 388. More generally, find an inversion which transforms the vertices A', B', C', D' of a cyclic quadrilateral into the vertices of a rectangle.

Show that the poles are the limit points (Exercise 152) of the inscribed circle and of the third diagonal of quadrilateral $A'B'C'D'$.

Solution. Suppose (*fig.* t388) that an inversion with pole P transforms cyclic quadrilateral $A'B'C'D'$ into rectangle $ABCD$. We first show that we can find an inversion such that $ABCD$ is in fact inscribed in the same circle as $A'B'C'D'$.

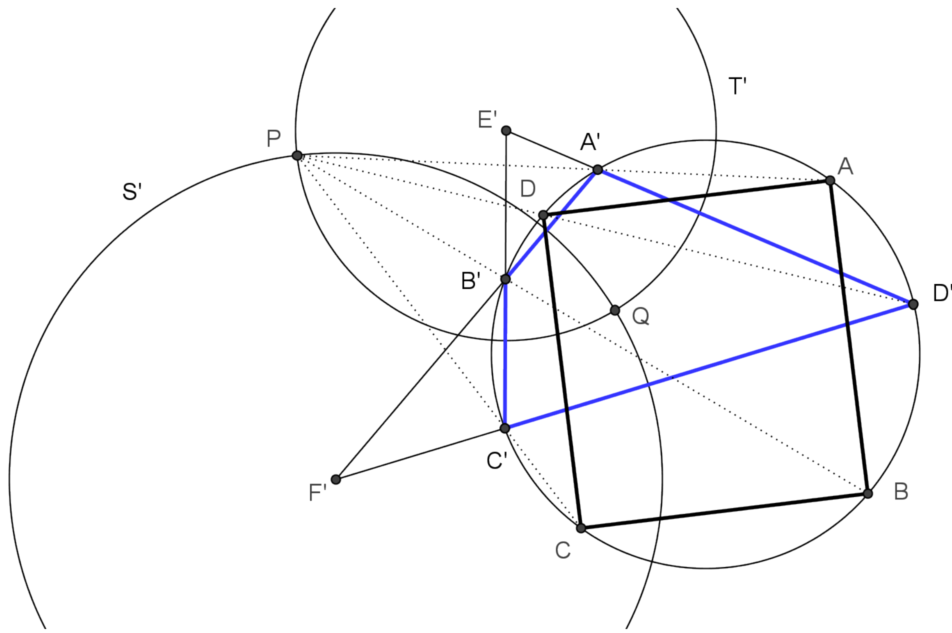

FIGURE t388

We follow the reasoning of exercise 387. As before, it is not hard to see that the line S that joins the midpoints of sides AB, CD must be orthogonal to any circle through A and B, and also to any circle through C and D. Therefore, under the inversion we seek, the image S' of S is a circle orthogonal to lines $A'B'$, $C'D'$, and also to circle $A'B'C'D'$. The center of S' must be the intersection E' of lines $A'B'$, $C'D'$, and its radius must be the length of the tangent from E' to circle $A'B'C'D'$.

Analogously, the line T joining the midpoints of sides AD, BC inverts into a circle T' whose center F' is the intersection of lines $A'D'$, $B'C'$, and whose radius is the length of the tangent from F' to circle $A'B'C'D'$. So we have enough information to construct circles S', T'.

Now the inversion we seek will take circles S', T' into lines S, T, so its pole must be one of the two intersections P, Q of the circles. And if we want $ABCD$ to be inscribed in circle $A'B'C'D'$, we need only take the power of the inversion to be the power of P or Q with respect to circle $A'B'C'D'$. But in fact, by **215** any inversion with P or Q as its pole will transform $A'B'C'D'$ into a rectangle, usually with a different circumcircle.

We now prove the statement in the problem concerning limit points. The third diagonal of $A'B'C'D'$ (considered as a complete quadrilateral) is simply $E'F'$. By construction (the lengths of their radii), circles S', T' are orthogonal to circle $A'B'C'D'$. And they are certainly orthogonal to line $E'F'$ (their common centerline). It follows (exercise 152) that P and Q are the limit points of line $E'F'$ and circle $A'B'C'D'$.

Problem 389. Still more generally, find an inversion which transforms four given points into the vertices of a parallelogram.

Solution. Suppose there is an inversion taking four given points A', B', C', D' into a parallelogram $ABCD$. If A', B', C', D' all lie on the same circle, then so do A, B, C, D, and we are led to the situation in exercise 388. So let us assume that A', B', C', D' do not lie on the same circle.

Since triangles ABC, ADC must be congruent, so must circles ABC, ADC. Line AC is the extension of their common chord, so the two circles are symmetric in line AC. That is, AC forms equal angles with both these circles.

Let us see what this implies for the original diagram, before inversion. Circle ABC is the inversion of circle $A'B'C'$. Circle ADC is the inversion of circle $A'D'C'$. Line AC is the image of some circle S' through the pole of inversion. Circle S' must form equal angles with circles $A'B'C'$, $A'D'C'$; that is, it must bisect the angle between arcs $\widehat{A'B'C'}$ and $\widehat{A'D'C'}$.

Likewise, the pole of inversion must lie on a circle T' bisecting the angle between arcs $\widehat{B'A'D'}$ and $\widehat{B'C'D'}$. Circles S' and T' intersect twice, and either point of intersection can be taken as the pole of the required inversion. As in exercise 388, the power of the inversion can be arbitrary.

Notes. In this argument, we chose arcs $\widehat{A'B'C'}, \widehat{A'D'C'}$ because we want A' and D' to lie on opposite sides of circle S', just as their images A, D lie on opposite sides of line AC.

Students can be given the auxiliary problem of constructing a circle forming equal angles with two given circles. See **227**.

Problem 390. Given two circles and a point A, find an inversion in which the point corresponding to A is a center of similarity of the transformed circles.

Lemma. A line intersecting two circles at equal angles must pass through one of their centers of similarity, and conversely.

Proof of lemma. Radii of the two circles, drawn to their points of intersection with the given line, are parallel in pairs. Thus the intersection of the given line with the common centerline of the two circles is a center of similarity of the two circles.

Conversely, if a line passes through a center of similarity of two circles, pairs of radii to the points of intersection are homothetic, thus parallel. It follows that the line makes equal angles with the two circles.

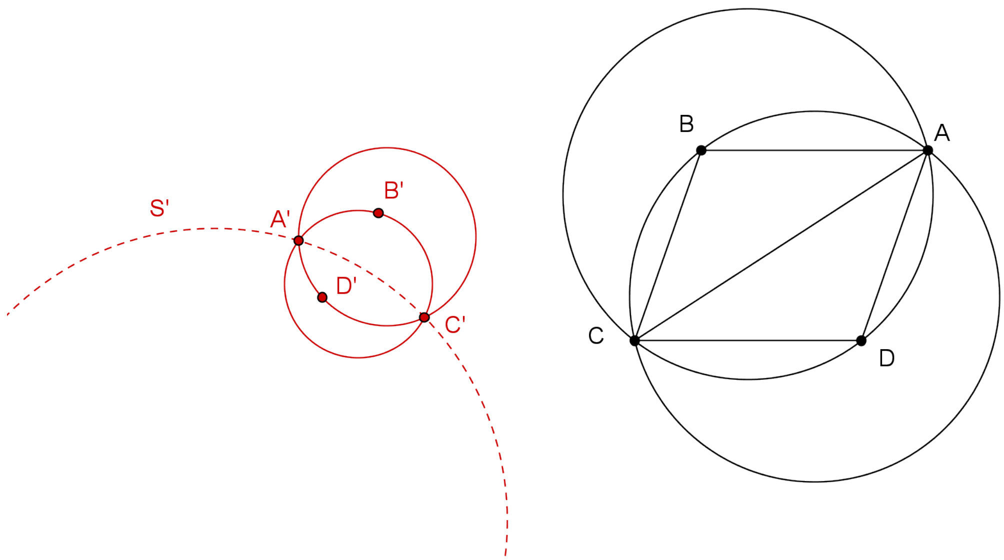

FIGURE t389

Note. This lemma tells us that we can characterize the centers of similarity of two circles as the point common to all the lines which intersect them at equal angles.

Students can fill in the details of the proof of this elementary lemma. See **227**.

Solution. Suppose the required inversion is I, and it takes point A onto point A'. Then the set of circles through A which intersect the given circles C_1, C_2 at equal angles are transformed into the set of lines passing through A'.

There are two inversions J and K which take C_1 onto C_2 (**227**). A circle intersecting C_1 and C_2 at equal angles will be its own image under either J or K. It follows that any circle through A which forms equal angles with C_1 and C_2 must pass either through point P_J, the inverse point to A under J, or through P_K, the inverse point to A under K. In order that these circles invert into lines, we must take either P_J or P_K as the pole. The power of inversion can be chosen arbitrarily.

Problem 391. A variable point M on a circle is joined to two fixed points A, B. The two lines intersect the circle again at P, Q. Denote by R the second intersection of the circle with the parallel to AB passing through P. Show that line QR intersects AB at a fixed point.

Use this result to find a method of inscribing a triangle in a given circle with two sides passing through given points, while the third is parallel to a given direction; or such that the three sides pass through given points. (These two questions are easily reduced to each other, and to 115).

Solve the analogous problem for a polygon with an arbitrary number of sides. (Another method is proposed in Exercise 253b.)

Note. We break this problem statement down into seven parts:

1°. The intersection of QR and AB is fixed.

2°. Inscribe a triangle in a given circle, with two sides passing through given points, while the third is parallel to a given direction.

3°. Inscribe a triangle in a given circle, whose three sides pass through three given points.

4°. Inscribe a polygon in a given circle with an *even* number of sides, such that one side passes through a given point while the others are parallel respectively to a set of given lines.

5°. Inscribe a polygon in a given circle with an *odd* number of sides, such that one side passes through a given point while the others are parallel respectively to a set of given lines.

6°. Inscribe a polygon in a given circle, such that a number of *consecutive* sides pass through a set of given points while the others are parallel respectively to a set of given lines.

7°. Inscribe a polygon in a give circle, such that a number of sides, located arbitrarily around the figure, pass through a a given set of points while the others are parallel respectively to a set of given lines.

Solution 1°**.** In figure t391a, point S is the intersection of lines QR and AB. We have $\widehat{RQM} = \frac{1}{2}\ \widehat{RM} = \widehat{RPM} = \widehat{BAM}$, and certainly $\widehat{MBA} = \widehat{SBQ}$, so triangles MBA, QBS are similar, so that $BS = \frac{BM \cdot BQ}{AB}$. The product $BM \cdot BQ$ (the numerator of this fraction) is just the power of B with respect to the circle, and so does not change as point M varies along the circle. And the denominator of the fraction is certainly constant, so BS is constant, and point S does not move as M varies.

Note. We can phrase this result slightly differently. For any two points A, B on the circle, there is an associated fixed point S with the property described in the problem statement.

Solution 2°**.** We will adapt the notation of figure t391a to address this construction. We show how to construct a triangle MPQ with sides passing through fixed points A, B and a third side parallel to some line KL.

We find the fixed point S associated (as in 1°) with points A and B. We then use the result of exercise 115 to construct auxiliary triangle PQR, inscribed in the given circle, with one side parallel to line AB and another to KL, and the third side (QR) passing through S. Indeed, by construction (of point S), line RS must intersect the circle again at a point collinear with M and B, which thus must be Q. Then triangle MPQ satisfies the required conditions.

Having constructed PQR, we draw line PA, and label its second intersection with the circle as M. Then we draw line MB. It is not hard to see that the second intersection of MB with the given circle must be Q. Indeed, the intersection of MB and the given circle must be collinear with RS (by $1°$), so it must be point Q.

Solution $3°$**.** We again use the notation of figure t391a, but in a different way. Let the three given points be A, B and C. Again, we determine point S, dependent on A and B, as indicated in the result to our main problem. Construction $2°$ above allows us to find a triangle PQR, two of whose sides pass through points C and S, and whose third side PR is parallel to line AB.

Having constructed triangle PQR, we determine M as the second intersection of line PA with the circle. As in the previous construction, line MB must intersect the circle again at Q, and triangle MPQ satisfies the conditions of the problem.

$4°$**.** We solve the problem for a hexagon. The generalization to any even number of sides is immediate.

Suppose the required figure is $PQRSTU$, that side PQ passes through a given point A, while the other sides are parallel respectively to lines Q_1R_1, R_2S_1, S_2T_1, T_2U_1, SU_2P_1.

We choose an arbitrary point Q' on the circle, and construct a polygon starting with Q' as one vertex by drawing lines parallel in turn to the given lines. We get polygon $P'Q'R'S'T'U'$ (*fig.* t391b), whose sides are parallel respectively to the required polygon. Because chords QR, $Q'R'$ are parallel, arcs $\overset{\frown}{QQ'}$, $\overset{\frown}{RR'}$ are equal, but oppositely oriented. Similarly, arcs $\overset{\frown}{R'R}$, $\overset{\frown}{SS'}$, $\overset{\frown}{T'T}$, $\overset{\frown}{UU'}$, $\overset{\frown}{P'P}$ are all equal, and consecutive arcs (in this list) are oppositely oriented. Since the number of sides of the polygon is even (here that number is 6), it follows that arcs $\overset{\frown}{P'P}$, $\overset{\frown}{QQ'}$ are equal and oppositely oriented. It follows that chords $P'Q'$, PQ are parallel. (Note that we began with only five given lines; chords PQ, $P'Q'$ are not parallel *a priori*).

We have constructed a polygon inscribed in the given circle, with sides parallel to the (five) given lines. It remains to arrange for the last side to pass through the given point. But this is easy. If the given point is A, we draw a parallel to PQ through A. Its intersection points P and Q with the given circle are one side of the required polygon, and the others are found by drawing parallels as before.

Notes. This construction, as well as the next, does not depend on the very first result in this exercise. We will not use that result until we come to statement $6°$. Thus students can be given this problem independent of the others in this sequence. Or, they can be asked first to solve the simpler problem of inscribing in a circle any hexagon (or polygon wtih evenly many sides) with all but one of its sides parallel to a set of lines. They will find, in the process, that this condition determines the direction of the sixth side of the hexagon (but not its length).

Solution $5°$**.** The argument is only slightly different from the proof of $4°$. We solve the problem for a pentagon. The generalization to an arbitrary odd number of sides is immediate.

Suppose the required polygon $PQRST$ (*fig.* t391c), is such that side PQ passes through a given point A, and the other sides are parallel respectively to lines $Q_1R_1,\ R_2S_1,\ S_2T_1,\ T_2P_1$.

We choose an arbitrary point Q' on the circle, and construct a polygon starting with Q' as one vertex by drawing lines parallel in turn to the given lines. We get polygon $P'Q'R'S'T'$ (*fig.* t391c), whose sides are parallel respectively to the required polygon. As before, arcs $\widehat{QQ'},\ \widehat{SS'},\ \widehat{T'T},\ \widehat{P'P}$ are all equal, but this time, because there are oddly many sides in the polygon, arcs $\widehat{P'P},\ \widehat{QQ'}$ are oriented in the same direction. It follows (by adding arc $\widehat{P'Q}$ to both) that arcs $\widehat{P'Q'},\ \widehat{PQ}$ are equal, so chords $P'Q',\ PQ$ are also equal. This construction, starting with an arbitrary point, gives us the length of PQ. Thus we can start our construction of the polygon by drawing a chord through the given point A and equal in length to PQ.

Notes. We have shown, within the argument, that the length of side $P'Q'$ of a polygon constructed as in the solution, does not depend on the choice of point Q'.

Students can complete the construction, by recalling how to draw a chord of a given length through a given point. They can recall that this construction is possible whenever the given length is between the diameter of the circle and the minimal length of a chord through A. The length of PQ is determined by the directions of the given lines, and students can think about when the construction indicated in 5° is possible.

In 4°, we saw that for even n, if we know the directions of $n-1$ sides of an n-gon, then the direction of the last side is determined, but not its length. Now we see that for odd n, if we know the directions of $n-1$ sides of a n-gon, then the length of the last side is determined, but not its direction. For this reason, the construction is not always possible when n is odd.

Solution 6°**.** If only one of the sides of the polygon is required to go through a given point, the problem is solved in 3° and 4°. We solve the problem for a hexagon. The generalization to any number of sides offers no new difficulties.

Suppose the required hexagon is $PMQUVW$ (*fig.* t391d), in which sides PM and MQ pass through given points A and B respectively, and choose these sides so that side WP is required to be parallel to a given line; that is, so that proceeding around the figure (in a clockwise direction, for figure 391d) , PM and MQ are the first two sides we encounter which are required to go through given points. We will show that if we can construct another hexagon $PRQUVW$ in which one fewer side is required to go through a fixed point, then we can also construct $PMQUVW$.

Indeed, we have essentially done this, in 2°. Figure t391d is labeled similarly to figure 391a, and in that figure, we know that if can construct triangle PQR, (with two sides in given direction and a third passing through a given point), we can also construct triangle PMQ. This shows us how to construct hexagon $PMQUVW$, if we have already constructed hexagon $PRQUVW$. That is, we have reduced by one the number of sides required to pass through a given point.

Because we have assumed that the sides passing through points are consecutive around the required figure, we can continue this process, finally arriving at the construction of 4° or 5°.

Solution 7°. We use auxiliary polygons to reduce this problem to the situation in 6°. That is, we show that the construction of our polygon can be made to depend on that of another polygon, in which the order of the sides satisfying two different sorts of conditions is reversed. Applying this result several times, we can arrange that all the sides of the auxiliary polygon which are required to pass through a given point are consecutive, which is the situation in 6°. We can thus construct the sequence of auxiliary polygons, and arrive a the required figure.

Suppose, for example, we are required to construct hexagon $PQRSTU$ (*fig.* t391e), in which side PQ must be parallel to a given line KL, and side QR must pass through a given point A. We will show that this construction depends on the construction of an auxiliary hexagon PQ_1RSTU, in which PQ_1 passes through a given point, and Q_1R is parallel to a given line.

We draw chord RQ_1 parallel to KL through point R, and let A_1 be the intersection of line PQ_1 with the line parallel to KL through A. Then $PQRQ_1$ is an isosceles trapezoid, and therefore so is ARQ_1A_1. Thus points A, A_1 are the same distance from the center O of the given circle, and we can construct point A_1, without knowing hexagon $PQRSTU$, as the intersection of a circle of radius OA with the parallel through A to KL. Now if we know how to construct polygon PQ_1RSTU, we can construct polygon $PQRSTU$.

But in PQ_1RSTU, side PQ_1 must go through a given point A_1, and side Q_1R must be parallel to the given line KL. That is, we have reversed the order of the sides satisfying different requirements. Following the plan given above, we can then construct polygon $PQRSTU$.

Note. In figure t391e, polygon PQ_1RSTU is not a 'proper' polygon, in the sense of **21**. Two of its sides intersect at a point which is not a vertex. We must in general allow such figures for our auxiliary polygons. If we want to say that the required polygon itself can always be constructed, we must likewise allow such re-entrant figures as solutions.

Problem 392. About a given circle, circumscribe a triangle whose vertices belong to given lines.

Solution. We use the method of poles and polars (see **206**). Since the polar of a tangent to a circle is its point of contact, (**204**), the polar of the circumscribed triangle is the inscribed triangle formed by the three points of intersection of its sides with the circle. Since the vertices of the original triangle lie on certain lines a, b, c, the sides of the new inscribed triangle must pass through the points A, B, C which are the polars of these lines.

Thus we have the following construction. We take the polars of the three given lines with respect to the circle, then use the result of exercise 391 to draw a triangle inscribed in the circle, whose sides pass through these three points. We then take the polar figure to this inscribed triangle to get the required circumscribed triangle.

Problem 393. Given two points A, B on a line, we draw two variable circles tangent to the line at these points, and also tangent to each other. These two circles have a second common (external) tangent $A'B'$. Show that as the two tangent circles vary, the circles on diameter $A'B'$ remain tangent to yet another (fixed) circle. Find the locus of the midpoint of $A'B'$.

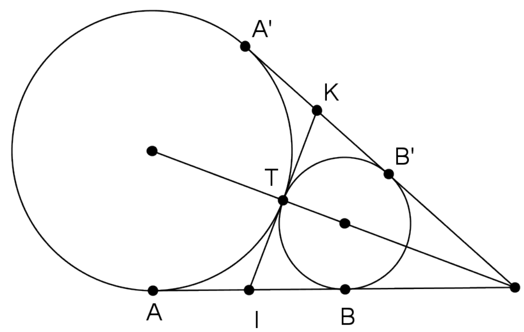

FIGURE t393

Solution. Suppose (*fig.* t393) T is the point of tangency of the two circles, and that IK is their common internal tangent, with I on line AB and K on line $A'B'$. Then we know (**92**) that $IT = IA = IB = \frac{1}{2}AB$. It follows that point I is the midpoint of AB, and is fixed (for any two such circles). We also know that $TK = KA' = KB'$. Since the common centerline of the two circles is a line of symmetry for the whole figure, we know that $IT = TK$, so $IK = 2IT = AB$, a fixed distance. Hence the locus of point K (the locus of midpoints of $A'B'$) is a circle centered at I with radius AB.

Now any circle on diameter $A'B'$ (for any two positions of the intial tangent circles) must have a radius equal to $A'K = B'K = TK = TI = AI = \frac{1}{2}AB$, and its center moves along the circle with center I described in the previous paragraph. Hence any such circle is tangent to a fixed circle centered at I, of radius $\frac{3}{2}AB$. And in fact, it is also tangent to another fixed circle, also centered at I, with radius $\frac{1}{2}AB$.

Problem 394. Two variable circles C, C_1 are tangent at a point M, and tangent to a given circle at given points A, B.

1^o. Find the locus of M;

Solution $1°$**.** In figure t394a, point P is the intersection of the tangents to the given circle O. We use the result of **139**. Line PA is the radical axis of circles O, C, and line PB is the radical axis of circles O, C_1, so P is the radical center of circles O, C, C_1, and lies on the radical axis of C, C_1. But this radical axis is simply the common tangent PM of these two circles. From **92**, it follows that $PA = PB = PM$. Since $PA = PB$ is constant, so is PM, and point M lies on a circle centered at P with radius PA.

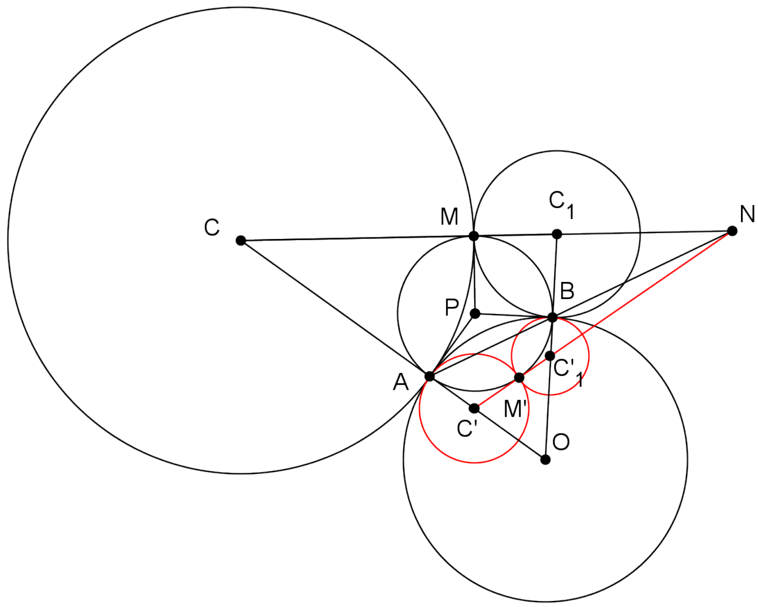

Figure t394a

Note. It is not hard to see that any point on this circle can serve as a position of point M. For some positions, circles C, $C_!$ are tangent *internally* to circle O, and when P coincides with A or B, one of the circles degenerates to a point.

2^o. Find the locus of the second center of similarity of $C,\ C_1$;

Solution. It follows from the discussion of **227** that points A and B are antihomologous points in circles C, C_1. Thus the second center N of similarity lies on line AB.

But the locus of N is not the whole line: N must lie outside circle P constructed in part 1°. To prove this, we note first that the common centerline of $C,\ C_1$ tangent to circle P. Indeed, M is the foot of the perpendicular to this common centerline, and is also the point of contact of the common tangent from P to the two circles. Now N lies on the common centerline of $C,\ C_1$ (**143**), which is tangent to circle P and this centerline is tangent to circle P.

It follows that N, being on line CC_1, must lie outside circle P and therefore must also lie outside circle O. So the locus of N is that part of line AB lying outside circle O. (The proof of this statement is actually completed in 3°)

Note. Within this proof, we have shown that if three circles are tangent externally, then their common centerlines are tangent to the circle centered at their radical center whose radius is the length of the tangent to any of the circles from the radical center. Students can be asked to prove this result independently of the rest of the problem.

3^o. To each point N of the preceding locus there correspond two pairs of circles $C, C_1;\ C', C_1'$ satisfying the given conditions, and therefore two points of tangency $M,\ M'$.

Solution. We can construct circle P independent of the choice of N. Then, for any position of N, we can draw tangents NM, NM' to this circle (*fig.* t394a). One pair of centers is given by the intersections of OA, OB with tangent NM, and the other pair of centers is the corresponding intersections with NM'.

Note. This statement provides the converse to 2° by showing that every point N on the locus claimed does in fact serve as the center of similarity of a pair of circles C, C_1; in fact, to two pairs of such circles.

Find the locus of the center of the circle circumscribing NMM', the locus of the circle inscribed in this triangle, and the locus of the intersection of its altitudes. Each common point of pairs of these loci belongs to the third.

Solution. We break this statement into three parts.

4°**.** To find the locus of the circumcenter ω of NMM', we note that angles $\widehat{NMP}$, $\widehat{NM'P}$ are right angles, so the circle on diameter NP passes through points M, M'; that is, this circle is the circumcircle of NMM', whose center ω is therefore the midpoint of segment NP. Since P is fixed, we can describe the locus of ω as the image of the locus of N under a homothecy centered at P with factor $\frac{1}{2}$.

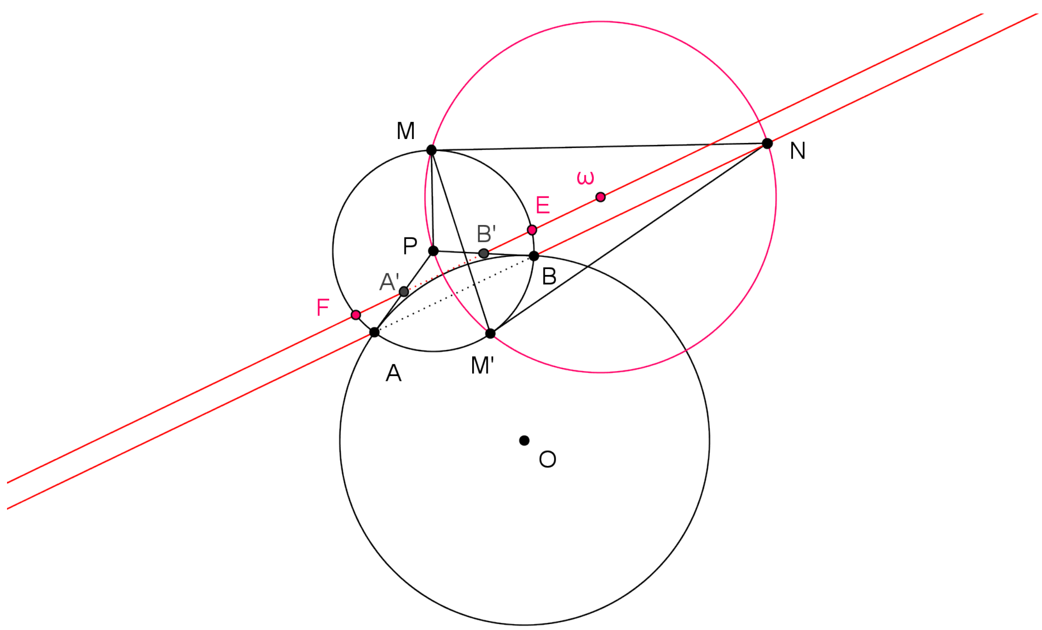

Figure t394b

Since the locus of N consists of the extensions outside circle P of segment AB, the locus of ω is the set of points on line $A'B'$, where A' and B' are the midpoints respectively of PA, PB, but outside of segment $A'B'$(*fig.* t394b).

5°**.** To find the locus of the incenter I of triangle NMM', we note that it is located on the bisector NP of angle $\widehat{MNM'}$, and also on the bisector of angle $\widehat{NMM'}$.

We now show that point I in fact lies on circle P. Indeed, we have $\widehat{PMI} = 90° - \widehat{NMI} = 90° - \widehat{IMM'} = \widehat{PIM}$, so triangle PIM is isoscele, with $PM = PI$. Thus I is on circle P. But (by symmetry in line PN), the extension of PI must pass through N. That is, point I lies on the intersections of PN with circle P. These points are those on two arcs lying between lines AB and the parallel to AB through P (*fig.* t394c).

6°**.** We will find the locus of the intersection of the altitudes (the orthocenter) of triangle NMM' in a rather indirect fashion. We will first find the locus of the midpoint K of base MM' of the isosceles triangle. Then we relate positions of point K to positions of point H using the result of exercise 70.

We start with some results from the theory of poles and polars. In figure 394d, we note that line MM' is the polar of point N with respect to circle P. Since line AB passes through point N, the pole of line AB lies on the polar of N (**205**). But the polar of AB with respect to circle P is just point O (as we proved in the note to 2°, OA and OB are the tangents to circle P at the endpoints of chord AB). So line MM' passes through point O.

Let K be the foot of the altitude from N in triangle NMM', and consider the circle Σ on diameter OP. Since angle $\widehat{PKO}$ is a right angle, point K lies on circle Σ. But OB is tangent to circle P at B, so angle $\widehat{PBO}$ is also a right angle, and B is also on Σ. We can show in the same way that A is on Σ. Note that points O, P do not vary with point N, so circle Σ also does not vary. And in fact it is easy to see that as N varies along line AB (but outside of circle P) , the locus of point K is arc $\widehat{APB}$ of circle Σ.

Now point P is equidistant from lines NM, NM', and so is on the bisector of angle $\widehat{N}$ of triangle NMM'. Since this triangle is isosceles, this angle bisector is also an altitude. So point P is the intersection of an altitude of triangle NMM' with its circumcircle. It follows from exercise 70 that the orthocenter H of the triangle is symmetric to P in line MM', or equivalently, in point K.

That is, $PH = 2PK$, and point H is the homothetic image of point K, with (fixed) point P as center and a factor of 2. Thus its locus is the homothetic image of an arc of circle Σ', homothetic to Σ with factor 2.

7°**.** We need to show that the loci of 4°, 5° and 6° are concurrent. Let us consider the first two loci separately. The locus of 4° is that part of line $A'B'$ lying outside circle P. The locus in 5° consists of two arcs of circle P. So these two loci cannot intersect in more than two points labeled E and F in figure t394b.

For positions of N where these loci coincide, say E, the incenter and circumcenter of triangle NMM' must also coincide with E, which means that the triangle is equilateral, and not just isosceles. But the orthocenter of an equilateral triangle also coincides with the incenter and circumcenter, and hence with point E. This means that point E is on circle Σ', the locus of H. Likewise, point F is on circle Σ'.

Problem 395. Two circles C, C' meet at A, and a common tangent meets them at P, P'. If we circumscribe a circle about triangle APP', show that the angle subtended by PP' at the center of this circumscribed circle is equal to the angle between circles C, C', and that the radius of this circle is the mean proportion between the radii of circles C, C' (which implies the result of Exercise 262, 3^o). Show that the ratio $\frac{AP}{AP'}$ is the square root of the ratio of these two radii.

Solution. The exercise asks us to prove three statements.

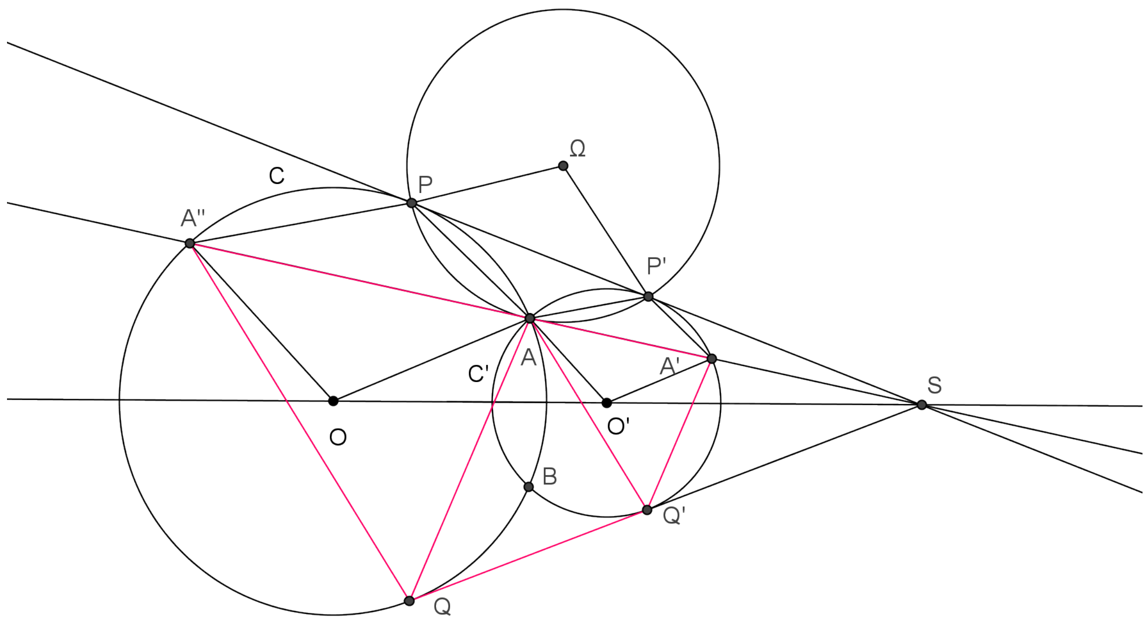

FIGURE t395

$1°$**.** We first prove the result about the angle subtended by PP' at the circumcenter of triangle APP'.

Let S be the external center of similitude for circles C, C', and let A'', A' be the second points of intersection of line SA with circles C, C' respectively (*fig.* t395). Then clearly triangles APA'', $A'P'A$ are homothetic with center S of homothecy. (In particular, they are similar.) And (**73, 74**) angles $\widehat{PA''A}$, $\widehat{APP'}$ are equal, as each is equal to half arc $\widehat{AP}$ in circle C. Likewise, $\widehat{P'A'A} = \widehat{AP'P} = \frac{1}{2}\,\widehat{AP'}$ in circle C'. It follows that triangles $A''PA$, PAP', $AP'A'$ are similar.

Now we note that the three circumcenters O, O', Ω of these triangles, considered as parts of the triangles, are corresponding points. Therefore quadrilaterals $A''PAO$, $PAP'\Omega$, $AP'A'O'$ are also similar.

From these three quadrilaterals we have: $\widehat{OAP} = \widehat{\Omega P'A}$ and $\widehat{O'AP'} = \widehat{\Omega PA}$. Therefore $\widehat{P\Omega P'} = 360° - \widehat{\Omega PA} - \widehat{PAP'} - \widehat{\Omega P'A} = 360° - \widehat{O'AP'} - \widehat{PAP'} - \widehat{OAP} = \widehat{OAO'}$. That is, the angle subtended by PP' at the center Ω of the circle circumscribing triangle APP' is equal to the angle between the radii of circles C, C' drawn to their point of intersection. It is not hard to see that this angle is equal to the angle between the tangents to these circles at their point of their intersection, which is the angle between the two circles themselves.

Note. Students may have trouble accepting the argument that quadrilaterals $A''PAO$, $PAP'\Omega$, $AP'A'O'$ are similar. They can avoid this difficulty, but lengthen the argument, by seeing that triangles $OA''A$, $\Omega PP'$, $O'AA'$ are also similar, then use sides of these triangles and combinations of their angles with the angles of similar triangles $A''PA$, PAP', $AP'A'$. This longer argument offers no new difficulties.

2°**.** Next we prove the result about the circumradius of triangle APP'.

Let $r,\ r', \omega$ be the radii of circles $C,\ C', \Omega$ respectively. From the similar quadrilaterals pointed out in 1° we have

$$r : \omega = AP : AP',\ \omega : r' = AP : AP'.$$

It follows from these two proportions above that $r : \omega = \omega : r'$, so ω is the mean proportion between r and r'.

4°**.** Finally, we get the result of exercises 262, 3°. Let B be the second intersection of circles $C,\ C'$, and let $Q,\ Q'$ be the points of contact of the second common tangent to those circles. Then it is clear from symmetry in line OO' that the circumradii of APP', BQQ' are equal.

We can repeat the argument of 1° to show that triangles $A''QA$, QAQ', $AQ'A'$ are similar, so that the quadrilaterals formed by those three triangles and their circumcenters are also similar, and therefore the circumradius of AQQ' is likewise the mean proportion between r and r'. By symmetry in line OO', the same is true of the circumradius of triangle BPP'. Thus we obtain the result of exercise 262, 3°.

Problem 396. What are necessary and sufficient conditions which four circles A, B; C, D must satisfy in order that they can be transformed by inversion so that the figure formed by the first two is congruent to that formed by the second two? (Using the terminology introduced in Note A, **289, 294**, what are the *invariants*, under the group of inversions, of the figure formed by two circles?)

1^o. If circles $A,\ B$ have a common point, it is necessary and sufficient that the angle of these two circles equal the angle of $C,\ D$; or, which is the same (by the preceding exercise), that the ratio of the common tangent to the geometric mean of the radii be the same in both cases;

Solution 1°**.** Two circles will always invert into lines or circles meeting at the same angle. So unless circles $A,\ B$ meet at the same angle as circles $C,\ D$, the second pair cannot be congruent to an image of the first pair under inversion. That is, the condition that the pairs of circles meet at the same angles is necessary.

Let us show that this condition is also sufficient. Suppose circles $A,\ B$ (*fig.* t396a) intersect at points $P,\ P'$, and circles $C,\ D$ intersect at points Q, Q', both pairs intersecting at the same angle α. If we invert the first pair of circles in pole P (with any power at all), we will get two lines $a,\ b$ which intersect at angle α (**221**). If we invert circles $C,\ D$ in pole Q (with any power at all), we get another pair of lines $c,\ d$, intersecting at angle α. By suitable rotation and/or translation, we can move the figure consisting of $C,\ D,\ c,\ d$ so that lines $c,\ d$ coincide with lines $a,\ b$. Let Q_1 be the image of Q under this series of rotation and translation.

Now we can invert $A,\ B$ into $a,\ b$ around P, then invert $a,\ b$ around Q_1 into two circles congruent to the figure formed by $C,\ D$. Assuming that P and Q_1 do

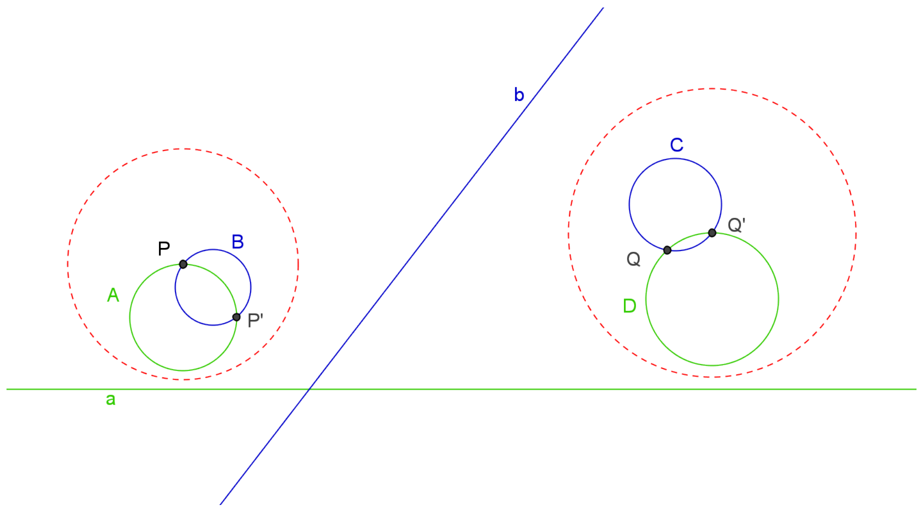

FIGURE figt396a

not coincide, these two inversions can be replaced by a single inversion, followed by a line reflection (see exercise 251, 2°). If we neglect the line reflection, then the figure formed by $A,\ B$ are inverted into a figure congruent to that formed by $C,\ D$.

If $P,\ Q_1$ coincide, we can repeat the argument, choosing Q' in place of Q, and obtaining a center Q_2 of inversion (by translation and rotation of $C,\ D$) which cannot coincide with P.

Thus the condition that the two pairs of circle meet at equal angles α is sufficient as well as necessary. The solution to exercise 395 shows that the value of α determines the value of angle $\widehat{P\Omega P'}$ in figure t395, which in turn determines the ratio $PP' : P\Omega$. This last ratio is the one referred to in the problem statement.

We must make some changes to this argument if the pairs of circles are tangent (i.e., intersect at an angle of 0°). Lines $a,\ b$ will be parallel, as will lines $c,\ d$. By adjusting the power of the inversion, we can make the distance between lines $c,\ d$ equal to the distance between $a,\ b$. Then we can follow the previous argument.

Note. Students can fill in details of the last paragraph, showing how the power of inversion determines the distance between the parallel lines. They can also consider the case in which both pairs of circles are tangent at the same point. (Translate one of the pairs away from the common point, so that the argument above applies.)

2°**.** If circles $A,\ B$ have no common point, it is necessary and sufficient that the ratio of the radii of the concentric circles into which they can be transformed by an inversion (Exercise 248) be the same as the ratio of the radii of the concentric circles into which $C,\ D$ can be transformed by an inversion (generally, a different inversion from the first). (Using the language of Note A, it is necessary and sufficient that the figures $(A,\ B)$ and $(C,\ D)$ have the same *reduced form* under inversion.)

This result can also be expressed as follows: the cross ratio (**212**) of the intersection points of $A,\ B$ with any of their common orthogonal circles is constant, and the same is true of the cross ratio of two of these points and the limit points.

The required condition is that this ratio have the same value for the circles $C,\ D$ as for $A,\ B$.

Finally, if $r,\ r'$ are the radii of $A,\ B$, and d is the distance between their centers, the quantity $\frac{d^2-r^2-r'^2}{rr'}$ must have the same value as the corresponding value calculated for the circles $C,\ D$.

We could also express this by saying that if the circles $A,\ B$ have a common tangent (for example, a common external tangent) of length t, and the same is true for $C,\ D$, then the ratio $\frac{t}{\sqrt{rr'}}$ must be the same in the two cases.

Solution 2°**.** We first prove that the given condition is necessary. Suppose some inversion S takes circles $A,\ B$ onto circles $A',\ B'$, which taken together form a figure congruent to that formed by circles $C,\ D$. If we translate and rotate circles $C,\ D$, they will then coincide with circles $A',\ B'$. So we can assume, without loss of generality, that circles $C,\ D$ are in fact the same as $A',\ B'$.

Recall (exercise 248) that any two non-intersecting circles can be inverted into concentric circles by using one of their limit points (exercise 152) as the pole. So we can invert $C,\ D$ around one of their limit points to get concentric circles $c,\ d$. We call this inversion T. (Since C and D have two limit points, we can choose the pole of T to be different from the pole of S.) The inversion S followed by the inversion T takes $A,\ B$ onto $c,\ d$. But the composition of these two inversions can be replaced by a single inversion S' and a line reflection (exercise 251, 2°). So S' (without the line reflection) takes $A,\ B$ into two concentric circles which are congruent to $c,\ d$. And T takes circles $C,\ D$ into concentric circles $c,\ d$. That is, if there is an inversion S taking $A,\ B$ onto a figure congruent to $C,\ D$, then the ratio of the concentric circles into which they can be inverted must be the same. The condition of the problem is necessary.

We next show that this condition is sufficient, using an argument similar to that in 1°. Suppose $A,\ B$ invert into concentric circles $a,\ b$, while $C,\ D$ invert into concentric circles $c,\ d$, and suppose that the radii of $c,\ d$ are proportional to those of $a,\ b$. We choose any power at all for the inversion taking $A,\ B$ onto $a,\ b$, and recall (**215**) that two figures which are inversions of the same figure with the same pole are homothetic to each other. This means that we can choose the power of the inversion taking $C,\ D$ onto $c,\ d$ in such a way that the figure formed by $c,\ d$ is congruent to that formed by $a,\ b$. (We use here the fact that the radii of the four circles are in proportion.) We can translate and rotate the figure formed by $C,\ D$ so that circles $a,\ b$ in fact coincide with circles $c,\ d$. As in 1°, we now see that $C,\ D$ can be obtained from $A,\ B$ by a sequence of two inversions. (We can avoid the situation where the poles of these inversions coincide by rotating $C,\ D$ around the common center of $a,\ b$.) As before, the sequence of two inversions can be replaced by a single inversion S' followed by a line reflection, and the inversion S' alone takes $A,\ B$ onto a figure congruent to $C,\ D$.

We now express this condition in terms of the cross ratios of the intersections of the given circles with the circles orthogonal to them. We will show, in the case of intersecting circles, that this cross ratio depends only on the angle at which the circles intersect, and not on the particular orthogonal circle. In the case of non-intersecting circles, we will show that this cross ratio depends only on the ratio of the concentric circles they invert into, and not on the particular orthogonal circle.

(The value of the cross ratio described here depends on the order in which the points are taken. Without loss of generality we can take the first pair of points to be the intersection of the orthogonal circle with one of the two given circles, and the second pair of points to be its intersection with the second of the given circles. The result of exercise 274 will tell us that the value λ of the cross ratio is then limited to two values, whose product is 1. We can then, again without loss of generality, take the points in order so that $|\lambda| \leq 1$.)

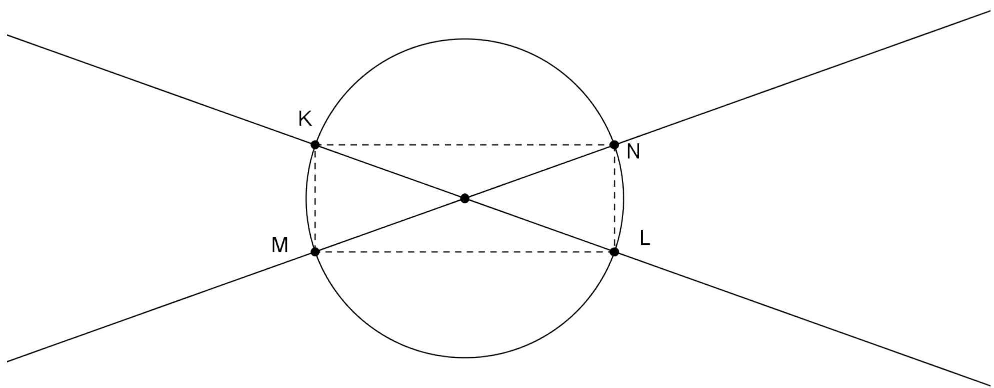

FIGURE t396b

We start with the case of two intersecting circles. These can be inverted into two intersecting lines, and the circles orthogonal to them into circles whose center is their point of intersection. One such circle, in figure t396b, is $KNLM$, where K and L belong to different semicircles. The cross ratio $\lambda = (KLMN) = -\frac{KM}{LM} : \frac{KN}{LN} = -(\frac{KM}{LM})^2$. But this last ratio depends only on the angle at which the two lines intersect, and not on the choice of orthogonal circle. Since inversion preserves both orthogonality and cross ratio (see exercise 273), the same is true of two intersecting circles.

In the case of two non-intersecting circles, we invert them into two concentric circles. Then the circles orthogonal to them invert into lines through the common center of the two circles. One such line, in figure t396c, is $MKOLN$. If the radii of the concentric circles are R and r, then the cross ratio $(KLMN) = \frac{KM}{LM} : \frac{KN}{LN} = \left(\frac{R-r}{R+r}\right)^2$. Dividing numerator and denominator of this last fraction by R, we find $\lambda = \left(\frac{1-\frac{r}{R}}{1+\frac{r}{R}}\right)^2$, and so depends only on the ratio of the radii of the concentric circles. Since inversion preserves both orthogonality and cross ratio, the same is true of the original two non-intersecting circles.

Note. If the circles are tangent, any circle orthogonal to both must pass through their point of tangency. The cross ratio of the four points described in

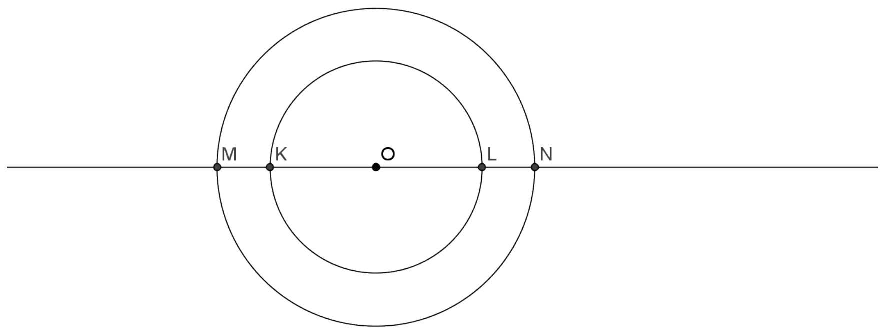

FIGURE t396c

the problem (and taken in order) is 1, as two of the points coincide. Students can fill in the details for this case.

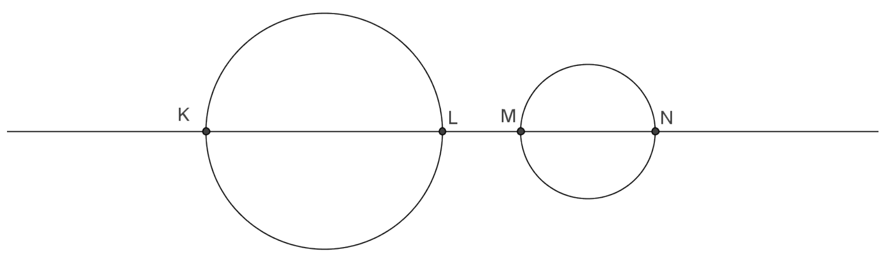

FIGURE t396d

We treat separately the assertions in the rest of the exercise.

3°. The quantity $\nu = \frac{d^2 - r^2 - r'^2}{rr'}$ must have the same value as the corresponding value calculated for the circles $C,\ D$ (with the variables as described in the problem statement).

Solution. We will show that ν is uniquely determined by the cross ratio λ. We already know that we can compute λ from an image under inversion of the original circles, so we choose, without losing generality, the situation where the circle orthogonal to the two original circles is simply their common centerline (*fig.* t396d), and we can do this whether or not the given circles have a point in common. Then we have

$$\lambda = (LKMN) = \frac{KM}{LM} : \frac{KN}{LN} = \frac{d+r-r'}{d-r-r'} : \frac{d+r+r'}{d-r+r'} = \frac{d^2-(r-r')^2}{d^2-(r+r')} =$$

$$= \frac{d^2-r^2-r'^2+2rr'}{d^2-r^2-r'^2-2rr'} = \frac{\nu+2}{\nu-2}.$$

Solving from ν in terms of λ, we find that $\nu = \frac{2\lambda+2}{\lambda-1}$. That is, the value of ν is determined uniquely by the value of λ. Since λ has the same value for the two pairs of circles considered, so must ν.

4°. If circles $A,\ B$ have a common tangent of length t, and the same is true for $C,\ D$, then the ratio $\frac{t}{\sqrt{rr'}}$ must be the same in the two cases.

Solution. Let $\tau = \frac{t}{\sqrt{rr'}}$. Then $t^2 = d^2-(r-r')^2 = d^2-r^2-r'^2-2rr'$. Comparing this to the definition of ν invites the direct computation $\nu+2 = \frac{d^2-r^2+2rr'}{rr}$, so that $\tau = \frac{t}{rr'} = \sqrt{\nu+2}$, As in 4°, this means that τ has the same value for the pairs of circles considered.

Problem 397. We are given two points $A,\ A'$ and two lines $D,\ D'$ parallel to, and at equal distance from, AA'.

1^o. Show that for every point P on D there corresponds a point P' on D' such that line PP' is tangent to the two circles $PAA',\ P'AA'$;

Solution 1°**.** For any point P on line D, we draw the circle through $P,\ A',\ A$, and also its tangent at P (*fig.* t397a). Let P' be the point of intersection of this tangent with D'. The centers of the circles through $A,\ A'$ lie on the perpendicular bisector of AA'. The centers of circles tangent to PP' lie on the line through P' perpendicular to PP'. Their intersection O' is the center of a circle through $P,\ A,\ A'$.

Indeed, if M is the intersection of AA' and PP', then M is the midpoint of PP' (**113**), and we have $MP'^2 = MP^2 = MA \cdot MA'$. By **132** (converse), this means that $P',\ A'$, and A are on the same circle, which must have its center at O'.

2^o. Prove that the product of the distances from $A,\ A'$ to line PP' is constant;

Solution. Let $H,\ H'$ (*fig.* t397a) be the feet of the perpendiculars from $A,\ A'$ respectively to line PP'. We know that (**132**) that

$$MA \cdot MA' = PM^2. \tag{1}$$

We use similar triangles to rewrite (1). Triangles $AHM,\ A'H'M$ are similar (their sides are parallel in pairs; see **43, 118**, first case). If we draw MM' perpendicular to D, we find that triangle PMM' is similar to the two triangles identified above (they are right triangles with two pairs of parallel sides; see **43, 118,** case I). That is, segments $AH,\ A'H',\ MM'$ are proportional to segments $AM,\ A'M,\ PM$.

Now we rewrite (1). We have $\frac{A'H'}{A'M} = \frac{MM'}{PM}$, so $A'M = A'H' \cdot \frac{PM}{MM'}$. Likewise, $AM = AH \cdot \frac{PM}{MM'}$, and we can rewrite (1) as $A'H' \cdot \frac{PM}{MM'} \cdot AH \cdot \frac{PM}{MM'} = PM^2$. Dividing this equation by $\left(\frac{MM'}{PM}\right)^2$, we find that

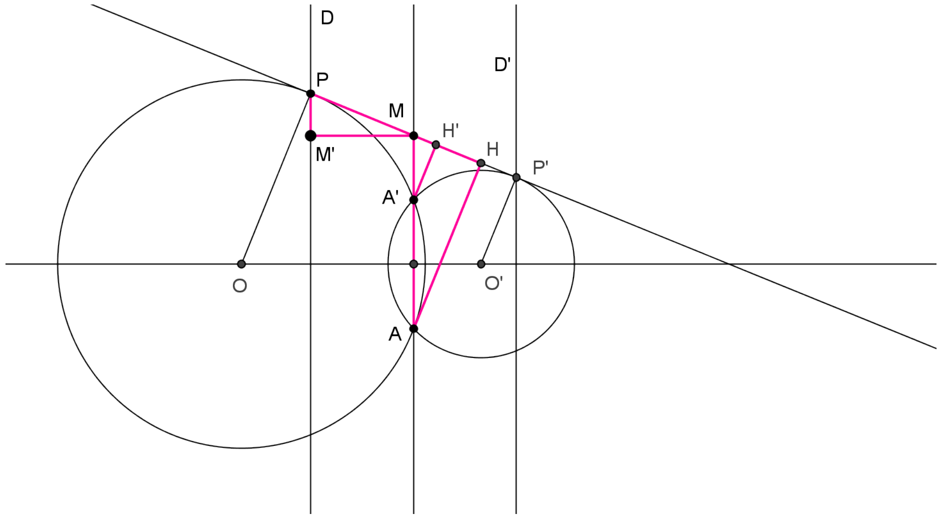

FIGURE t397a

(2) $$A'H' \cdot AH = MM'^2,$$

and this last length is constant, since $AA' \parallel D$.

3^o. Find the locus of the projection of A onto PP';

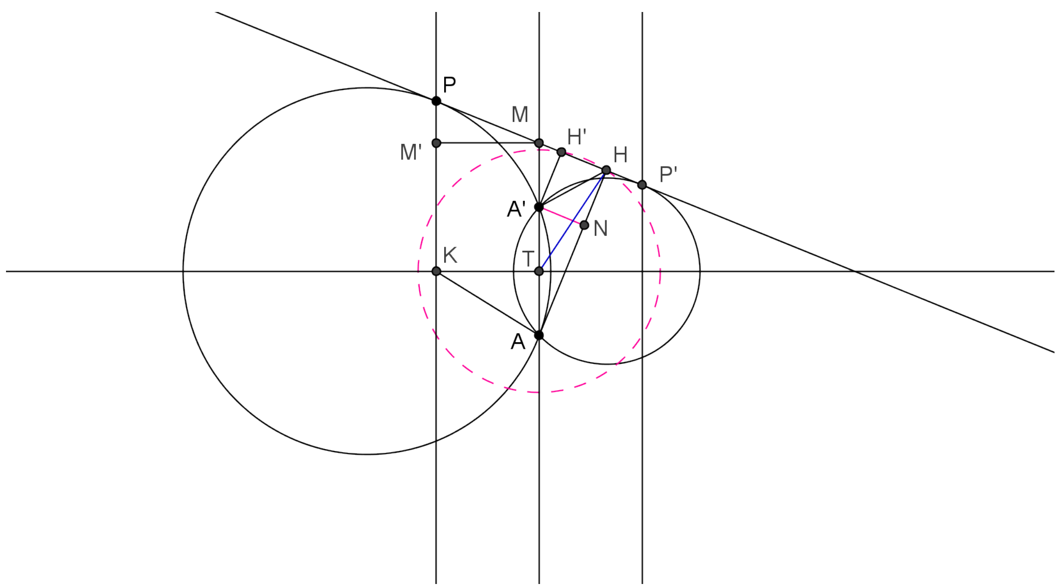

FIGURE t397b

Solution. We use the result of exercise 141, and also some ideas from its solution for the case $m = n = 1$, in the notation of that solution. (Students can be given this statement as a hint.) To apply this result, we examine the sum $HA^2 + HA'^2$, by looking at triangle HAA'.

In that triangle, we have $AA'^2 = A'H^2 + AH^2 - 2AH \cdot HN$ (**126**), where N is the foot of the perpendicular from A to line HA. But $HN = A'H'$ (from rectangle $A'NHH'$), so

$$HA^2 + HA'^2 = AA'^2 + 2AH \cdot AH', \tag{3}$$

and the result of 2° shows that this last quantity is constant. Thus the result of exercise 141 shows that H lies on a circle whose center is the midpoint T of AA'

To compute the radius of this circle, we proceed as in exercise 141 (or use **128),** 1°. From either of these results, we have

$$\begin{aligned} 4HT^2 &= 2A'H^2 + 2AH^2 - AA'^2 \\ &= 2(AA'^2 + 2AH \cdot AH') - AA'^2 (\text{from}(3)) \\ &= AA'^2 + 4MM'^2 .(\text{from}(2)) \end{aligned}$$

So the radius HT of the circle is equal to $\sqrt{\frac{1}{4}AA'^2 + MM'^2}$.

Notes. In figure t397b, right triangle ATK shows that radius HT of this circle is equal to the distance from A to point K, the point on line D which is equidistant from A and A'.

Analogous reasoning starting with triangle $A'H'A$ will show that point H' lies on the same circle.

4^o**.** Find a point P such that the line PP' passes through a given point Q;

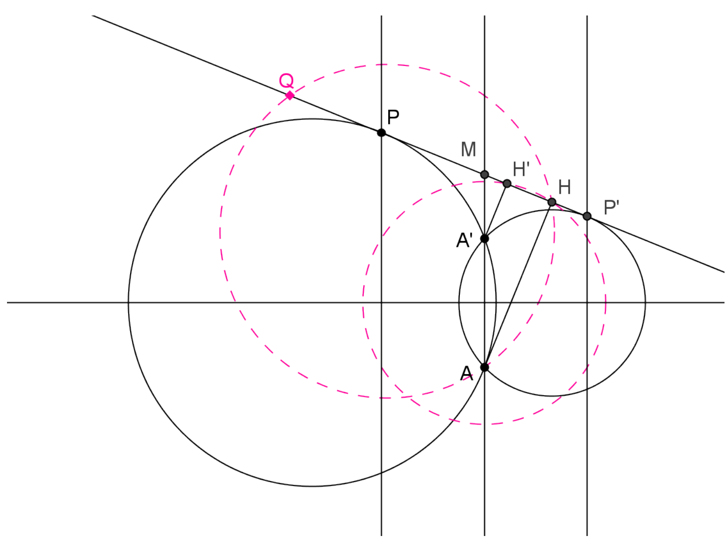

FIGURE t397c

Solution. If we start with any point Q, the associated point H must lie on a circle with diameter AQ (since angle $\widehat{QHA}$ is a right angle), and also on the circle described in 3°. The line connecting Q to any intersection point of these two circles will intersect D in a point P which satisfies the conditions of the problem.

5^o. Show that the angle of the circles PAA', $P'AA'$, and angle $\widehat{PA'P'}$, are constant.

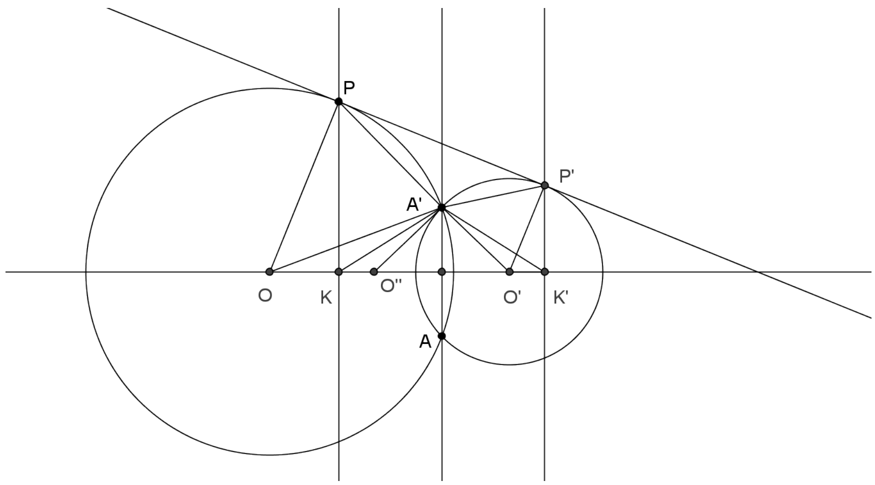

FIGURE t397d

Solution. We take the angle between the two circles to be equal to the angle between their radii at a point of intersection. In figure t397d, this is angle $\widehat{OA'O'}$, and we will show that it is equal to the constant angle $\widehat{KA'K'}$, where K, K' are the intersections of the two given parallel lines with the perpendicular bisector of segment AA'.

To this end, we lay off segment KO'' as in the figure so that $KO'' = K'O'$. Then triangles $A'KO''$, $A'K'O'$ are symmetric in line AA', so they are congruent, and

$$\widehat{KA'O''} = \widehat{K'A'O'}. \tag{4}$$

Next we show that $A'K$ bisects angle $\widehat{OA'O''}$. Indeed, triangles OPK, $O'P'K'$ are similar. (See **43**; their sides are parallel in pairs). So $OP : O'P' = OK : O'K'$. But $OP = OA'$, $O'P' = O'A' = O''A'$, $O'K' = KO''$, so this proportion is equivalent to $OA' : O''A' = OK : KO''$, and this last proportion shows (**115**) that $A'K$ is an angle bisector in triangle $A'OO''$.

Now from (4) we have $\widehat{O'A'K} = \widehat{KA'O''} = \widehat{K'A'O'}$. Adding angle $\widehat{KA'O'}$ to each, we find that $\widehat{OA'O'} = \widehat{KA'K'}$. The first angle is the angle between the two circles, and the second angle is constant. This proves the first assertion.

To show that $\widehat{PA'P'}$ is constant, we examine the angles around vertex A', and will show that the sum of the remaining angles is also contant. We know that

$\widehat{OA'O'}$ is constant from the previous paragraph. So we need to show that the sum $\widehat{OA'P} + \widehat{O'A'P'}$ is constant.

Since $\widehat{OA'O'}$ is constant, so is the sum $\widehat{A'OO'} + \widehat{A'O'O}$ (it is equal to $180° - \widehat{OA'O'}$). And since $OP \parallel O'P'$, we know that $\widehat{POO''} + \widehat{OO'P'} = 180°$, so the sum $\widehat{A'OP} + \widehat{A'O'P'} = (POO' + \widehat{OO'P'}) - (\widehat{A'OO'} + \widehat{A'O'O}) = \widehat{OA'O'}$.

Now we look at isosceles triangles POA', $P'O'A'$. The sum of their vertex angles is constant, so the sum of a pair of base angles is also constant: $\widehat{PA'O} + \widehat{P'A'O'} = \frac{1}{2}(180° - \widehat{POA'} + 180° - \widehat{P'O'A'}) = 180° - \frac{1}{2}(\widehat{OA'O'})$.

We have shown that the sum of the other angles around A' is constant, so $\widehat{PA'P'}$ must also be constant: $\widehat{PA'P'} = 360° - (\widehat{PA'O} + \widehat{P'A'O'}) - \widehat{OA'O'} = 180° - \frac{1}{2}\widehat{OA'O'} = 180° - \frac{1}{2}\widehat{KA'K'}$, which is constant.

Problem 398. Let C be a circle with diameter AB, and D a line perpendicular to this diameter, which intersects C. Let c, c' be the circles whose diameters are the two segments into which D divides AB. We draw a circle tangent to C, c, D, and another circle tangent to C, c', D. Show that these two circles are equal, and that their common radius is the fourth proportional to the radii of C, c, c'.

Solution. Let S be the circle tangent to c, D, and C, and let S' be the circle tangent to c', D, and C.

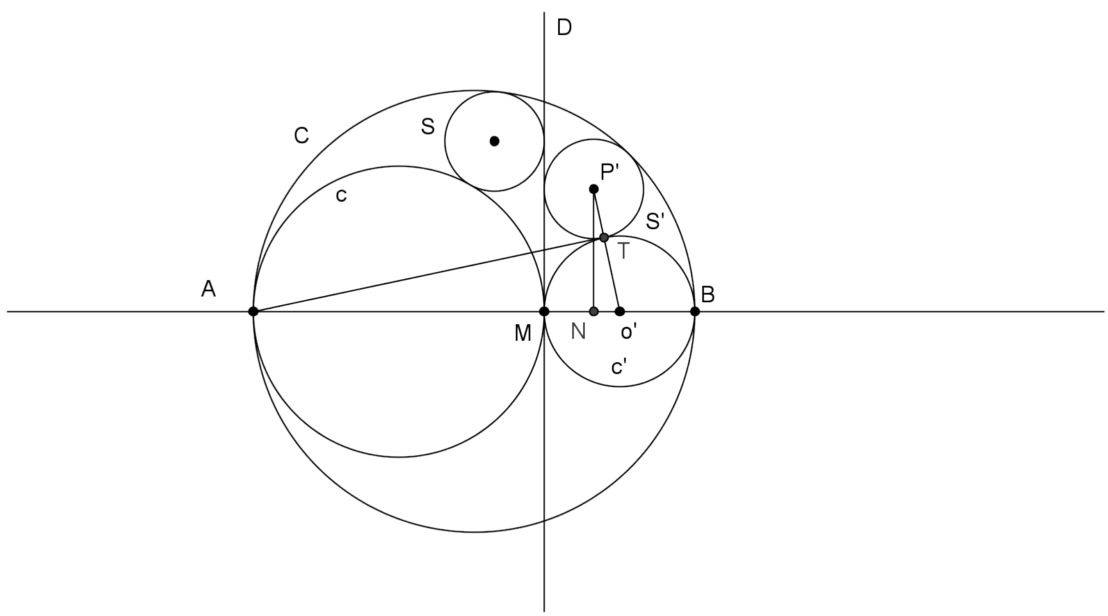

FIGURE t398

We consider the inversion with pole A and power $AM \cdot AB$. We will show that the image of line D is circle C. Indeed, the image is a circle through A (**220**) which is orthogonal to line AB (**219**). Points M and B are clearly images of each other, so the image of D must contain point B. That is, the image of D is a circle through A and B which is orthogonal to line AB. This must be circle C. And the image of circle C is of course then line D. Further, it is not hard to see circle c' is its own image. (The power of inversion is the power of A with respect to this circle.) Finally, the image of circle S' is a circle tangent to D, C, and c', and the only such circle (on the same side of AB as S') is S' itself.

It follows that the common tangent to c and S' at their point T of tangency must pass through the pole A of this inversion. Indeed, since c', S' are their own images under the inversion, their common point must be its own image as well. That is, AT^2 is the power of the inversion. But then $AT^2 = AM \cdot AB$ and by **132** (converse), AT is the tangent to c' (and therefore also to S') at T.

Let o', P' be the centers of circles c', S' respectively. Let N be the foot of the perpendicular from P' to diameter AB, and let R, r, r', ρ' be the respective radii of circles C, c, c', S'. Triangles ATo', $P'No'$ are both right triangles, and share angle $\widehat{P'o'A}$, so they are similar, and $Ao' : P'o' = To' : No'$ or $\frac{2R-r'}{\rho'+r'} = \frac{r'}{r'-\rho'}$.

Transforming this expression gives us $\rho' R = (R - r')r'$. Since $R = r + r'$, it follows that $\rho' R = rr'$. Thus the radius of circle S' is the fourth proportional to the radii of C, c, c'.

Similarly, we could compute the value of the radius r of circle S in terms of ρ' and R. But we need not carry out the argument: the expression we got for ρ' is symmetric in r and r', and we will just be changing the names of the objects in the argument, not their relationships to each other. Either way, we see that the radii of S, S' are equal.

Problem 399. (the Greek *Arbelos*) Let A, B be two tangent circles. Let C be a circle tangent to the first two; let C_1 be a circle tangent to A, B, C; let C_2 be a circle tangent to A, B, C_1; let C_3 be tangent to A, B, C_2, $\ldots$; and let C_n be a circle tangent to A, B, C_{n-1}. Consider the distance from any of the centers of C, $C_1, \ldots, C_n$ to the line of centers of A, B, and the ratio of this distance to the diameter of the corresponding circle. Show that this ratio varies by one unit in passing from any circle to the next one, at least in the case in which they are exterior (which always happens when circles A, B are tangent internally). Show how this statement must be modified when two consecutive circles C_{n-1}, C_n are tangent internally. (*Arbelos* is a Greek word meaning *sickle*)[1].

Solution. The problem describes a chain of circles, starting with a single circle C, which are all tangent to two larger circles, and each circle tangent to the previous as well. It will be convenient to discuss a slightly more general situation, in which the chain of circles is continued in both directions, so that there is an initial circle $C = C_0$, a chain of circles $C_1, \cdots C_n$, and also a chain of circles C_{-1}, $C_{-2}, \cdots$ as in figure t399a.

Let the center of circle C_n be O_n, and its radius as r_n. Let ℓ be the common centerline of circles A, B. We will also need to talk about the projection of O_n onto ℓ. We call this point P_n.

In that figure, T is point of tangency of circles A, B, and we invert the figure around T as the pole, using any power. Circles A, B invert into two parallel lines A', B' (**220**, corollary), and circles $\cdots$, C_{-2}, C_{-1}, C_0, $C_1, C_2, \cdots$ invert into

[1]This note is Hadamard's own. The usual translation of *arbelos* is *shoemaker's knife*. But see for instance Harold P. Boas, *Reflections On the Arbelos*, American Mathematical Monthly, 113, no. 3 (March 2006), 236-249. –transl.

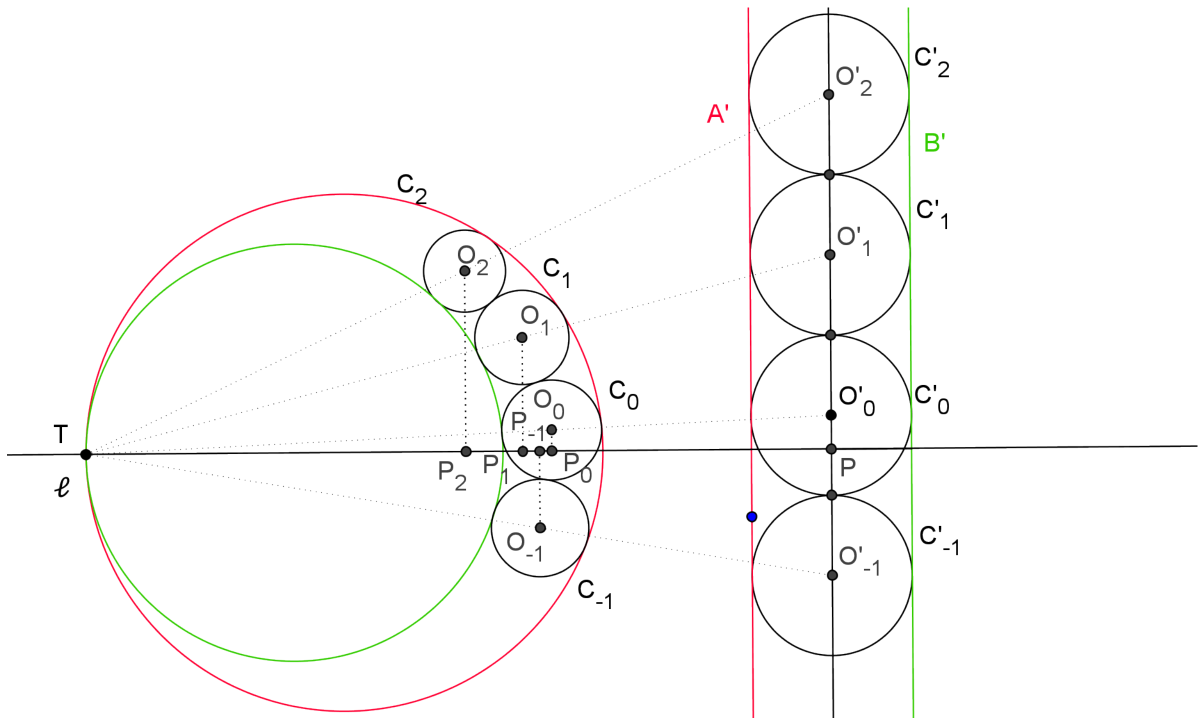

FIGURE t399a

circles $\cdots$, C'_{-2}, C'_{-1}, C'_0, $C'_1, C'_2, \cdots$ tangent to both lines, which therefore have equal radii. We denote this radius as r'.

Let the centers of the circles C'_n be the points O'_n. Because the circles are equal, these centers are collinear. Let P' be the intersection of this common centerline with ℓ.

For those circles whose centers lie on the same side of line ℓ, it is clear that $O'_{n-1}P' - O_nP' = 2r'$. We can write this as

$$\frac{O'_{n-1}P'}{2r'} - \frac{O'_nP'}{2r'} = 1. \tag{1}$$

For those circles whose centers lie on different sides of line ℓ, this relationship becomes $O'_{n-1}P' + O'_nP' = 2r'$, or

$$\frac{O'_{n-1}P'}{2r'} + \frac{O'_nP'}{2r'} = 1. \tag{2}$$

Now the pole of inversion is a center of similarity for any pair of circles which are inverses of each other, so (*fig.* t399b) we have $r' : r_n = TO'_n : TO_n = O'_nP' : O_nP_n$. Therefore $O'_nP' : 2r' = O_nP_n : 2r_n$. That is, the ratio of the distance from the centers of any of our circles C_n from the common centerline of A, B to the diameter of that same circle C_n does not change when we invert around pole T. Because of this, we can rewrite (1) and (2) as:

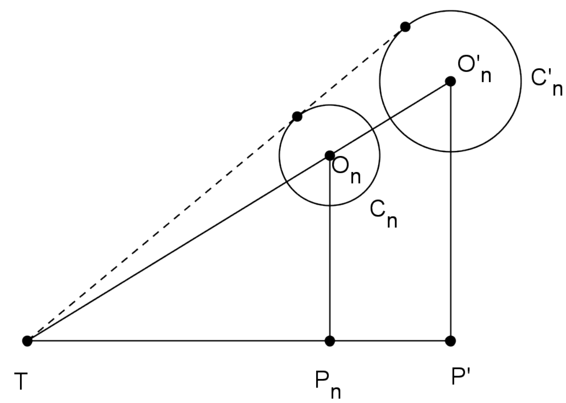

FIGURE t399b

$$\frac{O_{n+1}P_{n+1}}{2r_{n+1}} - \frac{O_nP_n}{2r_n} = 1. \tag{3}$$

$$\frac{O_{n+1}P_{n+1}}{2r_{n+1}} - \frac{O_nP_n}{2r_n} = 1. \tag{4}$$

These two equations prove the assertion of the problem.

In the case mentioned in the problem statement, where A and B are tangent internally, any two circles $C_n,\ C_{n+1}$ are tangent externally. Since circles $C_n,\ C_{n+1}$ are tangent externally, it follows that the centers of the inverted circles $C'_n,\ C'_{n+1}$ lie on the same or different sides of line ℓ according as the centers of $C_n,\ C_{n+1}$ lie on the same or different sides of ℓ. Thus in this case, we can say that equation (3) holds whenever the centers of the circles lie on the same side of ℓ, and equation (4) holds if these centers lie on opposite sides of ℓ.

If circles $A,\ B$ are tangent externally (*fig.* 399c), then circles $C_{-1},\ C_0$ and $C_0,\ C_1$ are tangent internally, while the others are tangent externally. This requires a small change in the concluding statement above. For those pairs of circles $C_n,\ C_{n+1}$ which are tangent internally, equation (3) holds if their centers are on opposite sides of ℓ, and equation (4) holds if their centers are on the same side of ℓ.

Note. We can state this result more generally if we allow for signed distances to line ℓ. If $x_n,\ x_{n+1}$ are the distances of consecutive circles from ℓ, and $r_n,\ r_{n+1}$ are their radii, then we can say that $\frac{x_n}{2r_n} - \frac{x_{n+1}}{2r_{n+1}} = 1$, when x_n is positive for circles on one side of ℓ and negative for circles on the other side.

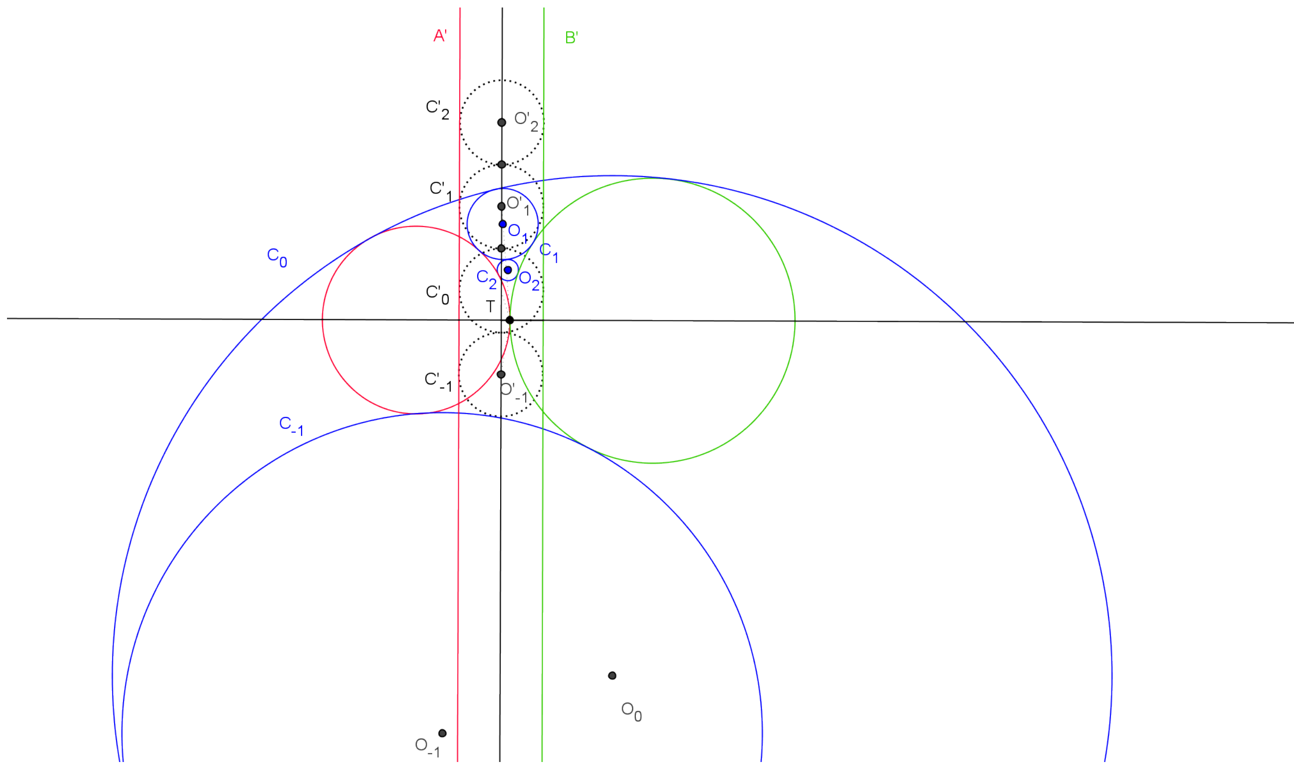

FIGURE t399c

For circles that are tangent externally, we have $\frac{x_n}{2r_n} + \frac{x_{n+1}}{2r_{n+1}} = 1$, with similar conventions of sign for $x_n,\ x_{n+1}$.

Problem 400. Let $A,\ B,\ C$ be three circles with centers at the vertices of a triangle, each pair of which are tangent externally (Exercise 91). Draw the circle externally tangent to these three circles, and also the circle internally tangent to these three circles. Calculate the radii of these circles knowing the sides a, b, c of the triangle (preceding exercise, Exercise 301).

Solution. If $a,\ b,\ c,\ s$ are the sides and semiperimeter of triangle ABC, then exercise 91 shows that the radii of the circles centered at $A,\ B,\ C$ are $s-a,\ s-b, s-c$ respectively.

There are many circles tangent to the three described in the problem, but only two (also mentioned in the problem) which are tangent to all three externally or to all three internally (see figure t400a). We first consider the circle which is externally tangent to all three (the small circle in figure t400a) which lies inside the curvilinear triangle formed by the three given circles. Let O be its center and ρ its radius.

We will apply the result of exercise 399 in various ways, with pairs of circles $A,\ B,\ C$ playing the role of the original circles given in that exercise. The common centerlines of these circles are the sides of the given triangle, so we will need the distances of these from O. Let these be $x,\ y,\ z$, and let $h,\ k,\ \ell$ be the lengths of the altitudes of the given triangle.

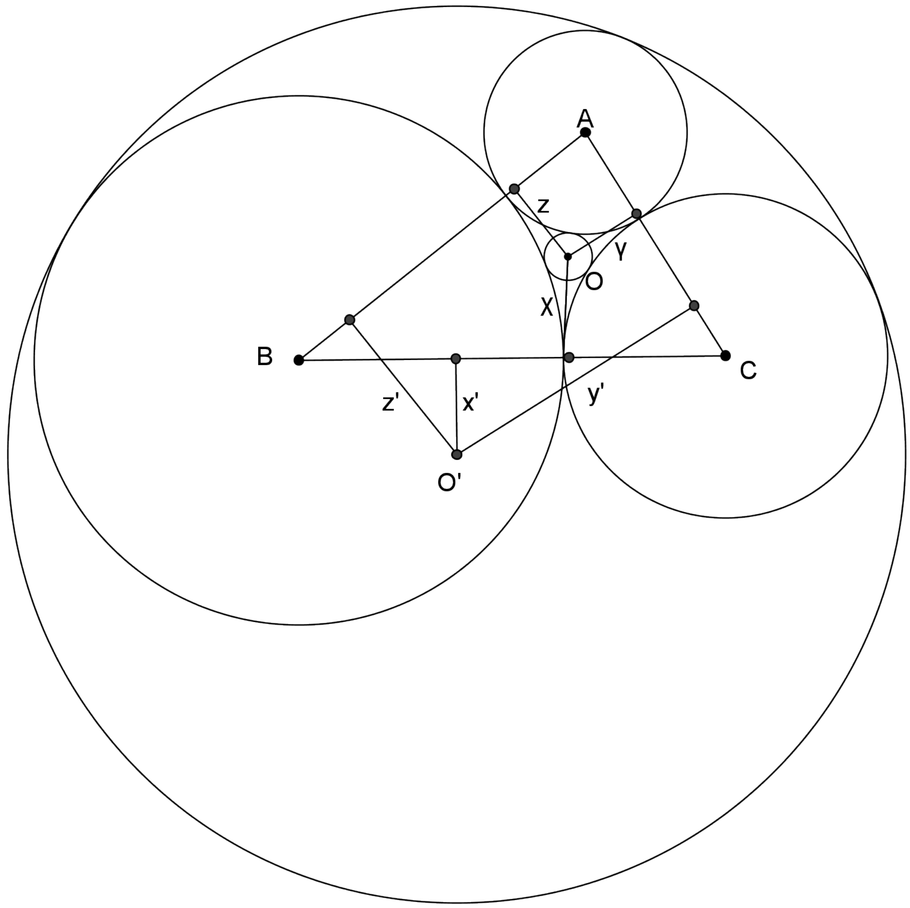

Figure t400a

We can now use formula (3) from exercise 399, applying it to circles B and C as the two given circles, and A and O as the (short) chain of circles. we have:

$$\frac{x}{2\rho} - \frac{h}{2(s-a)} = 1. \tag{1}$$

Analogously, if we start with A and B as the given circles, and $C,\ O$ the chain of circles, we have:

$$\frac{y}{2\rho} - \frac{k}{2(s-b)} = 1,$$

and starting with circles A and C we have:

$$\frac{z}{2\rho} - \frac{\ell}{2(s-c)} = 1.$$

Multiplying (1) by 2ρ and dividing by h, we get:

$$\frac{x}{h} - \frac{\rho}{s-a} = \frac{2\rho}{h},$$

and analogously:

$$\frac{y}{k} - \frac{\rho}{s-a} = \frac{2\rho}{k},$$
$$\frac{z}{\ell} - \frac{\rho}{s-c} = \frac{2\rho}{\ell}.$$

Adding, and using the result of exercise 301, we get

$$\rho\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{2}{h} + \frac{2}{k} + \frac{2}{\ell}\right) = 1.$$

This gives an expression for ρ, as required. We can transform it by introducing the area Δ of the triangle, and the radii $r_a,\ r_b,\ r_c$ of its escribed circles.

From exercise 299, we have $\frac{1}{s-a} = \frac{r_a}{\Delta},\ \frac{1}{s-b} = \frac{r_b}{\Delta},\ \frac{1}{s-c} = \frac{r_c}{\Delta}$. And from **249** $\frac{2}{h} = \frac{a}{\Delta},\ \frac{2}{k} = \frac{b}{\Delta},\ \frac{2}{\ell} = \frac{c}{\Delta}$, so we have:

$$\rho\left(\frac{r_a}{\Delta} + \frac{r_b}{\Delta} + \frac{r_c}{\Delta} + \frac{a}{\Delta} + \frac{b}{\Delta} + \frac{c}{\Delta}\right) = 1,$$

or

$$\rho\left(\frac{r_a}{\Delta} + \frac{r_b}{\Delta} + \frac{r_c}{\Delta} + \frac{a}{\Delta} + \frac{b}{\Delta} + \frac{c}{\Delta}\right) = 1,$$

or

$$\rho = \frac{\Delta}{a+b+c+r_a+r_b+r_c} = \frac{\Delta}{2s+r_a+r_b+r_c}.$$

Now suppose O' and ρ' are the center and radius of the second circle tangent to the three given circles. This circle can touch the others either internally (*fig.* t 400a) or externally (*fig.* t 400b). In the case where the new circle is tangent internally to the others, its center can lie either inside the given triangle or outside it.

Proceeding as before, we get the expression $\rho' = \pm\frac{\Delta}{2s-(r_a+r_b+r_c)}$.

Notes. We can explore the situation further if we introduce the convention of exercise 301 for the signed distance from a point to a line. Let x', y', z' are the distances from point O' to the sides of the triangle. In the case of a circle externally tangent to the others, we then have the system of equations:

$$\frac{h}{2r_a} + \frac{x'}{2\rho'} = 1;$$
$$\frac{k}{2r_b} + \frac{y'}{2\rho'} = 1;$$
$$\frac{\ell}{2r_c} + \frac{z'}{2\rho'} = 1.$$

In the case of a circle internally tangent to the others, we have:

$$\frac{h}{2r_a} - \frac{x'}{2\rho'} = 1;$$

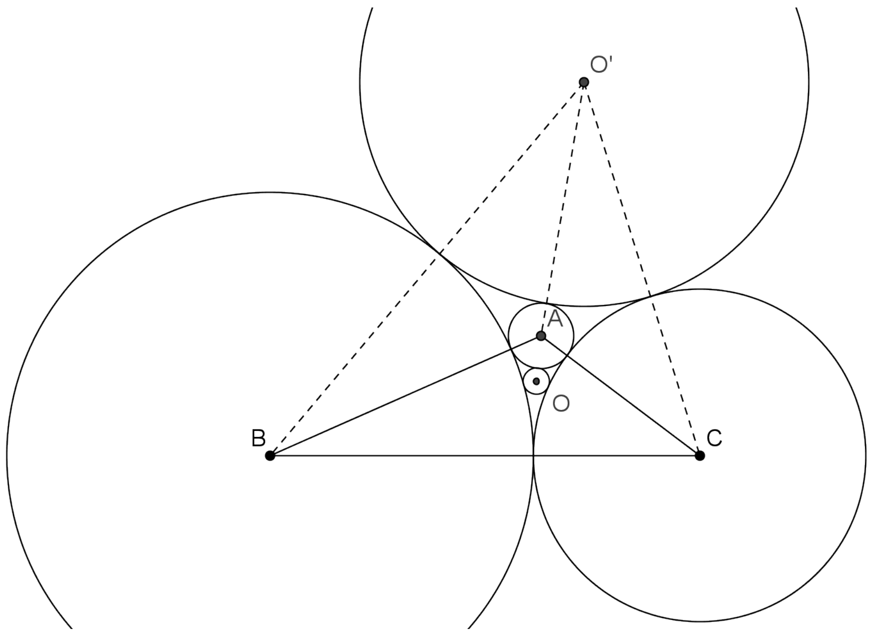

FIGURE t400b

$$\frac{k}{2r_b} - \frac{y'}{2\rho'} = 1;$$
$$\frac{\ell}{2r_c} - \frac{z'}{2\rho'} = 1.$$

Solving the first system, we get $\rho' = \frac{\Delta}{2s-(r_a+r_b+r_c)}$. From the second system, we get $\rho' = -\frac{\Delta}{2s-(r_a+r_b+r_c)}$. If $r_a + r_b + r_c = 2s$, circle O' becomes a straight line.

Problem 382. Let ABC be a triangle, and let $O,\ a,\ b,\ c$ be four arbitrary points. Construct (a) a triangle BCA' similar to triangle bcO and with the same orientation ($B,\ C$ being the points corresponding to $b,\ c$); (b) a triangle CAB' similar to caO with base CA, and (c) a triangle ABC' similar to abO with base AB. Show that triangle $A'B'C'$ is similar to, but with opposite orientation from, the triangle whose vertices are the inverses of the points $a,\ b,\ c$ with pole O.

Solution. In light of exercises 380-381, we introduce three figures $F_a,\ F_b,\ F_c$, each pair of which is similar and similarly oriented. Figure F_a includes point A; figure F_b includes point B as the point corresponding to A in F_a, and figure F_c includes point C as the point corresponding to B in F_b. We think of point A' as the fixed point of the similarity between figures $F_b,\ F_c$ and B' as the fixed point of the similarity between figures $F_c,\ F_a$. We can then show that C' is the fixed point of the similarity between figures $F_a,\ F_b$.

We first examine the angles between the three figures. The angle between figures F_b and F_c is equal to $\widehat{BA'C} = \widehat{bOc}$, and has the same orientation. Likewise, the angle between figures F_c and F_a will be $\widehat{CB'A} = \widehat{cOa}$. It follows that the angle between figures F_c and F_a is equal to $\widehat{aOb} = \widehat{AC'B}$.

Next we examine the ratios of corresponding sides in the three figures. This ratio, for figures F_b, F_c is $A'B : A'C = Ob : Oc$. For figures F_c, F_a it is $B'C : B'A = Oc : Oa$. It follows that the ratio of corresponding sides for figures F_a and F_b is equal to $Oa : Ob = C'A : C'B$. Therefore C' is the fixed point of the similarity taking F_a onto F_b.

Let a', b', c' be the points inverse to A, B, C respectively, with respect to some circle centered at O. We are now in a position to prove that triangles $A'B'C'$, $a'b'c'$ are similar but oppositely oriented. We do this by showing that they have equal but oppositely oriented angles. We introduce an auxiliary point A'', corresponding to A' in figure F_a.

Triangle $C'A''A'$ is similar to triangle $C'AB$, and has the same orientation, so that $\widehat{C'A'A''} = \widehat{C'BA} = \widehat{Oba}$ (both in magnitude and orientation). In the same way, triangles $B'A'A''$, $B'CA$ are similar, with the same orientation, so that $\widehat{B'A'A''} = \widehat{B'CA} = \widehat{Oca}$.

It follows that $\widehat{B'A'C'} = \widehat{B'A'A''} + \widehat{A''A'C'} = \widehat{Oca} + \widehat{abO} = \widehat{c'a'O} + \widehat{Oa'b'} = \widehat{c'a'b'}$, where a', b', c' are the points inverse to a, b, c respectively in a circle centered at O (**217**).

Thus we have, both in magnitude and sense of rotation, $\widehat{B'A'C'} = \widehat{c'a'b'}$. A similar argument gives the same result for the other two angles of triangle $A'B'C'$. Triangles $A'B'C'$, $a'b'c'$ are similar, but with opposite senses of rotation.

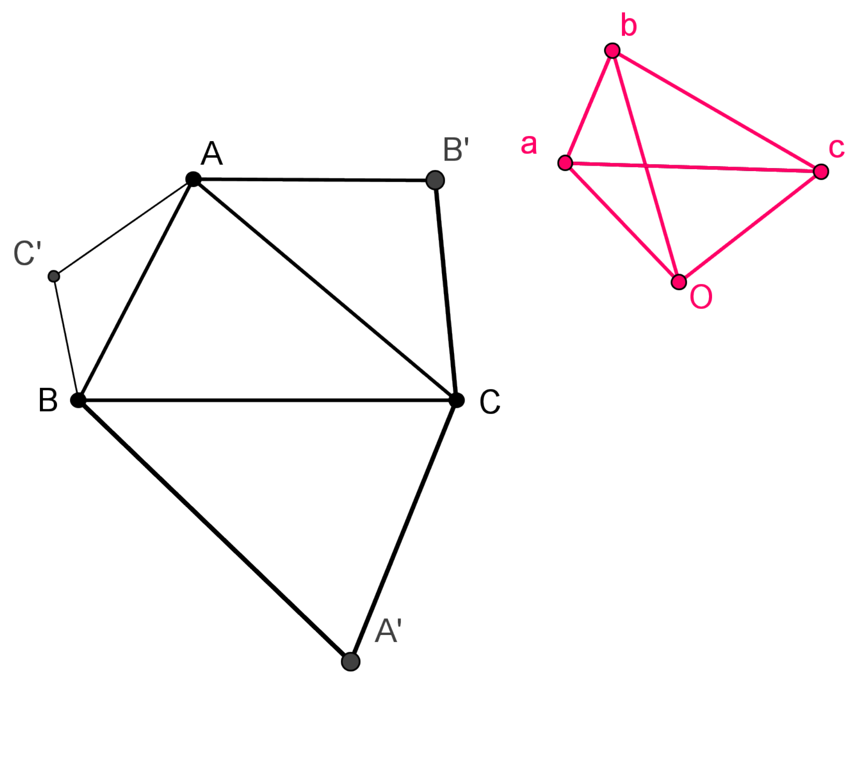

Figure t382

Problem 383. On two given segments as chords, construct circular arcs subtending the same arbitrary angle $\widehat{V}$. Show that, as $\widehat{V}$ varies, the radical axis of the two circles turns around a fixed point. (This point can be considered to be determined by the fact that the triangles with this vertex and with the two given segments as bases are equivalent, and they have the same angle at the common vertex.)

Solution. Suppose (*fig.* t383) the given segments are AB and CD, and P is the intersection of the lines they belong to. We assume, for now, that the angles we construct on AB and CD have the same orientation.

Following the hint in the problem statement, we will prove the following lemma:

Lemma. There exists exactly one point O with the following properties:

- (a) triangles OAB, OCD have the same area;
- (b) angles $\widehat{AOB}$, $\widehat{COD}$ have the same orientation;
- (c) angles $\widehat{AOB}$, $\widehat{COD}$ are equal.

We prove this lemma by constructing the point in question.

For condition (a) to be true, the distances from O to lines AB, CD must be in the ratio $CD : AB$ (note the reversal of the order AB, CD). By **157**, the locus of these points consists of two lines, both passing through P. It is not hard to see, from figure t383, that for any point X on one of these lines, angles $\widehat{AXB}$, $\widehat{CXD}$ have the same orientation, while for any point Y on the other line, angles $\widehat{AXB}$, $\widehat{CXD}$ have the opposite orientation. Thus point O satisfies conditions (a) and (b) if and only if it lines on line PX.

The set of points satisfying condition (c), in addition to (a) and (b), is a bit more difficult to describe. We will show that condition (c) requires point O to be on a second line, which is the locus of points such that $OM^2 - ON^2$ is constant, where M, N are the midpoints of segments AB, CD respectively. (See **128b**, corollary.) We do this by examining relationships within triangles AOB, COD.

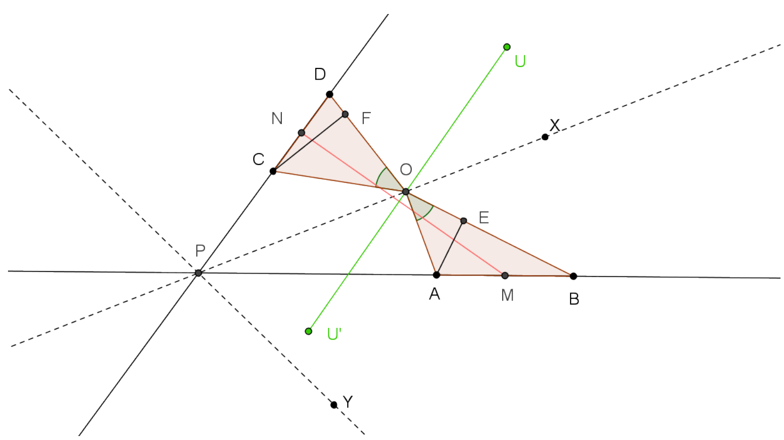

Figure t383

Let E be the foot of the perpendicular frm A to OB, and let F be the f of the perpendiculars from C to OD. From **126** we have:

$$AB^2 = OA^2 + OB^2 \pm 2OB \cdot OE, \tag{1}$$

$$CD^2 = OC^2 + OD^2 \pm 2OD \cdot OF. \tag{2}$$

Because $\widehat{AOB} = \widehat{COD}$, the ambiguous signs on the right are either both positive or both negative. And in fact we will show that these rightmost products in the equations above are equal. Indeed, from the equality of the same two angles we know that right triangles OAE, OCF are similar, so that

$$\frac{OA}{OE} = \frac{OC}{OF} \tag{3}$$

.

Now we use condition (a). Because triangles AOB, COD have the same area, and $\widehat{AOB} = \widehat{COD}$, we have (**256**):

$$OA \cdot OB = OC \cdot OD. \tag{4}$$

We now divide equation (4) by equation (3), to get

$$OB \cdot OE = OD \cdot OF. \tag{5}$$

This equation shows that the rightmost products in equations (1) and (2) are equal, and we have already noted that the signs are the same. This observation, together with the fact that we want to look at the difference of squares of certain segments, suggests that we subtract these first two equations. Doing so, we have:

$$AB^2 - CD^2 = OA^2 + OB^2 - OC^2 - OD^2. \tag{6}$$

We now use the result of **128** to write

$$OA^2 + OB^2 = 2OM^2 + \frac{1}{2}AB^2,$$

$$OC^2 + OD^2 = 2ON^2 + \frac{1}{2}CD^2.$$

Substituting these values into (6), we find:

$$OM^2 - ON^2 = \frac{1}{4}\left(AB^2 - CD^2\right). \tag{7}$$

By the result of **128b**, the locus of points satisfying (7) is some line UU', and point O must lie on this line. This argument shows that there is exactly one point O satisfying conditions (a), (b), and (c).

Note. For condition (c) to hold, it is necessary, but not sufficient, that point O be on line UU'. This is because we used condition (a) to define line UU'. It would have been more straightforward simply to find the locus of points satisfying condition (c) independent of the other conditions, but this problem is in general not an elementary one.

We now turn to the solution of the exercise itself. Most of the work has already been done, in proving our lemma. Suppose arcs AV_1B, CV_2D are the loci of points at which segments AB, CD both subtend angle V. Let B', D' be the second points of intersection of these arcs with lines OB, OD respectively (where O is the point located in our lemma). We have $\widehat{AOB} = \widehat{AB'B} \pm \widehat{OAB'}$, with the ambiguous sign depending on whether O lies inside or outside circle AV_1B. Analogously, $\widehat{COD} = \widehat{CD'D} \pm \widehat{OCD'}$. Since $\widehat{AB'B} = \widehat{CD'D} = V$ and $\widehat{AOB} = \widehat{COD}$ by construction, it follows that $\widehat{OAB'} = \widehat{OCD'}$, and therefore triangles OAB', OCD' are similar. Thus we have

$$OA : OB' = OC : OD'. \tag{8}$$

Dividing equation (4) by equation (8), we find that $OB \cdot OB' = OD \cdot OD'$, so O has the same power with respect to circles AV_1B and CV_2D. Thus O is always on the radical axis of these two circles. Since the position of O does not depend on the particular angle V (or equivalently, on the particular circles AV_1B, CV_2D, this proves the required statement.

Notes. We have assumed that angles $\widehat{AOB}$, $\widehat{COD}$ have the same orientation as well as being equal, and that the same is true for angles $\widehat{AV_1B}$, $\widehat{CV_2D}$. If these angles have opposite orientations, the same argument holds, but with the roles of points C and D reversed. Point O will be located at the intersection of line UU' with line PY (rather than PX).

Problem 384. A quadrilateral $ABCD$ (a *kite* or *rhomboid*) is such that the adjacent sides AD, AB are equal, and the other two sides are equal as well. Show that this quadrilateral is circumscribed about two circles. Find the locus of the centers of these circles if the quadrilateral is articulated, one of its sides remaining fixed.

Solution. Suppose the interior angle bisector at B in triangle ABC intersects AC at point O. Then (**115**) $AB : BC = AO : OC$. But from the problem statement, this is also the ratio $AD : DC$, which means that O is on the interior angle bisector at D of triangle ADC as well. It follows that O is equidistant from lines AB, BC, AD, DC, and so is the center of a circle tangent to lines AB, BC, CD, DA.

Similarly, the bisector of the exterior angle at B in triangle ABC intersects AC (extended) at point O' such that $AO' : O'C = AB : BC = AD : DC$, so O' is on the exterior angle bisector at D of triangle ADC, and so is the center of another circle tangent to lines AB, BC, CD, DA.

Now suppose segment AB remains fixed as the quadrilateral is articulated. Then point C describes a circle centered at B with radius BC. But the ratio $AO : OC = AB : BC$ remains fixed, as does point A, so O describes a circle homothetic to the one described by C, with center of homothecy at A and ratio

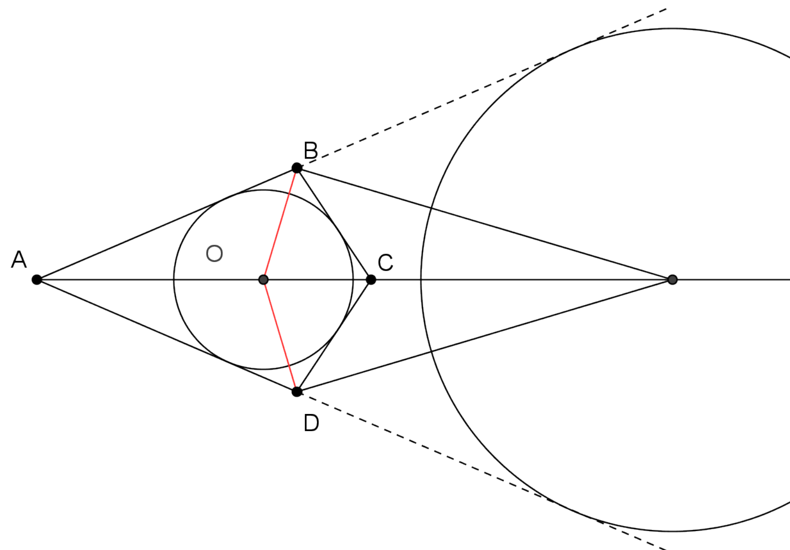

FIGURE t384

$AB : (AB{+}BC)$ (**142**). Similarly, O' describes a circle homothetic to that described by C, with center of homothecy at A and ratio $(AB - BC) : AB$.

Problem 385. More generally, if a quadrilateral $ABCD$ has an inscribed circle, and is articulated while the side AB remains fixed, then it has an inscribed circle in all its positions (Exercise 87). Find the locus of the center O of the inscribed circle.

(To make the situation definite, assume the inscribed circle is inside the polygon, and lay off lengths $AE = AD$ (in the direction of AB) and $BF = BC$ (in the direction of BA), both on side AB. Using Exercise 87, reduce the question to Exercise 257.)

Show that the ratio of the distances from O to two opposite vertices remains constant.

Solution. Quadrilateral $ABCD$ is circumscribed about a circle (*fig.* t385) if and only if $AD + BC = AC + BD$. As the quadrilateral is articulated, the lengths of its sides do not change, so this relationship either continues to hold or never holds. That is, the articulated quadrilateral always has an inscribed circle, if the original quadrilateral does, or never has an inscribed circle, if the original does not.

Next we find the locus of point O, the incenter, as the quadrilateral is articulated. To find this locus, we find E on line AB so that AE is in the same direction as AB and $AE = AD$. Similarly, we find F on line AB so that BF is in the same direction as BA and $BF = BC$. Note that E and F could both be inside segment AB, or one or both of them may lie outside AB. But in any case, $AE + BF = AD + BC > AB$, so segments AE and BF must overlap.

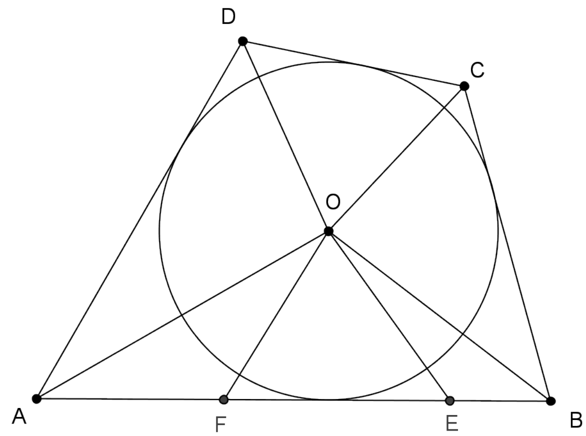

FIGURE t385

We will reduce the situation to that of exercise 257 by showing that segments AB, EF subtend supplementary angles at O.

We know that O is the intersection of the angle bisectors of $ABCD$, so triangles ODA, OEA are symmetric with respect to line AO and are therefore congruent. Hence $OD = OE$. Similarly, triangles OCB, OFB are symmetric with respect to line BO, so $OC = OF$.

We next note that $CD = EF$. Indeed, $AB + CD = AD + BC$, so $CD = AD + BC - AB = AE + BF - AB = EF$ (since AE and BF overlap). It follows that triangles OCD, OFE are congruent (**24,** case 3), so $\widehat{COD} = \widehat{EOF}$. But adding up the six angles in triangles AOB, COD we have:

$$\widehat{AOB} + \frac{1}{2}(\widehat{A} + \widehat{B}) + \widehat{COD} + \frac{1}{2}(\widehat{C} + \widehat{D})$$

$$= \widehat{AOB} + +\widehat{COD} + \frac{1}{2}(\widehat{A} + \widehat{B} + \widehat{C} + \widehat{D})$$

$$= \widehat{AOB} + \widehat{COD} + 180^\circ = 360^\circ.$$

So $\widehat{AOB} + \widehat{COD} = 180^\circ$, and

$$\widehat{AOB} + \widehat{EOF} = 180^\circ \tag{1}$$

as well. Therefore if O is a position of the incenter of $ABCD$, it lies on the locus of points at which AB, EF subtend supplementary angles. By the result of exercise 257, this locus is a circle whose center lies on line AB.

Conversely, any point on this locus is the incenter of some position of quadrilateral $ABCD$. Indeed, let point O satisfy condition (1), where E and F are constructed as above from some original position of articulated quadrilateral $ABCD$.

We can construct an articulated version of this quadrilateral such that O is its incenter, by finding triangles AOD, BOC congruent respectively to AOE, BOF. Then $\widehat{AOD}+\widehat{BOC}=\widehat{AOE}+\widehat{BOF}=\widehat{AOB}+\widehat{EOF}=180°$. Therefore $\widehat{AOB}+\widehat{COD}=360°-(\widehat{AOD}+\widehat{BOC}=180°$, so $\widehat{COD}=\widehat{EOF}$.

We know that O is the intersection of the angle bisectors at A and B of quadrilateral $ABCD$. So if $ABCD$ is circumscribed, then O must be its incenter. We now show that $ABCD$ is in fact circumscribed.

Now $OC=OF$ and $OD=OE$ by construction, so triangles COD, FOE are congrent (**24,** case 2), and $CD=EF$. Thus $AD+BC=AE+BF=AB+EF=AB+CD$, so that quadrilateral $ABCD$ is indeed circumscribed, and the locus of its incenter O coincides with the locus of points at which AB, EF subtend supplementary angles.

Finally, we show that the ratio $OA:OC$ is constant. Let S be the circle which is the locus of O. It is not obvious, but was shown in the solution to exercise 257, that when the endpoints of AB are inverted in S, their images are the endpoints of EF. In this case, A inverts onto F and B onto E. The result of exercise 242 tells us that the ratio of the distances of any point on circle S to two inverse points is constant. So, for example, the ratio $OA:OF$ is constant, and since $OF=OC$, the ratio $OA:OC$ is also constant. In the same way we can show that $OB:OD$ is also constant.

Notes. This generalization of exercise 384 is far from obvious, but it can be broken down into smaller sections which are not so hard, once the subgoal of each section is given. It is a bit of a challenge to construct a working model of the articulated quadrilateral $ABCD$ using dynamic geometry software.

Problem 386. Given four fixed points A, B, C, D on a circle, take an arbitrary point P in the plane, and denote by Q the second intersection point of the circles PAB and PCD. Find the locus of Q as P moves on a line or on a circle. Find the locus of points P such that Q coincides with P.

Solution. We observe first that lines AB, PQ, CD are the radical axes respectively of circles PAB, CAB, of circles PAB, PCD, and of circles ACD, PCD. Hence these three lines pass through the radical center O of these three circles (**139**). Since O is the intersection of lines AB, CD, it remains fixed as point P moves about the plane. This means that the product $OP\cdot OQ$ remains fixed: it is equal to the products $OA\cdot OB$ and $OC\cdot OD$, so its value is the power of O with respect to the original circle. This implies that P and Q are images of each other in an inversion about pole O with power equal to the power of O with respect to circle $ABCD$. This observation essentially solves the problem.

For instance, if point P describes a line not passing through O, then point Q describes a circle through O, and conversely (**220, 221**). If P describes a circle not passing through O, then Q describes a circle which is the inversion of that circle. If P describes a line or circle through O, Q likewise describes a line through O. Points P and Q coincide if and only if P lies on the circle of inversion.

Notes. The fact that point O remains fixed as points P varies is surprising in itself. Students might be shown this phenomenon with a dynamic sketch, and asked to explain it.

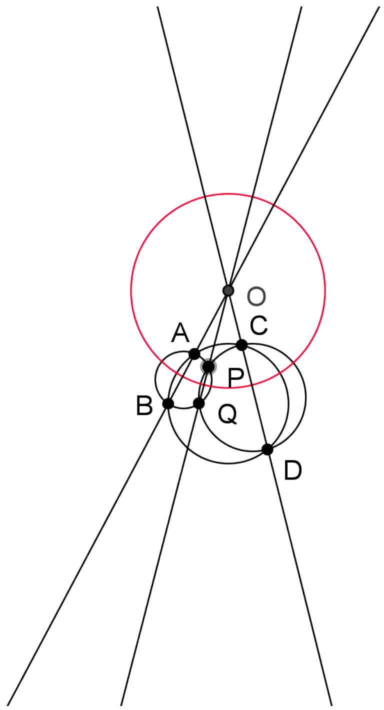

Figure t386

Problem 387. We join the vertices of a square $ABCD$ with an arbitrary point P in the plane. Let A', B', C', D' be the second points of intersection of these four lines with the circle circumscribed about $ABCD$. Show that $A'B' \times C'D' = A'D' \times B'C'$.

Conversely, let $A'B'C'D'$ be a cyclic quadrilateral such that $A'B' \times C'D' = A'D' \times B'C'$.

Find a point P such that the lines PA', PB', PC', PD' intersect the circumscribed circle in the vertices of a square.

Solution. In figure t367, we have (**131**) $PA \cdot PA' = PB \cdot PB' = PC \cdot PC' = PD \cdot PD' = p$, where p is the power of point P with respect to circle $ABCD$. That is, A', B,$'$, C', D' are the images of A, B, C, D under an inversion about pole P with power p. By **218**, then, we have:

$$A'B' = \frac{p \cdot BA}{PA \cdot PB}, \tag{1}$$

with analogous expressions for $B'C'$, $C'D'$, $D'A'$. By direct computation, taking into account that $AB = BC = CD = DA$, we have $A'B' \cdot C'D' = A'D' \cdot B'C' = \frac{p^2 c\dot{A}B^2}{PA \cdot PB \cdot PC \cdot PD}$.

We next consider the converse statement. Suppose cyclic quadrilateral $A'B'C'D'$ is such that $A'B' \cdot C'D' = A'D' \cdot B'C'$. (We assume that $A'B'C'D'$ is convex; that is, not self-intersecting.) Let us suppose that we know where point P (as described in the problem statement) lies. We consider the inversion around P as pole, with

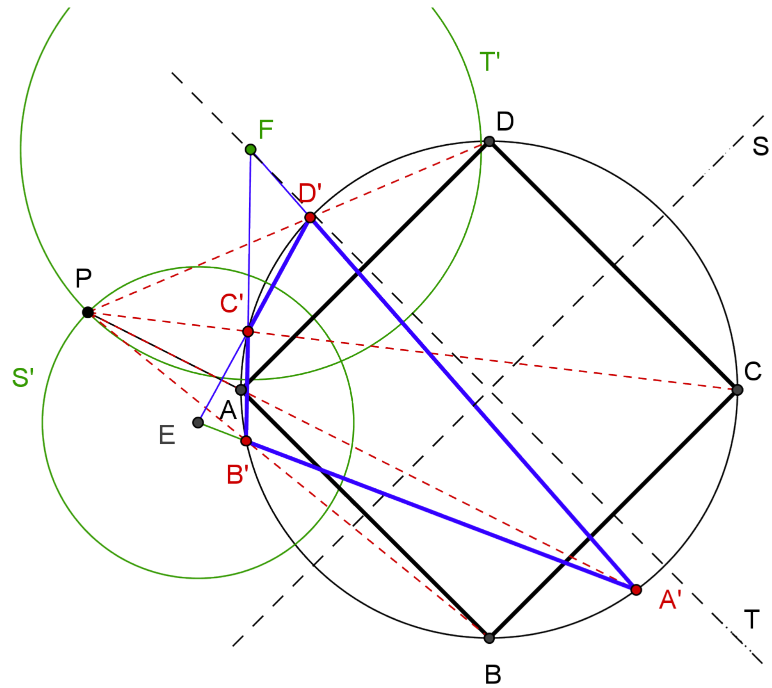

Figure t387

a power p equal to the power of P with respect to circle $A'B'C'D'$. That is, we assume that this inversion takes A', B', C', D' into A, B, C, D respectively, and these last points lie on circle $A'B'C'D'$. Note that this circle is its own image under this inversion.

Let us consider line S, the perpendicular bisector of AB. It is also the perpendicular bisector of CD (since we are assuming that $ABCD$ is a square), and we invert it, around pole P with power p. Line S is perpendicular to line AB, but also to any circle through A and B. Indeed, any such circle has its center on line S, so a tangent to S at their point of intersection will be perpendicular to the radius at that point; that is, perpendicular to S. The image of S will thus be a circle S' which is orthogonal to any circle through A' and B' (**219**), and in particular will be orthogonal to line $A'B'$. Likewise, S' will be orthogonal to every circle through C' and D', in particular to line $C'D'$. It follows that lines $A'B'$, $C'D'$ pass through the center E' of S'; that is, E' is the intersection of lines $A'B'$, $C'D'$. Finally, line S is certainly orthogonal to circle $ABCD$, so the images of these two objects are also orthogonal That is, circle S' is orthogonal to circle $A'B'C'D'$.

But does such a circle S' exist? Well, we have chosen quadrilateral $A'B'C'D'$ to be convex, so point E', the intersection of $A'B'$ and $C'D'$, lies outside the quadrilateral. Therefore S' is just the circle centered at E' with radius equal to the length of the tangent from E' to circle $A'B'C'D'$. We can now construct circle S'.

Likewise, line T, the perpendicular bisector of AD and BC, passes through point O, inverts into a circle T' whose center is at F', the intersection of lines $A'D'$, $B'C'$, and is orthogonal to circle $A'B'C'D'$. So we can construct circle T'.

Now under the inversion we seek, S', T' invert into lines S and T, so the pole that effects this inversion can only be one of their intersections P or Q of these two circles. These two circles must intersect, because one of them intersects arcs $\widehat{A'B'}$, $\widehat{C'D'}$, while the other intersects arcs $\widehat{A'D'}$, $\widehat{B'C'}$, of circle $A'B'C'D'$.

Finally, we find a pole P of inversion for which PA', PB, PC' PD' form a square. Let us look at an inversion with pole P and power equal to the power of P with respect to circle $A'B'C'D'$. We will show that in fact under this inversion, the image $ABCD$ of quadrilateral $A'B'C'D'$ is a square. Indeed, we know that any circle through A' and B' is orthogonal to S'. We look in particular at circle $PA'B'$, which inverts into line AB. Since circle $PA'B'$ is orthogonal to S', line AB is perpendicular to line S, the image of S'. For the same reason, line CD must be perpendicular to line S. Similar reasoning starting with circle T' shows that lines AD, BC must be perpendicular to line T. This reasoning shows that $ABCD$ is a parallelogram. And since A', B', C', D' line on a circle, their images A, B, C, D must lie on a circle (the image of the circle through $A'B'C'D'$). Thus parallelogram $ABCD$ must be a rectangle. Finally, the algebraic reasoning associated with equation (1) leads us from the relation $A'B' \cdot C'D' = A'D' \cdot B'C'$ to the relation $AB \cdot CD = AD \cdot BC$, and if we apply this formula to a rectangle (whose opposite sides must be equal), we quickly find that the rectangle is in fact a square.

The same reasoning applies to an inversion around point Q. These two points, constructed as indicated, give solutions to the problem: lines connecting them to A', B', C', D' intersect circle $A'B'C'D'$ again in the four vertices of a square.

Note. We have indicated in passing how to construct a circle with a given center and perpendicular to a given circle. Students can be asked to do this construction as an exercise, before undertaking this problem.

(This is a particular case of Exercise 270b, 5^o. However, the problem here admits of two solutions, while there is only one in the general case. What is the reason for this difference?)

Solution. The problem has two solutions because quadrilateral $A'B'C'D'$ is assumed to be cyclic. See, for example the solution to exercise 270, 5°, note 2.

Problem 388. More generally, find an inversion which transforms the vertices A', B', C', D' of a cyclic quadrilateral into the vertices of a rectangle.

Show that the poles are the limit points (Exercise 152) of the inscribed circle and of the third diagonal of quadrilateral $A'B'C'D'$.

Solution. Suppose (*fig.* t388) that an inversion with pole P transforms cyclic quadrilateral $A'B'C'D'$ into rectangle $ABCD$. We first show that we can find an inversion such that $ABCD$ is in fact inscribed in the same circle as $A'B'C'D'$.

We follow the reasoning of exercise 387. As before, it is not hard to see that the line S that joins the midpoints of sides AB, CD must be orthogonal to any circle through A and B, and also to any circle through C and D. Therefore, under the inversion we seek, the image S' of S is a circle orthogonal to lines $A'B'$, $C'D'$, and also to circle $A'B'C'D'$. The center of S' must be the intersection E' of lines $A'B'$, $C'D'$, and its radius must be the length of the tangent from E' to circle $A'B'C'D'$.

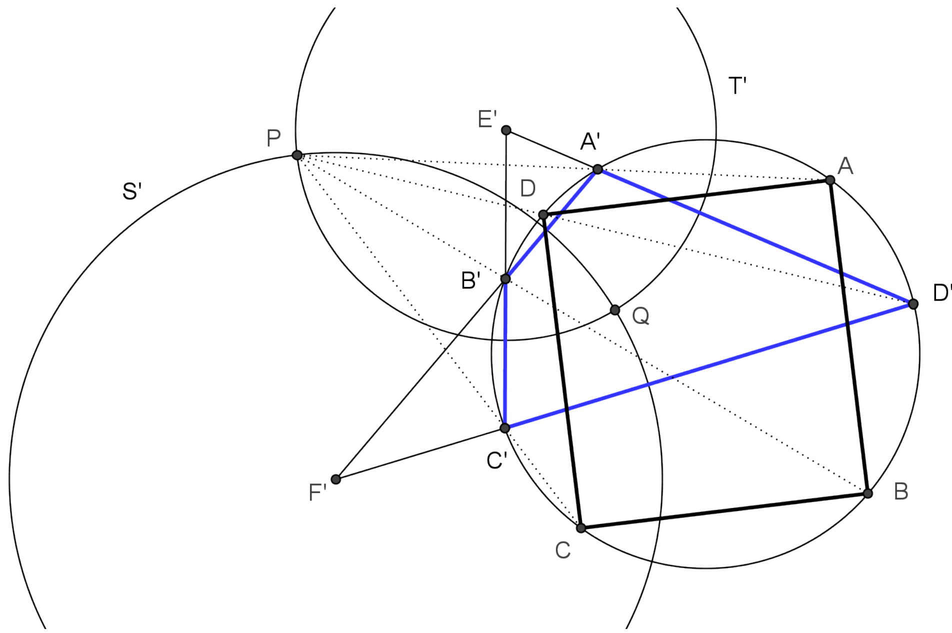

Figure t388

Analogously, the line T joining the midpoints of sides AD, BC inverts into a circle T' whose center F' is the intersection of lines $A'D'$, $B'C'$, and whose radius is the length of the tangent from F' to circle $A'B'C'D'$. So we have enough information to construct circles S', T'.

Now the inversion we seek will take circles S', T' into lines S, T, so its pole must be one of the two intersections P, Q of the circles. And if we want $ABCD$ to be inscribed in circle $A'B'C'D'$, we need only take the power of the inversion to be the power of P or Q with respect to circle $A'B'C'D'$. But in fact, by **215** any inversion with P or Q as its pole will transform $A'B'C'D'$ into a rectangle, usually with a different circumcircle.

We now prove the statement in the problem concerning limit points. The third diagonal of $A'B'C'D'$ (considered as a complete quadrilateral) is simply $E'F'$. By construction (the lengths of their radii), circles S', T' are orthogonal to circle $A'B'C'D'$. And they are certainly orthogonal to line $E'F'$ (their common center-line). It follows (exercise 152) that P and Q are the limit points of line $E'F'$ and circle $A'B'C'D'$.

Problem 389. Still more generally, find an inversion which transforms four given points into the vertices of a parallelogram.

Solution. Suppose there is an inversion taking four given points A', B', C', D' into a parallelogram $ABCD$. If A', B', C', D' all lie on the same circle, then so do A, B, C, D, and we are led to the situation in exercise 388. So let us assume that A', B', C', D' do not lie on the same circle.

Since triangles ABC, ADC must be congruent, so must circles ABC, ADC. Line AC is the extension of their common chord, so the two circles are symmetric in line AC. That is, AC forms equal angles with both these circles.

Let us see what this implies for the original diagram, before inversion. Circle ABC is the inversion of circle $A'B'C'$. Circle ADC is the inversion of circle $A'D'C'$. Line AC is the image of some circle S' through the pole of inversion. Circle S' must form equal angles with circles $A'B'C'$, $A'D'C'$; that is, it must bisect the angle between arcs $\widehat{A'B'C'}$ and $\widehat{A'D'C'}$.

Likewise, the pole of inversion must lie on a circle T' bisecting the angle between arcs $\widehat{B'A'D'}$ and $\widehat{B'C'D'}$. Circles S' and T' intersect twice, and either point of intersection can be taken as the pole of the required inversion. As in exercise 388, the power of the inversion can be arbitrary.

Notes. In this argument, we chose arcs $\widehat{A'B'C'}, \widehat{A'D'C'}$ because we want A' and D' to lie on opposite sides of circle S', just as their images A, D lie on opposite sides of line AC.

Students can be given the auxiliary problem of constructing a circle forming equal angles with two given circles. See **227**.

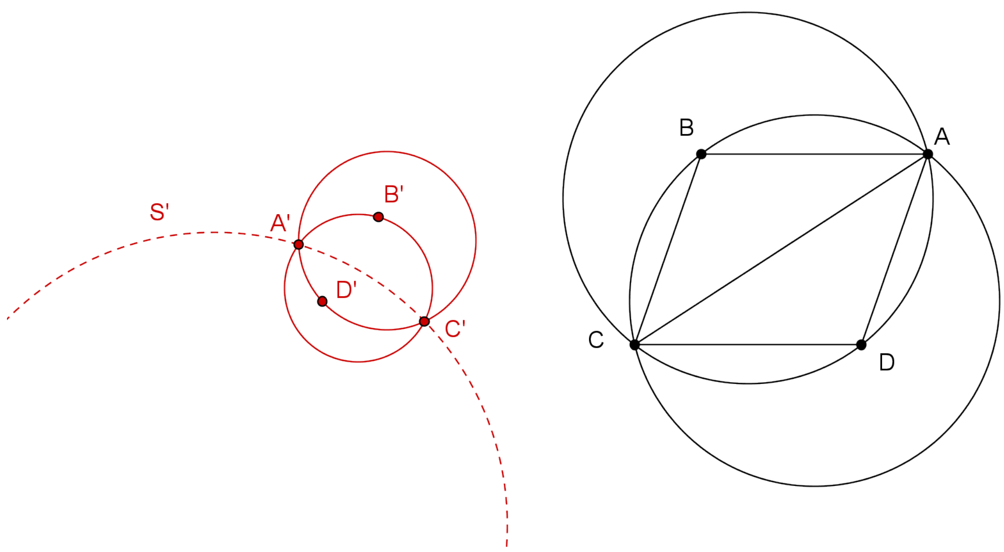

FIGURE t389

Problem 390. Given two circles and a point A, find an inversion in which the point corresponding to A is a center of similarity of the transformed circles.

Lemma. A line intersecting two circles at equal angles must pass through one of their centers of similarity, and conversely.

Proof of lemma. Radii of the two circles, drawn to their points of intersection with the given line, are parallel in pairs. Thus the intersection of the given line with the common centerline of the two circles is a center of similarity of the two circles.

Conversely, if a line passes through a center of similarity of two circles, pairs of radii to the points of intersection are homothetic, thus parallel. It follows that the line makes equal angles with the two circles.

Note. This lemma tells us that we can characterize the centers of similarity of two circles as the point common to all the lines which intersect them at equal angles.

Students can fill in the details of the proof of this elementary lemma. See **227**.

Solution. Suppose the required inversion is I, and it takes point A onto point A'. Then the set of circles through A which intersect the given circles C_1, C_2 at equal angles are transformed into the set of lines passing through A'.

There are two inversions J and K which take C_1 onto C_2 (**227**). A circle intersecting C_1 and C_2 at equal angles will be its own image under either J or K. It follows that any circle through A which forms equal angles with C_1 and C_2 must pass either through point P_J, the inverse point to A under J, or through P_K, the inverse point to A under K. In order that these circles invert into lines, we must take either P_J or P_K as the pole. The power of inversion can be chosen arbitrarily.

Problem 391. A variable point M on a circle is joined to two fixed points A, B. The two lines intersect the circle again at P, Q. Denote by R the second intersection of the circle with the parallel to AB passing through P. Show that line QR intersects AB at a fixed point.

Use this result to find a method of inscribing a triangle in a given circle with two sides passing through given points, while the third is parallel to a given direction; or such that the three sides pass through given points. (These two questions are easily reduced to each other, and to 115).

Solve the analogous problem for a polygon with an arbitrary number of sides. (Another method is proposed in Exercise 253b.)

Note. We break this problem statement down into seven parts:

1°. The intersection of QR and AB is fixed.

2°. Inscribe a triangle in a given circle, with two sides passing through given points, while the third is parallel to a given direction.

3°. Inscribe a triangle in a given circle, whose three sides pass through three given points.

4°. Inscribe a polygon in a given circle with an *even* number of sides, such that one side passes through a given point while the others are parallel respectively to a set of given lines.

5°. Inscribe a polygon in a given circle with an *odd* number of sides, such that one side passes through a given point while the others are parallel respectively to a set of given lines.

6°. Inscribe a polygon in a given circle, such that a number of *consecutive* sides pass through a set of given points while the others are parallel respectively to a set of given lines.

7°. Inscribe a polygon in a give circle, such that a number of sides, located arbitrarily around the figure, pass through a a given set of points while the others are parallel respectively to a set of given lines.

Solution 1°**.** In figure t391a, point S is the intersection of lines QR and AB. We have $\widehat{RQM} = \frac{1}{2}\ \widehat{RM} = \widehat{RPM} = \widehat{BAM}$, and certainly $\widehat{MBA} = \widehat{SBQ}$, so triangles MBA, QBS are similar, so that $BS = \frac{BM \cdot BQ}{AB}$. The product $BM \cdot BQ$ (the numerator of this fraction) is just the power of B with respect to the circle, and so does not change as point M varies along the circle. And the denominator of the fraction is certainly constant, so BS is constant, and point S does not move as M varies.

Note. We can phrase this result slightly differently. For any two points A, B on the circle, there is an associated fixed point S with the property described in the problem statement.

Solution 2°**.** We will adapt the notation of figure t391a to address this construction. We show how to construct a triangle MPQ with sides passing through fixed points A, B and a third side parallel to some line KL.

We find the fixed point S associated (as in 1°) with points A and B. We then use the result of exercise 115 to construct auxiliary triangle PQR, inscribed in the given circle, with one side parallel to line AB and another to KL, and the third side (QR) passing through S. Indeed, by construction (of point S), line RS must intersect the circle again at a point collinear with M and B, which thus must be Q. Then triangle MPQ satisfies the required conditions.

Having constructed PQR, we draw line PA, and label its second intersection with the circle as M. Then we draw line MB. It is not hard to see that the second intersection of MB with the given circle must be Q. Indeed, the intersection of MB and the given circle must be collinear with RS (by 1°), so it must be point Q.

Solution 3°**.** We again use the notation of figure t391a, but in a different way. Let the three given points be A, B and C. Again, we determine point S, dependent on A and B, as indicated in the result to our main problem. Construction 2° above allows us to find a triangle PQR, two of whose sides pass through points C and S, and whose third side PR is parallel to line AB.

Having constructed triangle PQR, we determine M as the second intersection of line PA with the circle. As in the previous construction, line MB must intersect the circle again at Q, and triangle MPQ satisfies the conditions of the problem.

4°**.** We solve the problem for a hexagon. The generalization to any even number of sides is immediate.

Suppose the required figure is $PQRSTU$, that side PQ passes through a given point A, while the other sides are parallel respectively to lines Q_1R_1, R_2S_1, S_2T_1, T_2U_1, SU_2P_1.

We choose an arbitrary point Q' on the circle, and construct a polygon starting with Q' as one vertex by drawing lines parallel in turn to the given lines. We get polygon $P'Q'R'S'T'U'$ (*fig.* t391b), whose sides are parallel respectively to the required polygon. Because chords QR, $Q'R'$ are parallel, arcs $\widehat{QQ'}$, $\widehat{RR'}$ are equal, but oppositely oriented. Similarly, arcs $\widehat{R'R}$, $\widehat{SS'}$, $\widehat{T'T}$, $\widehat{UU'}$, $\widehat{P'P}$ are all equal, and consecutive arcs (in this list) are oppositely oriented. Since the number of sides of the polygon is even (here that number is 6), it follows that arcs $\widehat{P'P}$, $\widehat{QQ'}$ are equal and oppositely oriented. It follows that chords $P'Q'$, PQ are parallel. (Note that we began with only five given lines; chords PQ, $P'Q'$ are not parallel *a priori*).

We have constructed a polygon inscribed in the given circle, with sides parallel to the (five) given lines. It remains to arrange for the last side to pass through the given point. But this is easy. If the given point is A, we draw a parallel to PQ through A. Its intersection points P and Q with the given circle are one side of the required polygon, and the others are found by drawing parallels as before.

Notes. This construction, as well as the next, does not depend on the very first result in this exercise. We will not use that result until we come to statement 6°. Thus students can be given this problem independent of the others in this sequence. Or, they can be asked first to solve the simpler problem of inscribing in a circle any hexagon (or polygon wtih evenly many sides) with all but one of its sides parallel to a set of lines. They will find, in the process, that this condition determines the direction of the sixth side of the hexagon (but not its length).

Solution 5°**.** The argument is only slightly different from the proof of 4°. We solve the problem for a pentagon. The generalization to an arbitrary odd number of sides is immediate.

Suppose the required polygon $PQRST$ (*fig.* t391c), is such that side PQ passes through a given point A, and the other sides are parallel respectively to lines Q_1R_1, R_2S_1, S_2T_1, T_2P_1.

We choose an arbitrary point Q' on the circle, and construct a polygon starting with Q' as one vertex by drawing lines parallel in turn to the given lines. We get polygon $P'Q'R'S'T'$ (*fig.* t391c), whose sides are parallel respectively to the required polygon. As before, arcs $\widehat{QQ'}$, $\widehat{SS'}$, $\widehat{T'T}$, $\widehat{P'P}$ are all equal, but this time, because there are oddly many sides in the polygon, arcs $\widehat{P'P}$, $\widehat{QQ'}$ are oriented in the same direction. It follows (by adding arc $\widehat{P'Q}$ to both) that arcs $\widehat{P'Q'}$, $\widehat{PQ}$ are equal, so chords $P'Q'$, PQ are also equal. This construction, starting with an arbitrary point, gives us the length of PQ. Thus we can start our construction of the polygon by drawing a chord through the given point A and equal in length to PQ.

Notes. We have shown, within the argument, that the length of side $P'Q'$ of a polygon constructed as in the solution, does not depend on the choice of point Q'.

Students can complete the construction, by recalling how to draw a chord of a given length through a given point. They can recall that this construction is possible whenever the given length is between the diameter of the circle and the minimal length of a chord through A. The length of PQ is determined by the directions of the given lines, and students can think about when the construction indicated in 5° is possible.

In 4°, we saw that for even n, if we know the directions of $n-1$ sides of an n-gon, then the direction of the last side is determined, but not its length. Now we see that for odd n, if we know the directions of $n-1$ sides of a n-gon, then the length of the last side is determined, but not its direction. For this reason, the construction is not always possible when n is odd.

Solution 6°**.** If only one of the sides of the polygon is required to go through a given point, the problem is solved in 3° and 4°. We solve the problem for a hexagon. The generalization to any number of sides offers no new difficulties.

Suppose the required hexagon is $PMQUVW$ (*fig.* t391d), in which sides PM and MQ pass through given points A and B respectively, and choose these sides so that side WP is required to be parallel to a given line; that is, so that proceeding around the figure (in a clockwise direction, for figure 391d) , PM and MQ are the first two sides we encounter which are required to go through given points. We will show that if we can construct another hexagon $PRQUVW$ in which one fewer side is required to go through a fixed point, then we can also construct $PMQUVW$.

Indeed, we have essentially done this, in 2°. Figure t391d is labeled similarly to figure 391a, and in that figure, we know that if can construct triangle PQR, (with two sides in given direction and a third passing through a given point), we can also construct triangle PMQ. This shows us how to construct hexagon $PMQUVW$, if we have already constructed hexagon $PRQUVW$. That is, we have reduced by one the number of sides required to pass through a given point.

Because we have assumed that the sides passing through points are consecutive around the required figure, we can continue this process, finally arriving at the construction of 4° or 5°.

Solution 7°**.** We use auxiliary polygons to reduce this problem to the situation in 6°. That is, we show that the construction of our polygon can be made to depend on that of another polygon, in which the order of the sides satisfying two different sorts of conditions is reversed. Applying this result several times, we can arrange that all the sides of the auxiliary polygon which are required to pass through a given point are consecutive, which is the situation in 6°. We can thus construct the sequence of auxiliary polygons, and arrive a the required figure.

Suppose, for example, we are required to construct hexagon $PQRSTU$ (*fig.* t391e), in which side PQ must be parallel to a given line KL, and side QR must pass through a given point A. We will show that this construction depends on the construction of an auxiliary hexagon PQ_1RSTU, in which PQ_1 passes through a given point, and Q_1R is parallel to a given line.

We draw chord RQ_1 parallel to KL through point R, and let A_1 be the intersection of line PQ_1 with the line parallel to KL through A. Then $PQRQ_1$ is an isosceles trapezoid, and therefore so is ARQ_1A_1. Thus points A, A_1 are the same distance from the center O of the given circle, and we can construct point A_1, without knowing hexagon $PQRSTU$, as the intersection of a circle of radius OA with the parallel through A to KL. Now if we know how to construct polygon PQ_1RSTU, we can construct polygon $PQRSTU$.

But in PQ_1RSTU, side PQ_1 must go through a given point A_1, and side Q_1R must be parallel to the given line KL. That is, we have reversed the order of the sides satisfying different requirements. Following the plan given above, we can then construct polygon $PQRSTU$.

Note. In figure t391e, polygon PQ_1RSTU is not a 'proper' polygon, in the sense of **21**. Two of its sides intersect at a point which is not a vertex. We must in general allow such figures for our auxiliary polygons. If we want to say that the required polygon itself can always be constructed, we must likewise allow such re-entrant figures as solutions.

Problem 392. About a given circle, circumscribe a triangle whose vertices belong to given lines.

Solution. We use the method of poles and polars (see **206**). Since the polar of a tangent to a circle is its point of contact, (**204**), the polar of the circumscribed triangle is the inscribed triangle formed by the three points of intersection of its sides with the circle. Since the vertices of the original triangle lie on certain lines a, b, c, the sides of the new inscribed triangle must pass through the points A, B, C which are the polars of these lines.

Thus we have the following construction. We take the polars of the three given lines with respect to the circle, then use the result of exercise 391 to draw a triangle inscribed in the circle, whose sides pass through these three points. We then take the polar figure to this inscribed triangle to get the required circumscribed triangle.

Problem 393. Given two points A, B on a line, we draw two variable circles tangent to the line at these points, and also tangent to each other. These two circles have a second common (external) tangent $A'B'$. Show that as the two tangent circles vary, the circles on diameter $A'B'$ remain tangent to yet another (fixed) circle. Find the locus of the midpoint of $A'B'$.

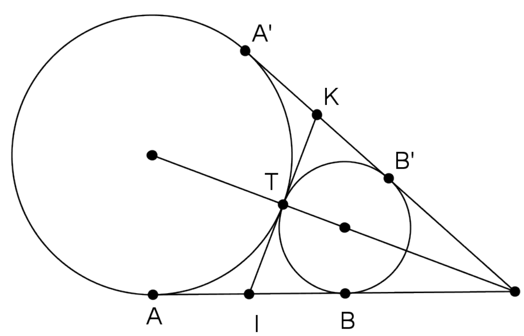

FIGURE t393

Solution. Suppose (*fig.* t393) T is the point of tangency of the two circles, and that IK is their common internal tangent, with I on line AB and K on line $A'B'$. Then we know (**92**) that $IT = IA = IB = \frac{1}{2}AB$. It follows that point I is the midpoint of AB, and is fixed (for any two such circles). We also know that $TK = KA' = KB'$. Since the common centerline of the two circles is a line of symmetry for the whole figure, we know that $IT = TK$, so $IK = 2IT = AB$, a fixed distance. Hence the locus of point K (the locus of midpoints of $A'B'$) is a circle centered at I with radius AB.

Now any circle on diameter $A'B'$ (for any two positions of the intial tangent circles) must have a radius equal to $A'K = B'K = TK = TI = AI = \frac{1}{2}AB$, and its center moves along the circle with center I described in the previous paragraph. Hence any such circle is tangent to a fixed circle centered at I, of radius $\frac{3}{2}AB$. And in fact, it is also tangent to another fixed circle, also centered at I, with radius $\frac{1}{2}AB$.

Problem 394. Two variable circles C, C_1 are tangent at a point M, and tangent to a given circle at given points A, B.

1^o. Find the locus of M;

Solution $1°$**.** In figure t394a, point P is the intersection of the tangents to the given circle O. We use the result of **139**. Line PA is the radical axis of circles O, C, and line PB is the radical axis of circles O, C_1, so P is the radical center of circles O, C, C_1, and lies on the radical axis of C, C_1. But this radical axis is simply the common tangent PM of these two circles. From **92**, it follows that $PA = PB = PM$. Since $PA = PB$ is constant, so is PM, and point M lies on a circle centered at P with radius PA.

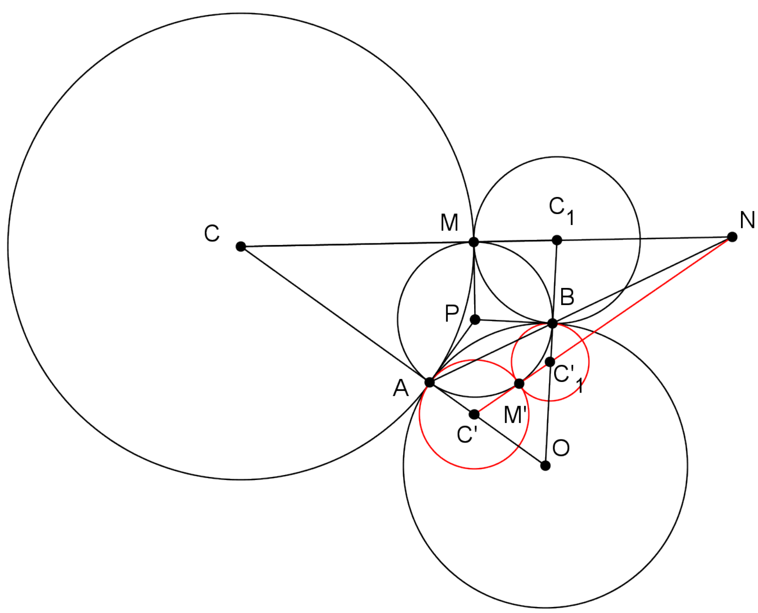

FIGURE t394a

Note. It is not hard to see that any point on this circle can serve as a position of point M. For some positions, circles C, $C_!$ are tangent *internally* to circle O, and when P coincides with A or B, one of the circles degenerates to a point.

2^o. Find the locus of the second center of similarity of C, C_1;

Solution. It follows from the discussion of **227** that points A and B are antihomologous points in circles C, C_1. Thus the second center N of similarity lies on line AB.

But the locus of N is not the whole line: N must lie outside circle P constructed in part 1°. To prove this, we note first that the common centerline of $C,\ C_1$ tangent to circle P. Indeed, M is the foot of the perpendicular to this common centerline, and is also the point of contact of the common tangent from P to the two circles. Now N lies on the common centerline of $C,\ C_1$ (**143**), which is tangent to circle P and this centerline is tangent to circle P.

It follows that N, being on line CC_1, must lie outside circle P and therefore must also lie outside circle O. So the locus of N is that part of line AB lying outside circle O. (The proof of this statement is actually completed in 3°)

Note. Within this proof, we have shown that if three circles are tangent externally, then their common centerlines are tangent to the circle centered at their radical center whose radius is the length of the tangent to any of the circles from the radical center. Students can be asked to prove this result independently of the rest of the problem.

3^o. To each point N of the preceding locus there correspond two pairs of circles $C, C_1;\ C', C_1'$ satisfying the given conditions, and therefore two points of tangency $M,\ M'$.

Solution. We can construct circle P independent of the choice of N. Then, for any position of N, we can draw tangents $NM,\ NM'$ to this circle (*fig.* t394a). One pair of centers is given by the intersections of $OA,\ OB$ with tangent NM, and the other pair of centers is the corresponding intersections with NM'.

Note. This statement provides the converse to 2° by showing that every point N on the locus claimed does in fact serve as the center of similarity of a pair of circles $C,\ C_1$; in fact, to two pairs of such circles.

Find the locus of the center of the circle circumscribing NMM', the locus of the circle inscribed in this triangle, and the locus of the intersection of its altitudes. Each common point of pairs of these loci belongs to the third.

Solution. We break this statement into three parts.

4°**.** To find the locus of the circumcenter ω of NMM', we note that angles $\widehat{NMP},\ \widehat{NM'P}$ are right angles, so the circle on diameter NP passes through points $M,\ M'$; that is, this circle is the circumcircle of NMM', whose center ω is therefore the midpoint of segment NP. Since P is fixed, we can describe the locus of ω as the image of the locus of N under a homothecy centered at P with factor $\frac{1}{2}$.

Since the locus of N consists of the extensions outside circle P of segment AB, the locus of ω is the set of points on line $A'B'$, where A' and B' are the midpoints respectively of $PA,\ PB$, but outside of segment $A'B'$(*fig.* t394b).

5°**.** To find the locus of the incenter I of triangle NMM', we note that it is located on the bisector NP of angle $\widehat{MNM'}$, and also on the bisector of angle $\widehat{NMM'}$.

We now show that point I in fact lies on circle P. Indeed, we have $\widehat{PMI} = 90^\circ - \widehat{NMI} = 90^\circ - \widehat{IMM'} = \widehat{PIM}$, so triangle PIM is isoscele, with $PM = PI$.

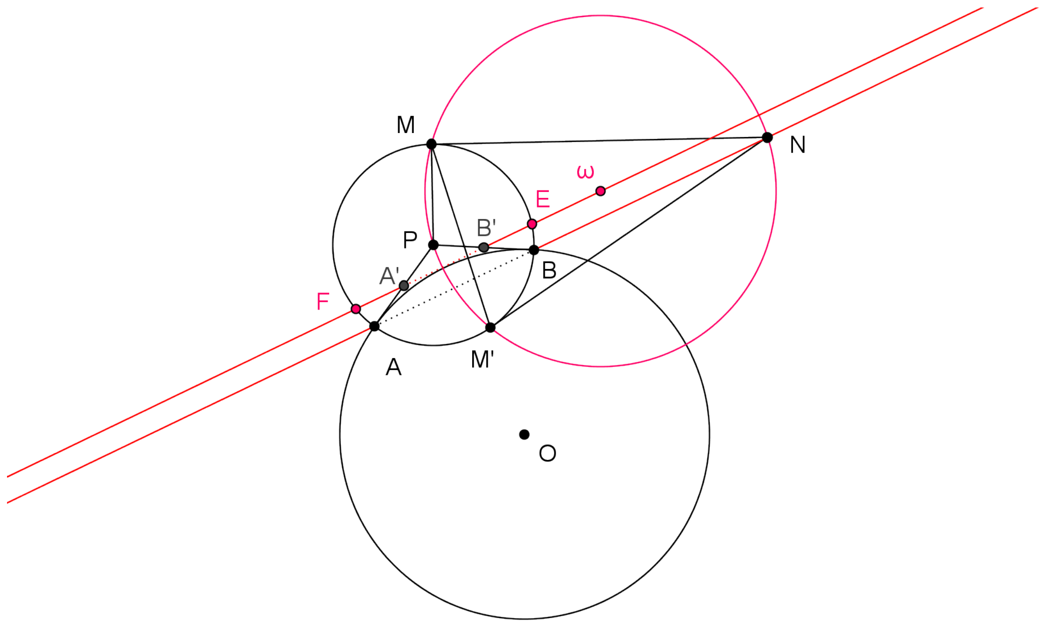

FIGURE t394b

Thus I is on circle P. But (by symmetry in line PN), the extension of PI must pass through N. That is, point I lies on the intersections of PN with circle P. These points are those on two arcs lying between lines AB and the parallel to AB through P (*fig.* t394c).

6°**.** We will find the locus of the intersection of the altitudes (the orthocenter) of triangle NMM' in a rather indirect fashion. We will first find the locus of the midpoint K of base MM' of the isosceles triangle. Then we relate positions of point K to positions of point H using the result of exercise 70.

We start with some results from the theory of poles and polars. In figure 394d, we note that line MM' is the polar of point N with respect to circle P. Since line AB passes through point N, the pole of line AB lies on the polar of N (**205**). But the polar of AB with respect to circle P is just point O (as we proved in the note to 2°, OA and OB are the tangents to circle P at the endpoints of chord AB). So line MM' passes through point O.

Let K be the foot of the altitude from N in triangle NMM', and consider the circle Σ on diameter OP. Since angle $\widehat{PKO}$ is a right angle, point K lies on circle Σ. But OB is tangent to circle P at B, so angle $\widehat{PBO}$ is also a right angle, and B is also on Σ. We can show in the same way that A is on Σ. Note that points $O,\ P$ do not vary with point N, so circle Σ also does not vary. And in fact it is easy to see that as N varies along line AB (but outside of circle P) , the locus of point K is arc $\widehat{APB}$ of circle Σ.

Now point P is equidistant from lines $NM,\ NM'$, and so is on the bisector of angle $\widehat{N}$ of triangle NMM'. Since this triangle is isosceles, this angle bisector is also an altitude. So point P is the intersection of an altitude of triangle NMM'

with its circumcircle. It follows from exercise 70 that the orthocenter H of the triangle is symmetric to P in line MM', or equivalently, in point K.

That is, $PH = 2PK$, and point H is the homothetic image of point K, with (fixed) point P as center and a factor of 2. Thus its locus is the homothetic image of an arc of circle Σ', homothetic to Σ with factor 2.

7°**.** We need to show that the loci of 4°, 5° and 6° are concurrent. Let us consider the first two loci separately. The locus of 4° is that part of line $A'B'$ lying outside circle P. The locus in 5° consists of two arcs of circle P. So these two loci cannot intersect in more than two points labeled E and F in figure t394b.

For positions of N where these loci coincide, say E, the incenter and circumcenter of triangle NMM' must also coincide with E, which means that the triangle is equilateral, and not just isosceles. But the orthocenter of an equilateral triangle also coincides with the incenter and circumcenter, and hence with point E. This means that point E is on circle Σ', the locus of H. Likewise, point F is on circle Σ'.

Problem 395. Two circles C, C' meet at A, and a common tangent meets them at P, P'. If we circumscribe a circle about triangle APP', show that the angle subtended by PP' at the center of this circumscribed circle is equal to the angle between circles C, C', and that the radius of this circle is the mean proportion between the radii of circles C, C' (which implies the result of Exercise 262, 3^o). Show that the ratio $\frac{AP}{AP'}$ is the square root of the ratio of these two radii.

Solution. The exercise asks us to prove three statements.

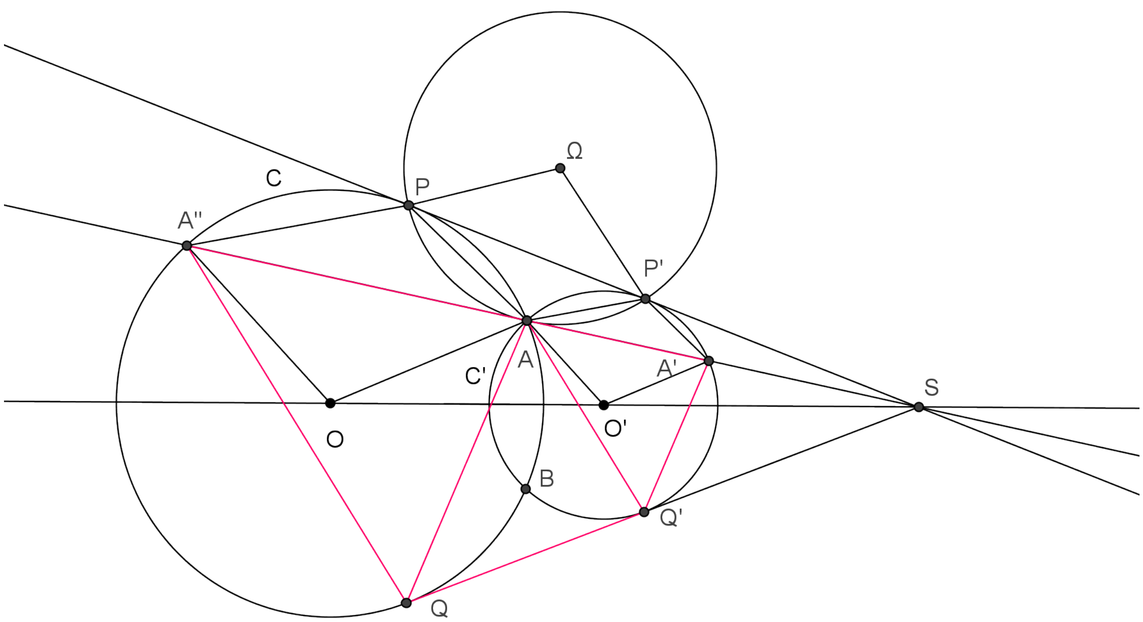

FIGURE t395

$1°$**.** We first prove the result about the angle subtended by PP' at the circumcenter of triangle APP'.

Let S be the external center of similitude for circles C, C', and let A'', A' be the second points of intersection of line SA with circles C, C' respectively (*fig.* t395). Then clearly triangles APA'', $A'P'A$ are homothetic with center S of homothecy. (In particular, they are similar.) And (**73, 74**) angles $\widehat{PA''A}$, $\widehat{APP'}$ are equal, as each is equal to half arc $\widehat{AP}$ in circle C. Likewise, $\widehat{P'A'A} = \widehat{AP'P} = \frac{1}{2}\ \widehat{AP'}$ in circle C'. It follows that triangles $A''PA$, PAP', $AP'A'$ are similar.

Now we note that the three circumcenters O, O', Ω of these triangles, considered as parts of the triangles, are corresponding points. Therefore quadrilaterals $A''PAO$, $PAP'\Omega$, $AP'A'O'$ are also similar.

From these three quadrilaterals we have: $\widehat{OAP} = \widehat{\Omega P'A}$ and $\widehat{O'AP'} = \widehat{\Omega PA}$. Therefore $\widehat{P\Omega P'} = 360° - \widehat{\Omega PA} - \widehat{PAP'} - \widehat{\Omega P'A} = 360° - \widehat{O'AP'} - \widehat{PAP'} - \widehat{OAP} = \widehat{OAO'}$. That is, the angle subtended by PP' at the center Ω of the circle circumscribing triangle APP' is equal to the angle between the radii of circles C, C' drawn to their point of intersection. It is not hard to see that this angle is equal to the angle between the tangents to these circles at their point of their intersection, which is the angle between the two circles themselves.

Note. Students may have trouble accepting the argument that quadrilaterals $A''PAO$, $PAP'\Omega$, $AP'A'O'$ are similar. They can avoid this difficulty, but lengthen the argument, by seeing that triangles $OA''A$, $\Omega PP'$, $O'AA'$ are also similar, then use sides of these triangles and combinations of their angles with the angles of similar triangles $A''PA$, PAP', $AP'A'$. This longer argument offers no new difficulties.

$2°$**.** Next we prove the result about the circumradius of triangle APP'.

Let r, r', ω be the radii of circles C, C', Ω respectively. From the similar quadrilaterals pointed out in $1°$ we have

$$r : \omega = AP : AP',\ \omega : r' = AP : AP'.$$

It follows from these two proportions above that $r : \omega = \omega : r'$, so ω is the mean proportion between r and r'.

$4°$**.** Finally, we get the result of exercises 262, $3°$. Let B be the second intersection of circles C, C', and let Q, Q' be the points of contact of the second common tangent to those circles. Then it is clear from symmetry in line OO' that the circumradii of APP', BQQ' are equal.

We can repeat the argument of $1°$ to show that triangles $A''QA$, QAQ', $AQ'A'$ are similar, so that the quadrilaterals formed by those three triangles and their circumcenters are also similar, and therefore the circumradius of AQQ' is likewise the mean proportion between r and r'. By symmetry in line OO', the same is true of the circumradius of triangle BPP'. Thus we obtain the result of exercise 262, $3°$.

Problem 396. What are necessary and sufficient conditions which four circles A, B; C, D must satisfy in order that they can be transformed by inversion so that the figure formed by the first two is congruent to that formed by the second two?

(Using the terminology introduced in Note A, **289, 294**, what are the *invariants*, under the group of inversions, of the figure formed by two circles?)

1^o. If circles $A,\ B$ have a common point, it is necessary and sufficient that the angle of these two circles equal the angle of $C,\ D$; or, which is the same (by the preceding exercise), that the ratio of the common tangent to the geometric mean of the radii be the same in both cases;

Solution $1°$**.** Two circles will always invert into lines or circles meeting at the same angle. So unless circles $A,\ B$ meet at the same angle as circles $C,\ D$, the second pair cannot be congruent to an image of the first pair under inversion. That is, the condition that the pairs of circles meet at the same angles is necessary.

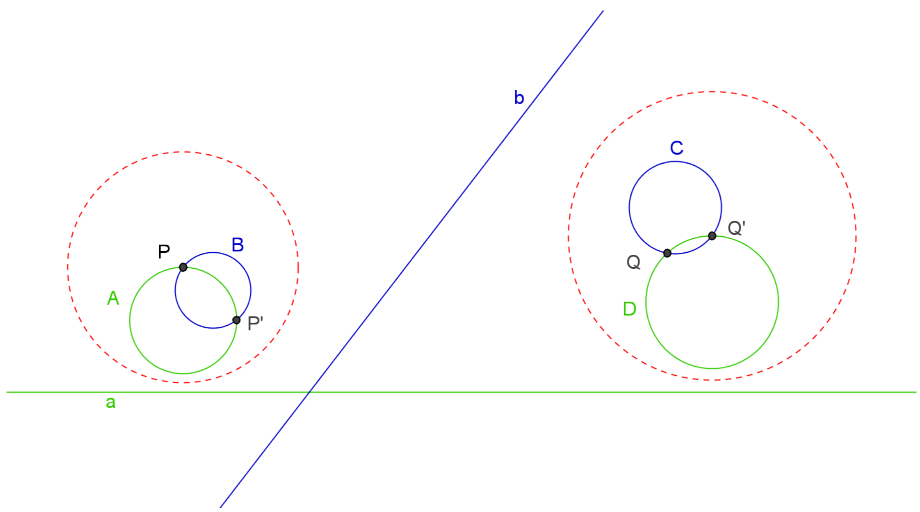

FIGURE figt396a

Let us show that this condition is also sufficient. Suppose circles $A,\ B$ (*fig.* t396a) intersect at points $P,\ P'$, and circles $C,\ D$ intersect at points Q, Q', both pairs intersecting at the same angle α. If we invert the first pair of circles in pole P (with any power at all), we will get two lines $a,\ b$ which intersect at angle α (**221**). If we invert circles $C,\ D$ in pole Q (with any power at all), we get another pair of lines $c,\ d$, intersecting at angle α. By suitable rotation and/or translation, we can move the figure consisting of $C,\ D,\ c,\ d$ so that lines $c,\ d$ coincide with lines $a,\ b$. Let Q_1 be the image of Q under this series of rotation and translation.

Now we can invert $A,\ B$ into $a,\ b$ around P, then invert $a,\ b$ around Q_1 into two circles congruent to the figure formed by $C,\ D$. Assuming that P and Q_1 do not coincide, these two inversions can be replaced by a single inversion, followed by a line reflection (see exercise 251, 2°). If we neglect the line reflection, then the figure formed by $A,\ B$ are inverted into a figure congruent to that formed by $C,\ D$.

If $P,\ Q_1$ coincide, we can repeat the argument, choosing Q' in place of Q, and obtaining a center Q_2 of inversion (by translation and rotation of $C,\ D$) which cannot coincide with P.

Thus the condition that the two pairs of circle meet at equal angles α is sufficient as well as necessary. The solution to exercise 395 shows that the value of α

determines the value of angle $\widehat{P\Omega P'}$ in figure t395, which in turn determines the ratio $PP' : P\Omega$. This last ratio is the one referred to in the problem statement.

We must make some changes to this argument if the pairs of circles are tangent (i.e., intersect at an angle of $0°$). Lines a, b will be parallel, as will lines c, d. By adjusting the power of the inversion, we can make the distance between lines c, d equal to the distance between a, b. Then we can follow the previous argument.

Note. Students can fill in details of the last paragraph, showing how the power of inversion determines the distance between the parallel lines. They can also consider the case in which both pairs of circles are tangent at the same point. (Translate one of the pairs away from the common point, so that the argument above applies.)

2°**.** If circles A, B have no common point, it is necessary and sufficient that the ratio of the radii of the concentric circles into which they can be transformed by an inversion (Exercise 248) be the same as the ratio of the radii of the concentric circles into which C, D can be transformed by an inversion (generally, a different inversion from the first). (Using the language of Note A, it is necessary and sufficient that the figures (A, B) and (C, D) have the same *reduced form* under inversion.)

This result can also be expressed as follows: the cross ratio (**212**) of the intersection points of A, B with any of their common orthogonal circles is constant, and the same is true of the cross ratio of two of these points and the limit points. The required condition is that this ratio have the same value for the circles C, D as for A, B.

Finally, if r, r' are the radii of A, B, and d is the distance between their centers, the quantity $\frac{d^2-r^2-r'^2}{rr'}$ must have the same value as the corresponding value calculated for the circles C, D.

We could also express this by saying that if the circles A, B have a common tangent (for example, a common external tangent) of length t, and the same is true for C, D, then the ratio $\frac{t}{\sqrt{rr'}}$ must be the same in the two cases.

Solution 2°**.** We first prove that the given condition is necessary. Suppose some inversion S takes circles A, B onto circles A', B', which taken together form a figure congruent to that formed by circles C, D. If we translate and rotate circles C, D, they will then coincide with circles A', B'. So we can assume, without loss of generality, that circles C, D are in fact the same as A', B'.

Recall (exercise 248) that any two non-intersecting circles can be inverted into concentric circles by using one of their limit points (exercise 152) as the pole. So we can invert C, D around one of their limit points to get concentric circles c, d. We call this inversion T. (Since C and D have two limit points, we can choose the pole of T to be different from the pole of S.) The inversion S followed by the inversion T takes A, B onto c, d. But the composition of these two inversions can be replaced by a single inversion S' and a line reflection (exercise 251, 2°). So S' (without the line reflection) takes A, B into two concentric circles which are congruent to c, d. And T takes circles C, D into concentric circles c, d. That is, if there is an inversion S taking A, B onto a figure congruent to C, D, then the ratio of the concentric circles into which they can be inverted must be the same. The condition of the problem is necessary.

We next show that this condition is sufficient, using an argument similar to that in 1°. Suppose $A,\ B$ invert into concentric circles $a,\ b$, while $C,\ D$ invert into concentric circles $c,\ d$, and suppose that the radii of $c,\ d$ are proportional to those of $a,\ b$. We choose any power at all for the inversion taking $A,\ B$ onto $a,\ b$, and recall (**215**) that two figures which are inversions of the same figure with the same pole are homothetic to each other. This means that we can choose the power of the inversion taking $C,\ D$ onto $c,\ d$ in such a way that the figure formed by $c,\ d$ is congruent to that formed by $a,\ b$. (We use here the fact that the radii of the four circles are in proportion.) We can translate and rotate the figure formed by $C,\ D$ so that circles $a,\ b$ in fact coincide with circles $c,\ d$. As in 1°, we now see that $C,\ D$ can be obtained from $A,\ B$ by a sequence of two inversions. (We can avoid the situation where the poles of these inversions coincide by rotating $C,\ D$ around the common center of $a,\ b$.) As before, the sequence of two inversions can be replaced by a single inversion S' followed by a line reflection, and the inversion S' alone takes $A,\ B$ onto a figure congruent to $C,\ D$.

We now express this condition in terms of the cross ratios of the intersections of the given circles with the circles orthogonal to them. We will show, in the case of intersecting circles, that this cross ratio depends only on the angle at which the circles intersect, and not on the particular orthogonal circle. In the case of non-intersecting circles, we will show that this cross ratio depends only on the ratio of the concentric circles they invert into, and not on the particular orthogonal circle.

(The value of the cross ratio described here depends on the order in which the points are taken. Without loss of generality we can take the first pair of points to be the intersection of the orthogonal circle with one of the two given circles, and the second pair of points to be its intersection with the second of the given circles. The result of exercise 274 will tell us that the value λ of the cross ratio is then limited to two values, whose product is 1. We can then, again without loss of generality, take the points in order so that $|\lambda| \leq 1$.)

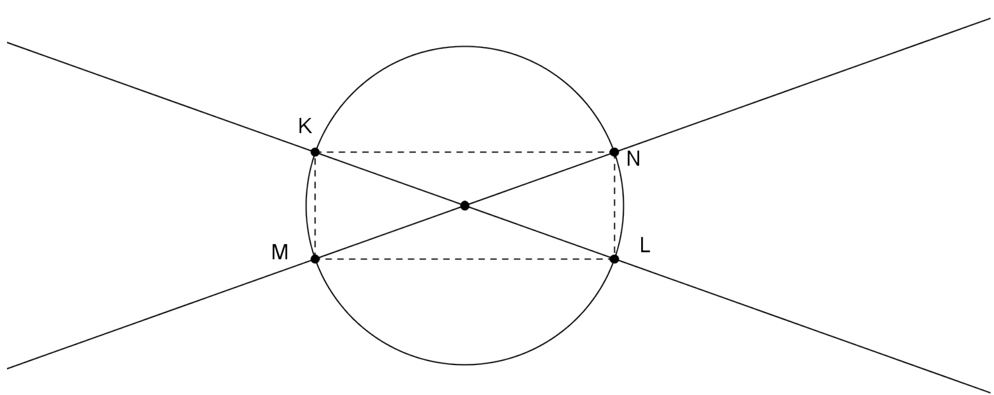

FIGURE t396b

We start with the case of two intersecting circles. These can be inverted into two intersecting lines, and the circles orthogonal to them into circles whose center is their point of intersection. One such circle, in figure t396b, is $KNLM$, where K and L belong to different semicircles. The cross ratio $\lambda = (KLMN) = -\frac{KM}{LM} : \frac{KN}{LN} = -(\frac{KM}{LM})^2$. But this last ratio depends only on the angle at which the two lines intersect, and not on the choice of orthogonal circle. Since inversion preserves both orthogonality and cross ratio (see exercise 273), the same is true of two intersecting circles.

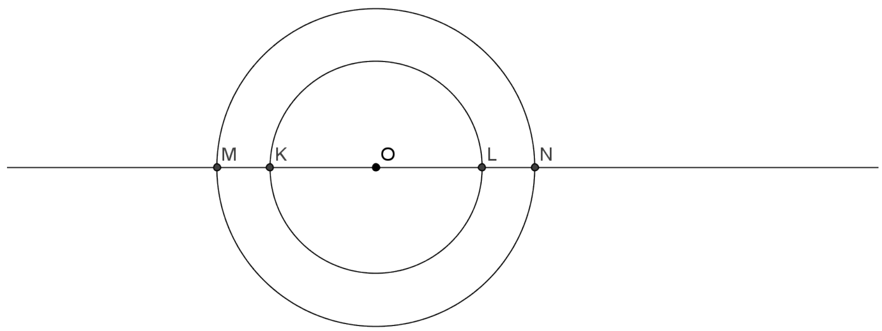

FIGURE t396c

In the case of two non-intersecting circles, we invert them into two concentric circles. Then the circles orthogonal to them invert into lines through the common center of the two circles. One such line, in figure t396c, is $MKOLN$. If the radii of the concentric circles are R and r, then the cross ratio $(KLMN) = \frac{KM}{LM} : \frac{KN}{LN} = \left(\frac{R-r}{R+r}\right)^2$. Dividing numerator and denominator of this last fraction by R, we find $\lambda = \left(\frac{1-\frac{r}{R}}{1+\frac{r}{R}}\right)^2$, and so depends only on the ratio of the radii of the concentric circles. Since inversion preserves both orthogonality and cross ratio, the same is true of the original two non-intersecting circles.

Note. If the circles are tangent, any circle orthogonal to both must pass through their point of tangency. The cross ratio of the four points described in the problem (and taken in order) is 1, as two of the points coincide. Students can fill in the details for this case.

We treat separately the assertions in the rest of the exercise.

3°**.** The quantity $\nu = \frac{d^2-r^2-r'^2}{rr'}$ must have the same value as the corresponding value calculated for the circles $C,\ D$ (with the variables as described in the problem statement).

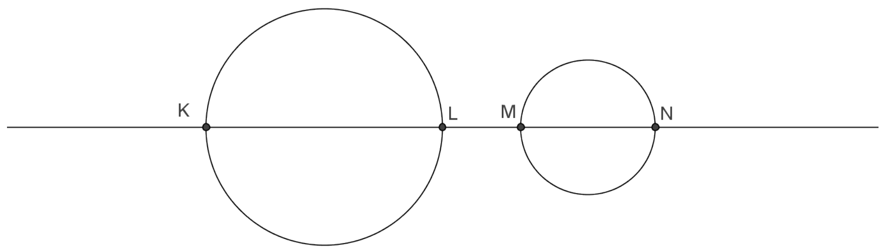

FIGURE t396d

Solution. We will show that ν is uniquely determined by the cross ratio λ. We already know that we can compute λ from an image under inversion of the original circles, so we choose, without losing generality, the situation where the circle orthogonal to the two original circles is simply their common centerline (*fig.* t396d), and we can do this whether or not the given circles have a point in common. Then we have

$$\lambda = (LKMN) = \frac{KM}{LM} : \frac{KN}{LN} = \frac{d+r-r'}{d-r-r'} : \frac{d+r+r'}{d-r+r'} = \frac{d^2-(r-r')^2}{d^2-(r+r')} =$$
$$= \frac{d^2-r^2-r'^2+2rr'}{d^2-r^2-r'^2-2rr'} = \frac{\nu+2}{\nu-2}.$$

Solving from ν in terms of λ, we find that $\nu = \frac{2\lambda+2}{\lambda-1}$. That is, the value of ν is determined uniquely by the value of λ. Since λ has the same value for the two pairs of circles considered, so must ν.

4°. If circles $A,\ B$ have a common tangent of length t, and the same is true for $C,\ D$, then the ratio $\frac{t}{\sqrt{rr'}}$ must be the same in the two cases.

Solution. Let $\tau = \frac{t}{\sqrt{rr'}}$. Then $t^2 = d^2 - (r-r')^2 = d^2 - r^2 - r'^2 - 2rr'$. Comparing this to the definition of ν invites the direct computation $\nu + 2 = \frac{d^2-r^2+2rr'}{rr}$, so that $\tau = \frac{t}{rr'} = \sqrt{\nu+2}$, As in 4°, this means that τ has the same value for the pairs of circles considered.

Problem 397. We are given two points $A,\ A'$ and two lines $D,\ D'$ parallel to, and at equal distance from, AA'.

1^o. Show that for every point P on D there corresponds a point P' on D' such that line PP' is tangent to the two circles $PAA',\ P'AA'$;

Solution 1°**.** For any point P on line D, we draw the circle through $P,\ A',\ A$, and also its tangent at P (*fig.* t397a). Let P' be the point of intersection of this tangent with D'. The centers of the circles through $A,\ A'$ lie on the perpendicular bisector of AA'. The centers of circles tangent to PP' lie on the line through

P' perpendicular to PP'. Their intersection O' is the center of a circle through P, A, A'.

Indeed, if M is the intersection of AA' and PP', then M is the midpoint of PP' (**113**), and we have $MP'^2 = MP^2 = MA \cdot MA'$. By **132** (converse), this means that P', A', and A are on the same circle, which must have its center at O'.

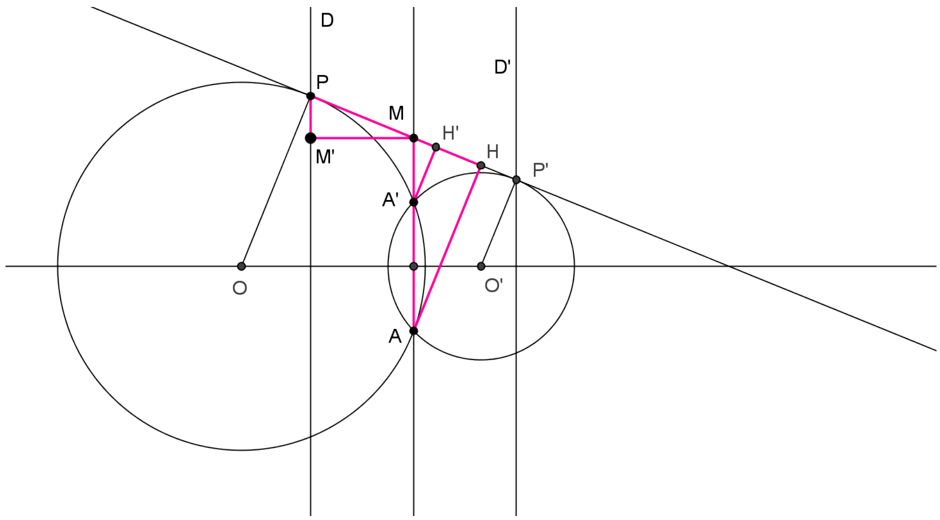

FIGURE t397a

2^o. Prove that the product of the distances from A, A' to line PP' is constant;

Solution. Let H, H' (*fig.* t397a) be the feet of the perpendiculars from A, A' respectively to line PP'. We know that (**132**) that

$$(1) \qquad MA \cdot MA' = PM^2.$$

We use similar triangles to rewrite (1). Triangles AHM, $A'H'M$ are similar (their sides are parallel in pairs; see **43, 118**, first case). If we draw MM' perpendicular to D, we find that triangle PMM' is similar to the two triangles identified above (they are right triangles with two pairs of parallel sides; see **43, 118,** case I). That is, segments AH, $A'H'$, MM' are proportional to segments AM, $A'M$, PM.

Now we rewrite (1). We have $\frac{A'H'}{A'M} = \frac{MM'}{PM}$, so $A'M = A'H' \cdot \frac{PM}{MM'}$. Likewise, $AM = AH \cdot \frac{PM}{MM'}$, and we can rewrite (1) as $A'H' \cdot \frac{PM}{MM'} \cdot AH \cdot \frac{PM}{MM'} = PM^2$. Dividing this equation by $\left(\frac{MM'}{PM}\right)^2$, we find that

$$(2) \qquad A'H' \cdot AH = MM'^2,$$

and this last length is constant, since $AA' \parallel D$.

3^o**.** Find the locus of the projection of A onto PP';

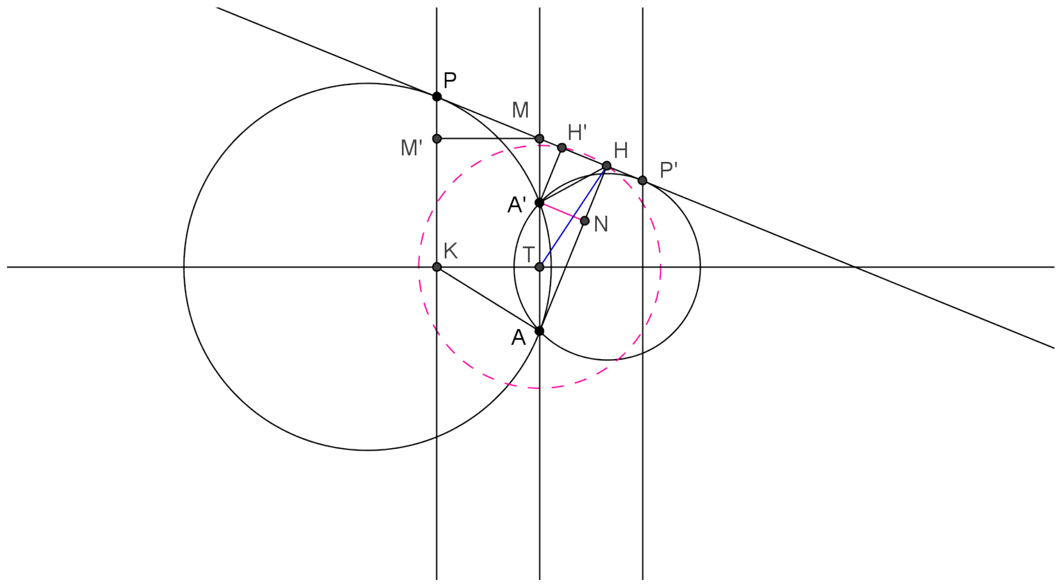

FIGURE t397b

Solution. We use the result of exercise 141, and also some ideas from its solution for the case $m = n = 1$, in the notation of that solution. (Students can be given this statement as a hint.) To apply this result, we examine the sum $HA^2 + HA'^2$, by looking at triangle HAA'.

In that triangle, we have $AA'^2 = A'H^2 + AH^2 - 2AH \cdot HN$ (**126**), where N is the foot of the perpendicular from A to line HA. But $HN = A'H'$ (from rectangle $A'NHH'$), so

$$HA^2 + HA'^2 = AA'^2 + 2AH \cdot AH', \tag{3}$$

and the result of 2° shows that this last quantity is constant. Thus the result of exercise 141 shows that H lies on a circle whose center is the midpoint T of AA'

To compute the radius of this circle, we proceed as in exercise 141 (or use **128),** 1°. From either of these results, we have

$$\begin{aligned} 4HT^2 &= 2A'H^2 + 2AH^2 - AA'^2 \\ &= 2(AA'^2 + 2AH \cdot AH') - AA'^2 (\text{from}(3)) \\ &= AA'^2 + 4MM'^2 . (\text{from}(2)) \end{aligned}$$

So the radius HT of the circle is equal to $\sqrt{\frac{1}{4}AA'^2 + MM'^2}$.

Notes. In figure t397b, right triangle ATK shows that radius HT of this circle is equal to the distance from A to point K, the point on line D which is equidistant from A and A'.

Analogous reasoning starting with triangle $A'H'A$ will show that point H' lies on the same circle.

4^o**.** Find a point P such that the line PP' passes through a given point Q;

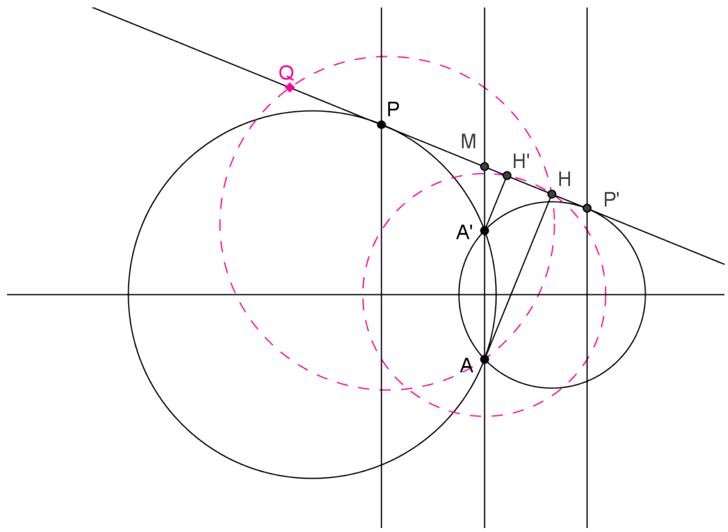

FIGURE t397c

Solution. If we start with any point Q, the associated point H must lie on a circle with diameter AQ (since angle $\widehat{QHA}$ is a right angle), and also on the circle described in 3°. The line connecting Q to any intersection point of these two circles will intersect D in a point P which satisfies the conditions of the problem.

5^o. Show that the angle of the circles PAA', $P'AA'$, and angle $\widehat{PA'P'}$, are constant.

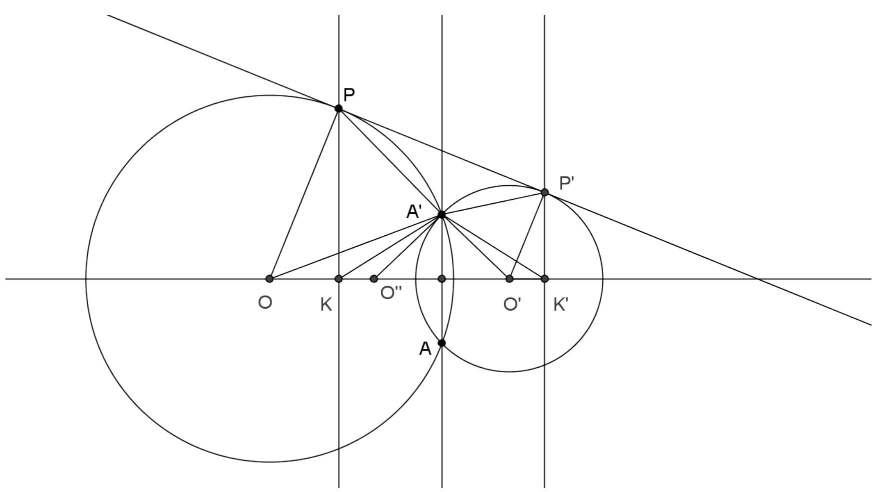

FIGURE t397d

Solution. We take the angle between the two circles to be equal to the angle between their radii at a point of intersection. In figure t397d, this is angle $\widehat{OA'O'}$, and we will show that it is equal to the constant angle $\widehat{KA'K'}$, where $K,\ K'$ are the intersections of the two given parallel lines with the perpendicular bisector of segment AA'.

To this end, we lay off segment KO'' as in the figure so that $KO'' = K'O'$. Then triangles $A'KO''$, $A'K'O'$ are symmetric in line AA', so they are congruent, and

$$\widehat{KA'O''} = \widehat{K'A'O'}. \tag{4}$$

Next we show that $A'K$ bisects angle $\widehat{OA'O''}$. Indeed, triangles OPK, $O'P'K'$ are similar. (See **43**; their sides are parallel in pairs). So $OP : O'P' = OK : O'K'$. But $OP = OA'$, $O'P' = O'A' = O''A'$, $O'K' = KO''$, so this proportion is equivalent to $OA' : O''A' = OK : KO''$, and this last proportion shows (**115**) that $A'K$ is an angle bisector in triangle $A'OO''$.

Now from (4) we have $\widehat{O'A'K} = \widehat{KA'O''} = \widehat{K'A'O'}$. Adding angle $\widehat{KA'O'}$ to each, we find that $\widehat{OA'O'} = \widehat{KA'K'}$. The first angle is the angle between the two circles, and the second angle is constant. This proves the first assertion.

To show that $\widehat{PA'P'}$ is constant, we examine the angles around vertex A', and will show that the sum of the remaining angles is also contant. We know that $\widehat{OA'O'}$ is constant from the previous paragraph. So we need to show that the sum $\widehat{OA'P} + \widehat{O'A'P'}$ is constant.

Since $\widehat{OA'O'}$ is constant, so is the sum $\widehat{A'OO'} + \widehat{A'O'O}$ (it is equal to $180° - \widehat{OA'O'}$). And since $OP \parallel O'P'$, we know that $\widehat{POO''} + \widehat{OO'P'} = 180°$, so the sum $\widehat{A'OP} + \widehat{A'O'P'} = (POO' + \widehat{OO'P'}) - (\widehat{A'OO'} + \widehat{A'O'O}) = \widehat{OA'O'}$.

Now we look at isosceles triangles POA', $P'O'A'$. The sum of their vertex angles is constant, so the sum of a pair of base angles is also constant: $\widehat{PA'O} + \widehat{P'A'O'} = \frac{1}{2}(180° - \widehat{POA'} + 180° - \widehat{P'O'A'}) = 180° - \frac{1}{2}(\widehat{OA'O'})$.

We have shown that the sum of the other angles around A' is constant, so $\widehat{PA'P'}$ must also be constant: $\widehat{PA'P'} = 360° - (\widehat{PA'O} + \widehat{P'A'O'}) - \widehat{OA'O'} = 180° - \frac{1}{2}\widehat{OA'O'} = 180° - \frac{1}{2}\widehat{KA'K'}$, which is constant.

Problem 398. Let C be a circle with diameter AB, and D a line perpendicular to this diameter, which intersects C. Let c, c' be the circles whose diameters are the two segments into which D divides AB. We draw a circle tangent to C, c, D, and another circle tangent to C, c', D. Show that these two circles are equal, and that their common radius is the fourth proportional to the radii of C, c, c'.

Solution. Let S be the circle tangent to c, D, and C, and let S' be the circle tangent to c', D, and C.

We consider the inversion with pole A and power $AM \cdot AB$. We will show that the image of line D is circle C. Indeed, the image is a circle through A (**220**) which is orthogonal to line AB (**219**). Points M and B are clearly images of each other, so the image of D must contain point B. That is, the image of D is a circle through A and B which is orthogonal to line AB. This must be circle C. And the image of circle C is of course then line D. Further, it is not hard to see circle c' is its own image. (The power of inversion is the power of A with respect to this circle.) Finally, the image of circle S' is a circle tangent to D, C, and c', and the only such circle (on the same side of AB as S') is S' itself.

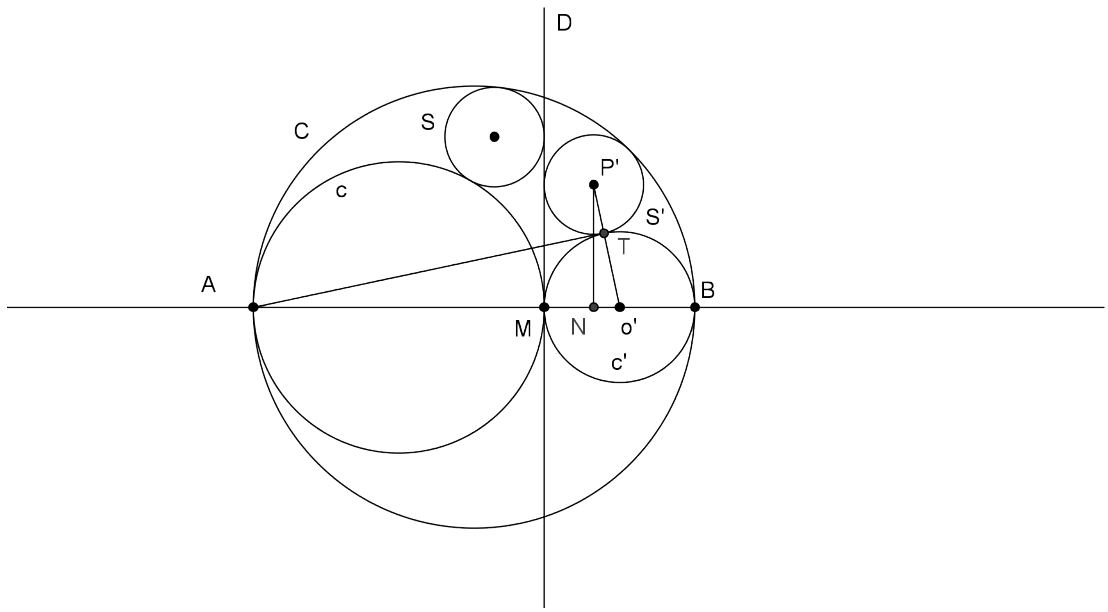

FIGURE t398

It follows that the common tangent to c and S' at their point T of tangency must pass through the pole A of this inversion. Indeed, since $c',\ S'$ are their own images under the inversion, their common point must be its own image as well. That is, AT^2 is the power of the inversion. But then $AT^2 = AM \cdot AB$ and by **132** (converse), AT is the tangent to c' (and therefore also to S') at T.

Let $o',\ P'$ be the centers of circles $c',\ S'$ respectively. Let N be the foot of the perpendicular from P' to diameter AB, and let $R,\ r,\ r',\ \rho'$ be the respective radii of circles $C,\ c,\ c',\ S'$. Triangles ATo', $P'No'$ are both right triangles, and share angle $\widehat{P'o'A}$, so they are similar, and $Ao' : P'o' = To' : No'$ or $\frac{2R-r'}{\rho'+r'} = \frac{r'}{r'-\rho'}$.

Transforming this expression gives us $\rho' R = (R - r')r'$. Since $R = r + r'$, it follows that $\rho' R = rr'$. Thus the radius of circle S' is the fourth proportional to the radii of $C,\ c,\ c'$.

Similarly, we could compute the value of the radius r of circle S in terms of ρ' and R. But we need not carry out the argument: the expression we got for ρ' is symmetric in r and r', and we will just be changing the names of the objects in the argument, not their relationships to each other. Either way, we see that the radii of $S,\ S'$ are equal.

Problem 399. (the Greek *Arbelos*) Let $A,\ B$ be two tangent circles. Let C be a circle tangent to the first two; let C_1 be a circle tangent to $A,\ B,\ C$; let C_2 be a circle tangent to $A,\ B,\ C_1$; let C_3 be tangent to $A,\ B,\ C_2,\ \ldots$; and let C_n be a circle tangent to $A,\ B,\ C_{n-1}$. Consider the distance from any of the centers of $C,\ C_1, \ldots,\ C_n$ to the line of centers of $A,\ B$, and the ratio of this distance to the diameter of the corresponding circle. Show that this ratio varies by one unit in passing from any circle to the next one, at least in the case in which they are exterior (which always happens when circles $A,\ B$ are tangent internally). Show

how this statement must be modified when two consecutive circles C_{n-1}, C_n are tangent internally. (*Arbelos* is a Greek word meaning *sickle*)[2].

Solution. The problem describes a chain of circles, starting with a single circle C, which are all tangent to two larger circles, and each circle tangent to the previous as well. It will be convenient to discuss a slightly more general situation, in which the chain of circles is continued in both directions, so that there is an initial circle $C = C_0$, a chain of circles $C_1, \cdots C_n$, and also a chain of circles $C_{-1},\ C_{-2}, \cdots$ as in figure t399a.

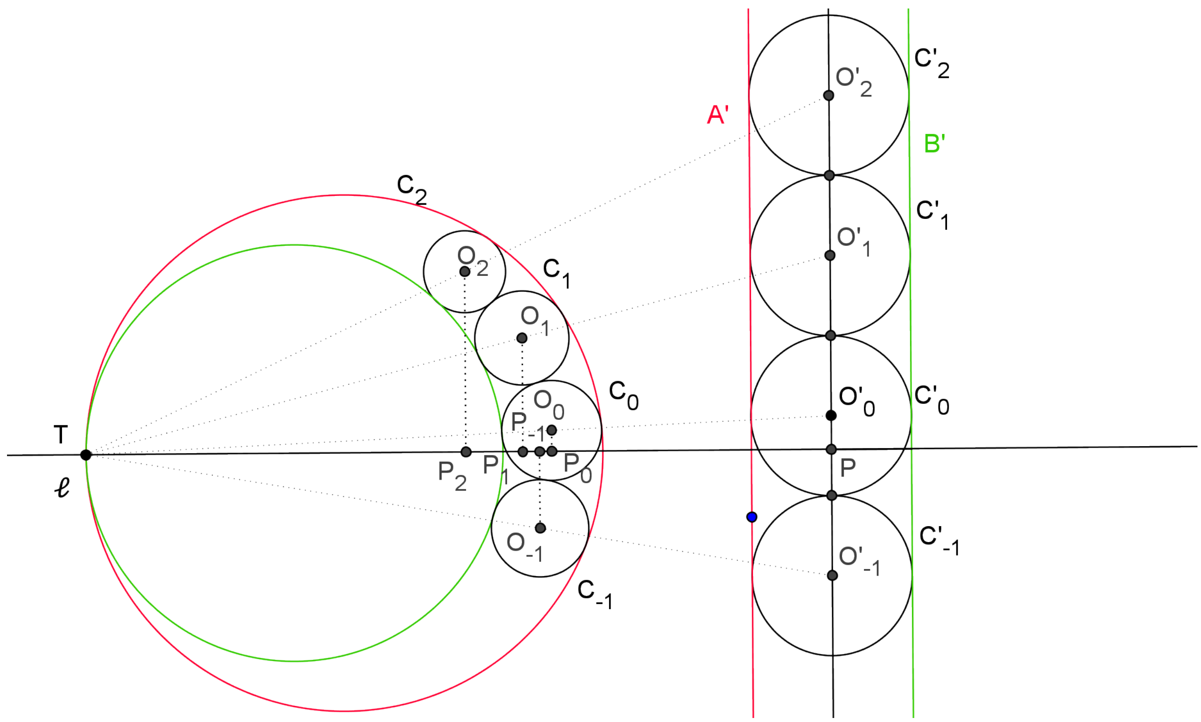

Figure t399a

Let the center of circle C_n be O_n, and its radius as r_n. Let ℓ be the common centerline of circles $A,\ B$. We will also need to talk about the projection of O_n onto ℓ. We call this point P_n.

In that figure, T is point of tangency of circles $A,\ B$, and we invert the figure around T as the pole, using any power. Circles $A,\ B$ invert into two parallel lines $A',\ B'$ (**220**, corollary), and circles $\cdots,\ C_{-2},\ C_{-1},\ C_0,\ C_1, C_2, \cdots$ invert into circles $\cdots,\ C'_{-2},\ C'_{-1},\ C'_0,\ C'_1, C'_2, \cdots$ tangent to both lines, which therefore have equal radii. We denote this radius as r'.

Let the centers of the circles C'_n be the points O'_n. Because the circles are equal, these centers are collinear. Let P' be the intersection of this common centerline with ℓ.

[2]This note is Hadamard's own. The usual translation of *arbelos* is *shoemaker's knife*. But see for instance Harold P. Boas, *Reflections On the Arbelos*, American Mathematical Monthly, 113, no. 3 (March 2006), 236-249. –transl.

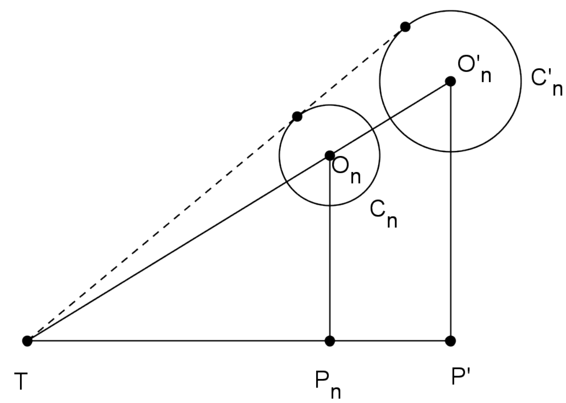

FIGURE t399b

For those circles whose centers lie on the same side of line ℓ, it is clear that $O'_{n-1}P' - O_nP' = 2r'$. We can write this as

$$\frac{O'_{n-1}P'}{2r'} - \frac{O'_nP'}{2r'} = 1. \tag{1}$$

For those circles whose centers lie on different sides of line ℓ, this relationship becomes $O'_{n-1}P' + O'_nP' = 2r'$, or

$$\frac{O'_{n-1}P'}{2r'} + \frac{O'_nP'}{2r'} = 1. \tag{2}$$

Now the pole of inversion is a center of similarity for any pair of circles which are inverses of each other, so (*fig.* t399b) we have $r' : r_n = TO'_n : TO_n = O'_nP' : O_nP_n$. Therefore $O'_nP' : 2r' = O_nP_n : 2r_n$. That is, the ratio of the distance from the centers of any of our circles C_n from the common centerline of A, B to the diameter of that same circle C_n does not change when we invert around pole T. Because of this, we can rewrite (1) and (2) as:

$$\frac{O_{n+1}P_{n+1}}{2r_{n+1}} - \frac{O_nP_n}{2r_n} = 1. \tag{3}$$

$$\frac{O_{n+1}P_{n+1}}{2r_{n+1}} - \frac{O_nP_n}{2r_n} = 1. \tag{4}$$

These two equations prove the assertion of the problem.

In the case mentioned in the problem statement, where A and B are tangent internally, any two circles $C_n,\ C_{n+1}$ are tangent externally. Since circles $C_n,\ C_{n+1}$

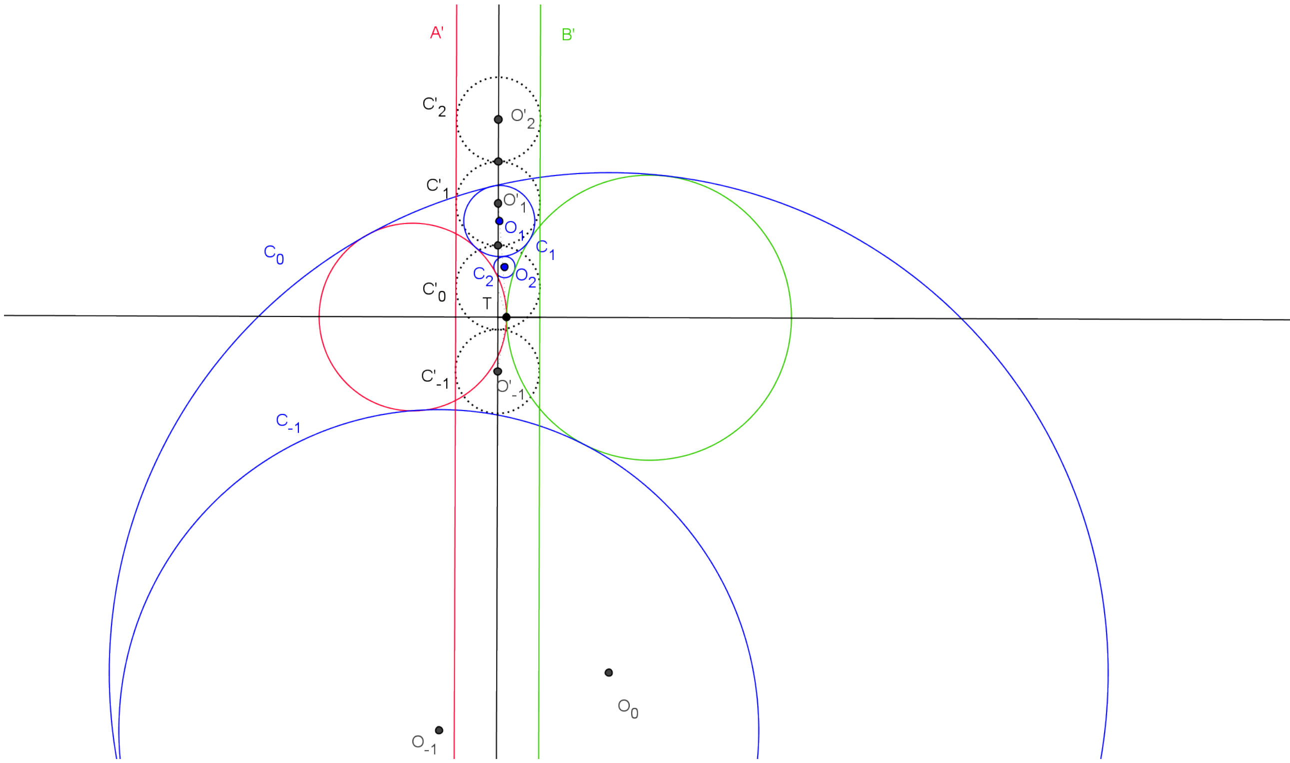

FIGURE t399c

are tangent externally, it follows that the centers of the inverted circles $C'_n,\ C'_{n+1}$ lie on the same or different sides of line ℓ according as the centers of $C_n,\ C_{n+1}$ lie on the same or different sides of ℓ. Thus in this case, we can say that equation (3) holds whenever the centers of the circles lie on the same side of ℓ, and equation (4) holds if these centers lie on opposite sides of ℓ.

If circles $A,\ B$ are tangent externally (*fig.* 399c), then circles $C_{-1},\ C_0$ and $C_0,\ C_1$ are tangent internally, while the others are tangent externally. This requires a small change in the concluding statement above. For those pairs of circles $C_n,\ C_{n+1}$ which are tangent internally, equation (3) holds if their centers are on opposite sides of ℓ, and equation (4) holds if their centers are on the same side of ℓ.

Note. We can state this result more generally if we allow for signed distances to line ℓ. If $x_n,\ x_{n+1}$ are the distances of consecutive circles from ℓ, and $r_n,\ r_{n+1}$ are their radii, then we can say that $\frac{x_n}{2r_n} - \frac{x_{n+1}}{2r_{n+1}} = 1$, when x_n is positive for circles on one side of ℓ and negative for circles on the other side.

For circles that are tangent externally, we have $\frac{x_n}{2r_n} + \frac{x_{n+1}}{2r_{n+1}} = 1$, with similar conventions of sign for $x_n,\ x_{n+1}$.

Problem 400. Let $A,\ B,\ C$ be three circles with centers at the vertices of a triangle, each pair of which are tangent externally (Exercise 91). Draw the circle externally tangent to these three circles, and also the circle internally tangent to these three circles. Calculate the radii of these circles knowing the sides a, b, c of the triangle (preceding exercise, Exercise 301).

Solution. If $a,\ b,\ c,\ s$ are the sides and semiperimeter of triangle ABC, then exercise 91 shows that the radii of the circles centered at $A,\ B,\ C$ are $s-a,\ s-b, s-c$ respectively.

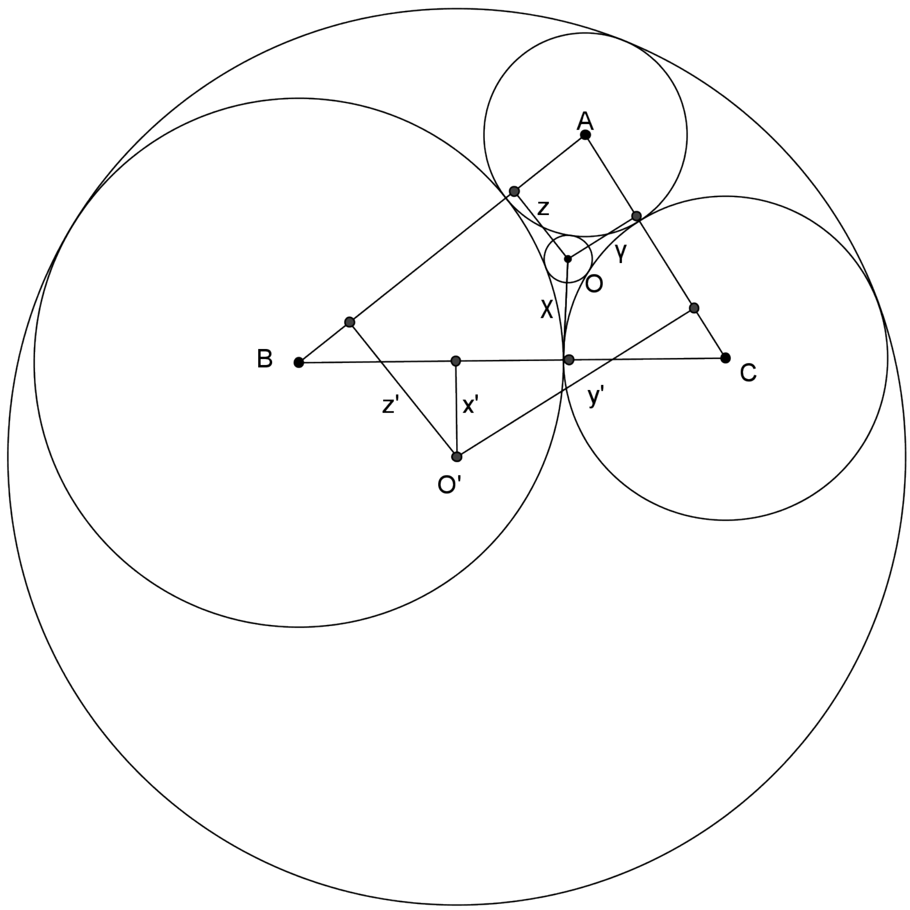

Figure t400a

There are many circles tangent to the three described in the problem, but only two (also mentioned in the problem) which are tangent to all three externally or to all three internally (see figure t400a). We first consider the circle which is externally tangent to all three (the small circle in figure t400a) which lies inside the curvilinear triangle formed by the three given circles. Let O be its center and ρ its radius.

We will apply the result of exercise 399 in various ways, with pairs of circles $A,\ B,\ C$ playing the role of the original circles given in that exercise. The common centerlines of these circles are the sides of the given triangle, so we will need the distances of these from O. Let these be $x,\ y,\ z$, and let $h,\ k,\ \ell$ be the lengths of the altitudes of the given triangle.

We can now use formula (3) from exercise 399, applying it to circles B and C as the two given circles, and A and O as the (short) chain of circles. we have:

$$\frac{x}{2\rho} - \frac{h}{2(s-a)} = 1. \tag{1}$$

Analogously, if we start with A and B as the given circles, and C, O the chain of circles, we have:

$$\frac{y}{2\rho} - \frac{k}{2(s-b)} = 1,$$

and starting with circles A and C we have:

$$\frac{z}{2\rho} - \frac{\ell}{2(s-c)} = 1.$$

Multiplying (1) by 2ρ and dividing by h, we get:

$$\frac{x}{h} - \frac{\rho}{s-a} = \frac{2\rho}{h},$$

and analogously:

$$\frac{y}{k} - \frac{\rho}{s-a} = \frac{2\rho}{k},$$
$$\frac{z}{\ell} - \frac{\rho}{s-c} = \frac{2\rho}{\ell}.$$

Adding, and using the result of exercise 301, we get

$$\rho\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{2}{h} + \frac{2}{k} + \frac{2}{\ell}\right) = 1.$$

This gives an expression for ρ, as required. We can transform it by introducing the area Δ of the triangle, and the radii $r_a,\ r_b,\ r_c$ of its escribed circles.

From exercise 299, we have $\frac{1}{s-a} = \frac{r_a}{\Delta},\ \frac{1}{s-b} = \frac{r_b}{\Delta},\ \frac{1}{s-c} = \frac{r_c}{\Delta}$. And from **249** $\frac{2}{h} = \frac{a}{\Delta},\ \frac{2}{k} = \frac{b}{\Delta},\ \frac{2}{\ell} = \frac{c}{\Delta}$, so we have:

$$\rho\left(\frac{r_a}{\Delta} + \frac{r_b}{\Delta} + \frac{r_c}{\Delta} + \frac{a}{\Delta} + \frac{b}{\Delta} + \frac{c}{\Delta}\right) = 1,$$

or

$$\rho\left(\frac{r_a}{\Delta} + \frac{r_b}{\Delta} + \frac{r_c}{\Delta} + \frac{a}{\Delta} + \frac{b}{\Delta} + \frac{c}{\Delta}\right) = 1,$$

or

$$\rho = \frac{\Delta}{a+b+c+r_a+r_b+r_c} = \frac{\Delta}{2s+r_a+r_b+r_c}.$$

Now suppose O' and ρ' are the center and radius of the second circle tangent to the three given circles. This circle can touch the others either internally (*fig.* t 400a) or externally (*fig.* t 400b). In the case where the new circle is tangent internally to the others, its center can lie either inside the given triangle or outside it.

Proceeding as before, we get the expression $\rho' = \pm\frac{\Delta}{2s-(r_a+r_b+r_c)}$.

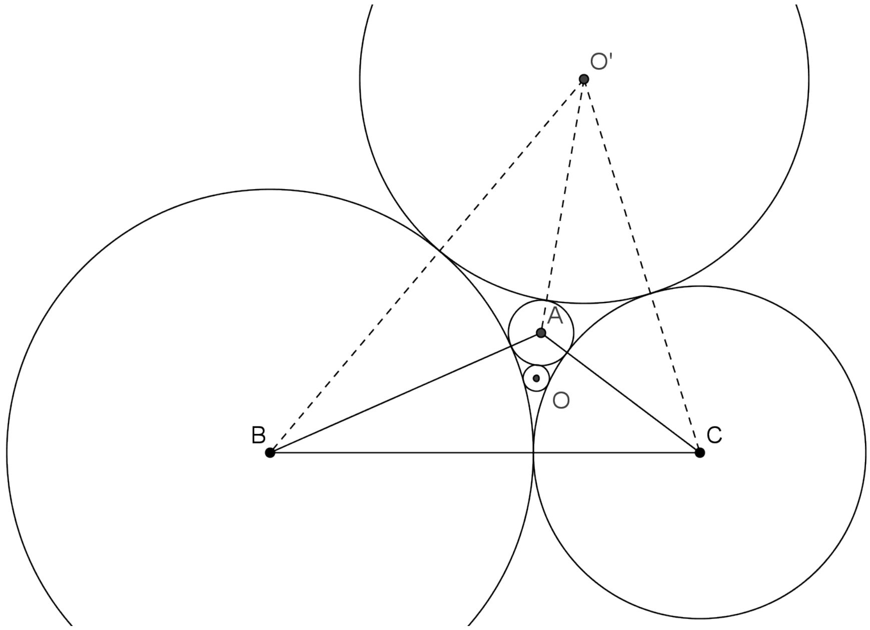

FIGURE t400b

Notes. We can explore the situation further if we introduce the convention of exercise 301 for the signed distance from a point to a line. Let x', y', z' are the distances from point O' to the sides of the triangle. In the case of a circle externally tangent to the others, we then have the system of equations:

$$\frac{h}{2r_a} + \frac{x'}{2\rho'} = 1;$$

$$\frac{k}{2r_b} + \frac{y'}{2\rho'} = 1;$$

$$\frac{\ell}{2r_c} + \frac{z'}{2\rho'} = 1.$$

In the case of a circle internally tangent to the others, we have:

$$\frac{h}{2r_a} - \frac{x'}{2\rho'} = 1;$$

$$\frac{k}{2r_b} - \frac{y'}{2\rho'} = 1;$$

$$\frac{\ell}{2r_c} - \frac{z'}{2\rho'} = 1.$$

Solving the first system, we get $\rho' = \frac{\Delta}{2s-(r_a+r_b+r_c)}$. From the second system, we get $\rho' = -\frac{\Delta}{2s-(r_a+r_b+r_c)}$. If $r_a + r_b + r_c = 2s$, circle O' becomes a straight line.

Problem 401. Given three circles with centers A, B, C and radii $a,\ b,\ c,$ let H be the radical center of the circles concentric with the first and with radii $a+h,\ b+h,\ c+h$, and let N be taken on AH such that $\frac{AN}{AH} = \frac{a}{a+h}$. Show that, as h varies, points H and N move on two straight lines, the first of which passes through the centers of the tangent circles (with contacts of the same kind) of the three given circles, and the second of which passes through the points of contact of these circles with circle A. Give an analogous statement which allows one to find the circles which have different kinds of contact with circles $A,\ B,\ C$.

Solution. We divide the problem into five statements:

(i). The locus of H is a line.

(ii). The locus of N is also line.

(iii). The locus of H passes through the center of the circle tangent externally to the three given circles.

(iv). The locus of N passes through the points of contact of the circle tangent externally to the three given circles with circle A.

(v). The locus of H and that of N pass through the analogous points for the circle tangent internally to the three given circles.

We consider both positive and negative values of the increment h, and circles $A_h,\ B_h,\ C_h$ with radii equal to $|a+h|,\ |b+h|,\ |c+h|$. Using this convention, the circles $A,\ B,\ C$ referred to in the problem statement can also be called $A_0,\ B_0,\ C_0$.

(i). We use the result of exercise 124 to find the path of H. Let K be the projection of point H on line AB, so that HK is the radical axis of circles $A_h,\ B_h$. If D is the midpoint of segment AB (*fig.* t401a), then we have (**136**) $DK = \frac{(a+h)^2-(b+h)^2}{2AB}$. When $h=0$, point H coincides with the radical center H_0 of circles $A_0,\ B_0,\ C_0$, point K assumes the position K_0 which is the projection of H_0 onto line AB, and we have $DK_0 = \frac{a^2-b^2}{2AB}$. It follows that

$$K_0K = DK - DK_0 = \frac{(a-b)h}{AB} \tag{1}$$

.

We now make the same computation with point H and line AC. If $L,\ L_0$ are the projections respectively of $H,\ H_0$ on line AC, the same argument gives us $L_0L = \frac{(a-c)h}{AC}$. Thus $\frac{K_0K}{L_0L} = \frac{a-b}{a-c} \cdot \frac{AC}{AB}$.

Thus we can describe point H using the following construction: We locate points $K_0,\ L_0$ on lines $AB,\ AC$ respectively, and lay off variable segments $K_0K,\ L_0L$, which maintain the same ratio as they vary. At K and L we erect perpendiculars to AB and AC respectively. Point H is then the intersection of these perpendiculars. It follows from the result of exercise 124 that the locus of point H is a line.

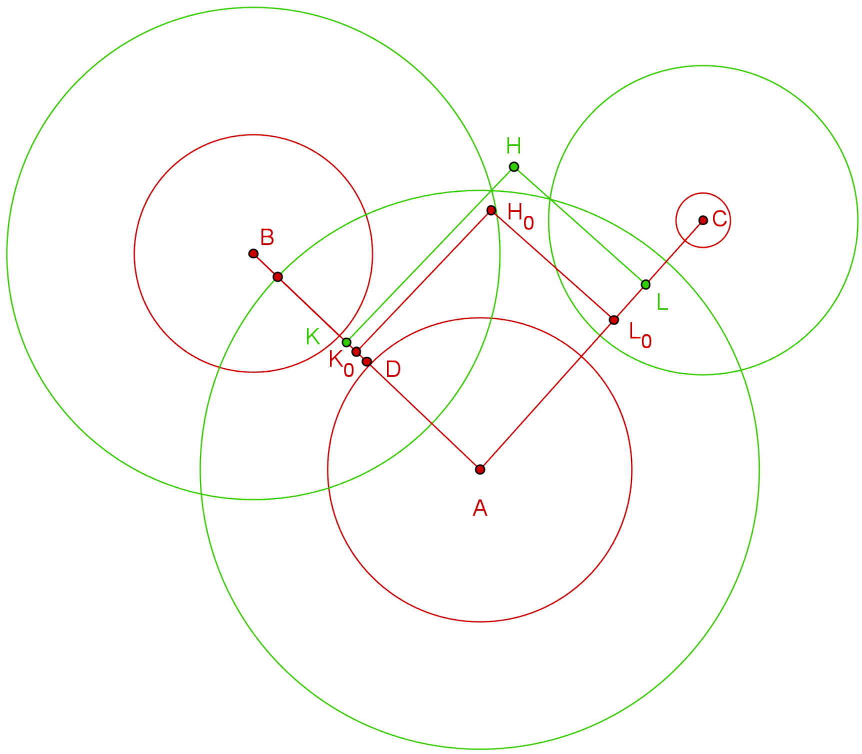

FIGURE t401a

(ii). We use Menelaus' theorem(**193**) to get the locus of point N. We fix some position H' of H, with the associated value h' of h (*fig.* t401b), and let K' be the projection of H' onto line AB, We have, by analogy to (1) above,

$$K_0K' = \frac{(a-b)h'}{AB}. \tag{2}$$

from (1) and (2) it follows that

$$H_0H : H_0H' = K_0K : K_0K' = h : h'. \tag{3}$$

Suppose points $N,\ N'$ (in figure t401b) divide segments $AH,\ AH'$ in the ratios $AN : AH = a : (a+h)$ and $AN' : AH' = a : (a+h')$. Then we have

$$AN : NH = a : h, \tag{4}$$

$$AN' : N'H' = a : h'. \tag{5}$$

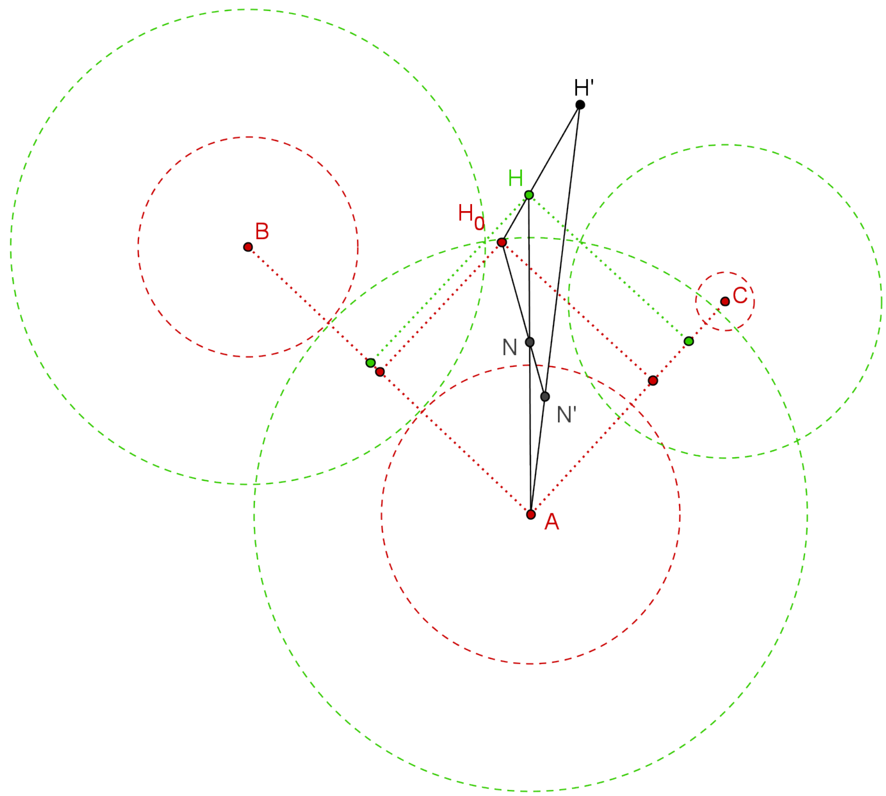

FIGURE t401b

Relations (3), (4), (5) imply that $\frac{H_0H}{H_0H'} \cdot \frac{N'H'}{N'A} \cdot \frac{NA}{NH} = 1$, If we then apply Menelaus' Theorem to these three points and triangle AHH', we find that $H_0,\ N,\ N'$ lie on the same line.

In other words, the locus of N is the line H_0N', where N' is the point corresponding to some particular value of h.

(iii). We take (*fig.* t401c) h equal to the radius r of circle Σ which is tangent externally to the three given circles $A_0,\ B_0,\ C_0$. Then clearly the circles $A_r,\ B_r,\ C_r$ (with radius $a+r,\ b+r,\ c+r$) will all pass through the center O of circle Σ, which is therefore the radical center of these three circles, and a position of H corresponding to $h - r$. Thus the line which is the locus of point H must pass through O.

(iv). The point N_r corresponding to $h = r$ will (by definition) divide segment AO in the ratio $a : (a + r)$ (since for this value of h, point H_r coincides with point O), and so must coincide with the point of tangency of circles $A_0,\ \Sigma$.

(v). Analogously, we can take h equal to $-r'$, where r' is the radius of the circle Σ' tangent internally to circles $A,\ B,\ C$. Then the circles $A_{-r},\ B_{-r},\ C_{-r}$

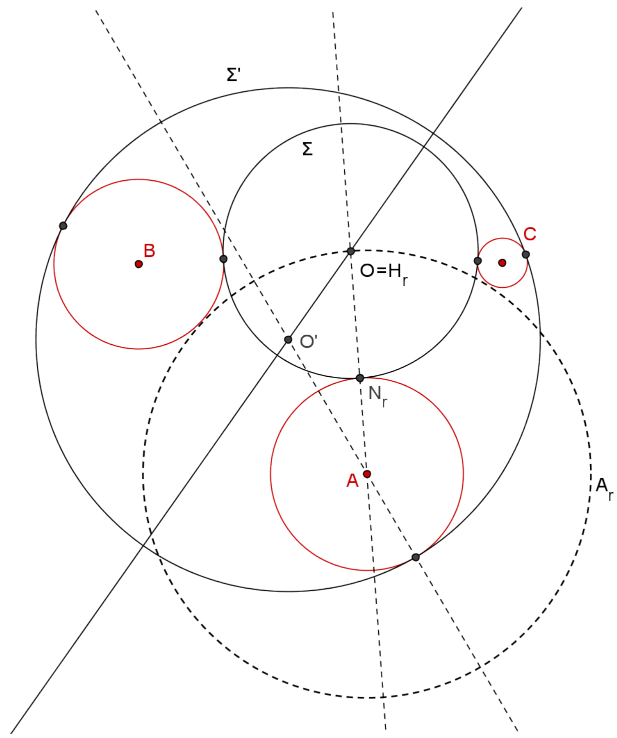

Figure t401c

will all pass through the center O' of circle Σ', which is therefore the radical center of these three circles. Thus the line which is the locus of point H must pass through O'.

Similarly, the locus of point N must pass through the point of contact of circle Σ' with circle A_0.

Notes. We can proceed in the same way in the case of circles tangent internally and externally to $A_0,\ B_0,\ C_0$ in different combinations. For example, if we want to examine the circle tangent externally to A_0, and internally to B_0 and C_0, we take circles centered at $A,\ B,\ C$, with radii $|a-h|,\ |b+h|,\ |c+h|$. Other cases can be treated analogously.

This exercise gives a new construction of circles $\Sigma,\ \Sigma'$, the circles tangent externally and internally to the three given circles. (See **231 -236** and **309 - 312b**). We first construct the radical center H_0 of the three given circles, then the radical center H of circles $A_h,\ B_h,\ C_h$ for any value of h. Next we construct the point N_h corresponding to this value of h. The intersections of line H_0N_h with circle A_0 will be the points of contact of $\Sigma,\ \Sigma'$ with circle A_0. We then draw the two lines connecting these points of contact with point A. Their intersections with line HH_0 will be the centers of Σ and Σ'.

Problem 402. Find a circle which intersects four given circles at equal angles.

Lemma. Construction: Find a circle which intersects three given circles at equal angles.

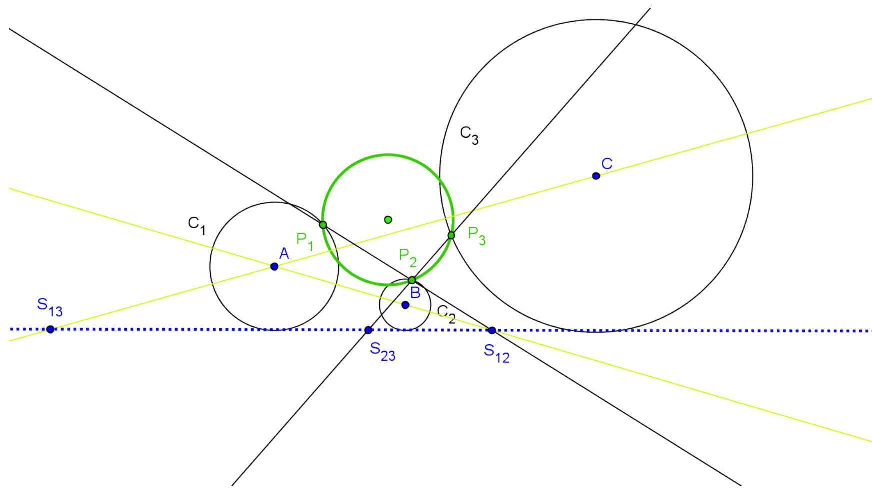

FIGURE t402a

Solution (for lemma). Suppose the circles are $C_1,\ C_2,\ C_3$ (*fig.* t402a). In **227** we saw that the circles making equal angles with $C_1,\ C_2$ are exactly those which pass through a pair of antihomologous points, relative to one of the two centers of symmetry of the two circles. If S_{12} is one of these centers, we can draw any common secant to $C_1,\ C_2$ through it, and find a pair $P_1,\ P_2$ of antihomologous points in circles $C_1,\ C_2$. Then we can choose a center of symmetry S_{23} of circles $C_2,\ C_3$, and draw line P_2S_{23}. Its intersection with circle C_3 will determine a point P_3 which is antihomologous to P_2 with respect to S_{23}. From **227** it follows that the circle through $P_1,\ P_2,\ P_3$ will make equal angles with circles $C_1,\ C_2,\ C_3$. The construction can clearly be done in infinitely many ways.

Solution (to exercise 402). Let the four given circles be $C_1,\ C_2,\ C_3,\ C_4$, and let σ be the required circle, which makes equal angles with them.

It was shown in **227, 309-310** that the circles constructed in Lemma 1, those which make equal angles with any *three* given circles, form four families, corresponding to the four axes of similarity of the circles (**145**). Each axis of similarity serves as the common radical axis of the circles in one of the families. These circles all have a common centerline, which is the perpendicular from the radical center of the three circles to the radical axis of the circles in the family considered.

We apply these observations first to the three circles $C_1,\ C_2,\ C_3$ and then to the three circles $C_1,\ C_2,\ C_4$ to give us two conditions for the location of the center O of the required circle σ. First, O must lie on the perpendicular from the radical center I_4 of circles $C_1,\ C_2,\ C_3$ to one of their axes s_4 of similarity. Second, the center of σ must lie on the perpendicular from the radical center I_3 of circles $C_1,\ C_2,\ C_4$ to one of their axes s_3 of similarity.

We have one more clue. if σ intersects $C_1,\ C_2$ in two pairs of antihomologous points, the the center of similarity S_{12} for which these pairs of points are antihomologous must lie on s_3 and also on s_4. This limits our choice of s_3 to two of the four axes of similarity of circles $C_1,\ C_2,\ C_4$.

These considerations allow us to find the center O of σ. We first choose any axis s_4 (*fig.* t402b) of the four axes of similarity of circles $C_1,\ C_2,\ C_3$. We then choose a center of similarity S_{12} of circles $C_1,\ C_2$ which lies on s_4. Next we choose one of the two axes of similarity s_3 of circles $C_1,\ C_2,\ C_4$ which passes through S_{12}. We drop perpendiculars from I_4 and I_3 (the radical centers of circles $C_1,\ C_2,\ C_3$ and $C_1,\ C_2,\ C_4$ respectively) onto $s_3,\ s_4$ (note the reversal of subscripts). The center of the required circle must be at the intersection of these two perpendiculars.

Each of the four axes of similarity of circles $C_1,\ C_2,\ C_3$ can be paired with two of the axes of similarity of $C_1,\ C_2$, making eight possible centers for the required circle.

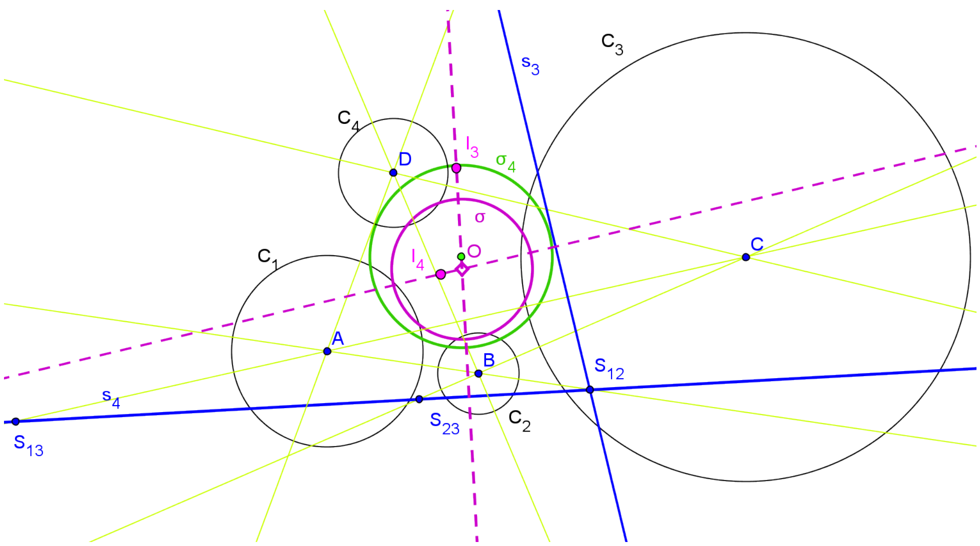

Figure t402b

We choose one of these points O and show how to construct the circle σ centered at O and making equal angles with $C_1,\ C_2,\ C_3,\ C_4$. First (using the result of our lemma) we draw any circle σ_4 making equal angles with $C_1,\ C_2,\ C_3$ (*fig.* t402a) and belonging to the family associated with s_4, the axis of similarity chosen above. Thus s_4 is the radical axis of circles $\sigma,\ \sigma_4$. Thus we can construct σ as the circle centered at the known point O and whose radical axis with circle σ_4 is line s_4.

This last task is not difficult. The radical axis of two circles is the locus of centers of circles orthogonal to both. So we just draw any circle c whose center lies on s_4 and which is orthogonal to circle σ_4. Then σ is the circle centered at O and orthogonal to c.

This circle σ makes equal angles with $C_1,\ C_2,\ C_3$ by construction. But we must prove that it makes the same angle with C_4. To investigate this issue, we draw any circle σ_3 making equal angles with $C_1,\ C_2,\ C_4$, and its associated axis s_3, of similarity for these circles. (Circle σ_3 and s_3 are not shown in figure t402b.)

The common centerline of σ and σ_3 is the perpendicular from I_3 to line s_3. If S_{12} is the intersection of the axes of similarity $s_3,\ s_4$, then S_{12} has the same power with respect to circles σ and σ_3, equal to the power of the inversion about pole S_{12} which takes circle C_1 onto C_2. (This follows from the fact that each of circles $\sigma,\ \sigma_3$ intersects C_1 and C_2 in pairs of antihomologous points with respect to S_{12}). From the facts that (a) the common centerline of $\sigma,\ \sigma_3$ is perpendicular to s_3, and (b) that point S_{12} of line s_3 has the same power with respect to these two circles, it follows that that s_3 is the radical axis of these two circles. Therefore circle σ intersects $C_1,\ C_2,\ C_4$ (and therefore all four circles) at the same angle.

As noted above, there are eight choices for point O, and so, in general, at most eight solutions to the problem. A given choice of point O leads to a solution if O lies outside circle c. Note (see exercise 152) that any choice of point O will either always lie inside, always lie on, or always lie outside, any choice of circle c.

Notes. Students may see right away that the construction of σ depends on the construction of σ_4, and, as our lemma shows, there are infinitely many possible circles σ_4. But all these circles have a common radical axis (the line S_4), so σ is actually determined by its center (which is uniquely fixed) and its radical axis with any of the circles σ_4.

Likewise, the construction of σ depends on the choice of circle c, orthogonal to σ_4 with its center on S_4. It is not hard to see (but difficult to prove directly) that there can be only one such circle σ.

Finally, students may notice that the construction of circle σ depends on the initial choice of three circles to be labeled $C_1,\ C_2,\ C_3$, then on the choice another triple of circles, including two of the first three and also C_4. It would appear that this makes for more choices for point O. But following the construction for other such choices quickly shows that in fact we end up with only distinct eight pairs of axes of similarity, and therefore only eight possibilities for point O.

It may be useful to look at the very special case of four each circles with centers at the vertices of a square. In that case there are infinitely many solutions.

Problem 402b. Find a circle which intersects three given circles at given angles. (We know (Exercise 256) the angle at which the required circle intersects any circle having the same radical center as the given circles. Among these, determine (Note C, **311**) three for which this angle is zero, so as to reduce the problem to the problem of tangent circles; or two[1] for which this angle is a right angle, thus reducing the question to Exercise 259.)

Lemma 1. We are given two circles A and B (with centers at points $A,\ B$), a point M on circle A, and two angles α and β. Construct a circle O which intersects A at M, and which intersects A and B at angles equal to $\alpha,\ \beta$ respectively.

Solution for Lemma 1. Let N (*fig.* t402bi) be one of the points of intersection of circles O and B. We rotate point B about O through an angle equal to $\widehat{NOM}$. Let K be the image of point B under this rotation. We consider triangle OMK. It is congruent to ONB, and has the same orientation. Therefore, point O

[1](1)The problem considered in no. **311** does not always have a solution, since the point α mentioned there might be inside the given circles; this situation can actually occur in this problem, even when the problem has a solution. One should show that this inconvenience can always be avoided by an appropriate combination of the two methods we indicate.

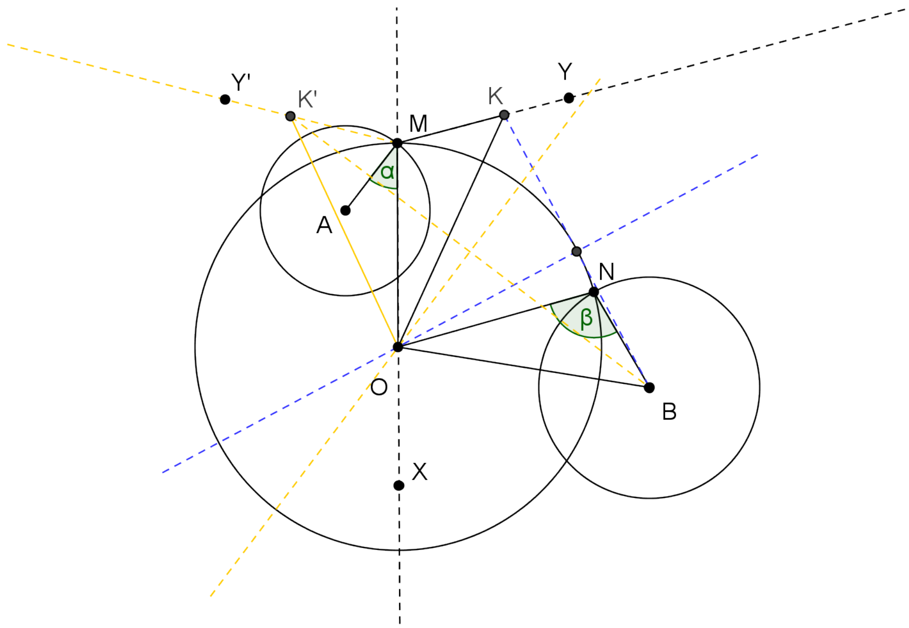

FIGURE t402bi

is equidistant from points K and B. Also AM meets OM at an angle equal to α, and MK meets OM at an angle equal to $\widehat{BNO}$, which is equal to β.

Thus we have the following construction. We draw line MX, forming an angle with AM equal to α, and line MY, forming an angle with XM equal to β. We then lay off segment MK along MY, equal to the radius of circle B. Finally, we draw the perpendicular bisector of segment BK. Its intersection with MX gives us point O.

We can choose for MX either of two lines forming an angle equal to α with MA. In general, the problem has two solutions.

Notes. We get no new solutions if we take for MY the other line forming an angle equal to β with OM. In this case, we will get a point K' symmetric to K in line OM. Then $OB = OK = OK'$, so points $B,\ K,\ K'$ all lie on a circle centered at O, and the perpendicular bisector of any chord of that circle (such as BK') will pass through point O.

In the solution to exercise 256, we noted that the phrase 'angle between two circles' is ambiguous. It could refer to either of the two angles (one obtuse and one acute) made by the tangents to the circles at their point of intersection. Equivalently, it could refer to either of the two angles formed by the lines extending the radii of the two circles drawn to a point of intersection.

So if a circle meets two circles at 'equal angles', we can construe these angles to be $\alpha,\ \beta$, or $180° - \alpha,\ 180° - \beta$, or $\alpha,\ 180° - \beta$, or $180° - \alpha,\ \beta$. In the present situation, the first two conditions describe a single family of circles, and the second two conditions describe a different family of circles.

The construction described will give us circles of either family, depending on how we choose point X. If we take point X such that angle $\widehat{AMX}$ is equal to

α (and not $180^\circ - \alpha$) , and lay off segment MK so that $\widehat{XMK} = \beta$ (and not $180^\circ - \beta$), then we get a circle of the first family. Indeed, if point O lies on ray MX, the resulting circle will clearly intersect the given circles in angles equal to α and β. But if O lies on the extension of line MX past M, then the resulting circle will intersect the given circles in angles equal to $180^\circ - \alpha$ and $180^\circ - \beta$.

As special cases, we can use this lemma to construct circles which are tangent (internally or externally) to two given circles, and passing through a given point on one of the circles (its point of tangency with that circle).

In the proof of the main problem, we will not use the full force of this lemma. We will need only to construct a circle meeting two given circles at two given angles. This easier problem can be solved in infinitely many ways, as we can pick point M on circle A arbitrarily.

Lemma 2. Construct a circle having a common radical axis with each of two given circles, and orthogonal to a third given circle.

Solution to Lemma 2. Let the first two given circles be A and B, and let C be the third given circle. Using **138**, we can construct two circles C', C'' each orthogonal to both A and B. The centers of C', C'' will be on the radical axis of A and B, which is also the radical axis of the circle we need. Hence (by the note to **139**e), each of circles C', C'' must be orthogonal to the circle we need. Thus, if the required circle exists, we can construct it using **139**, by constructing the circle orthogonal to C, C', and C''. The center of such a circle is the radical center of these three circles, and its radius is the tangent from the radical center to one of the three.

Solution to Exercise 402b. Suppose we must construct a circle σ which intersects circles A, B, C at angles equal to α, β, γ respectively.

As noted in exercise 256, and also in the discussion of lemma 1, the phrase ‘angle between two circles’ is ambiguous. Taking this ambiguity into consideration, the required circle can belong to one of four families of circles, intersecting the given circles at angles equal to:

i) $\alpha,\ \beta,\ \gamma$ or $180^\circ - \alpha,\ 180^\circ - \beta,\ 180^\circ - \gamma$;
ii) $180^\circ - \alpha,\ \beta,\ \gamma$ or $\alpha,\ 180^\circ - \beta,\ 180^\circ - \gamma$;
iii) $\alpha,\ 180^\circ - \beta,\ \gamma$ or $180^\circ - \alpha,\ \beta,\ 180^\circ - \gamma$;
iv) $\alpha,\ \beta,\ 180^\circ - \gamma$ or $180^\circ - \alpha,\ 180^\circ - \beta,\ \gamma$.

We examine the first case, and make the further assumption that the angles between the radii of the circle are strictly equal to α, β, γ. We indicate in parentheses the changes necessary if these angles are replaced by their supplements.

We first use Lemma 1 to construct some circle σ_1 (*fig.* t402bii) which intersects circles A and B at angles equal to α and β. (Alternatively, we could construct a circle σ_1' which intersects circles A and B at angles equal to $180^\circ - \alpha$ and $180^\circ - \beta$.) The required circle σ and circle σ_1 (or σ_1') will make equal angles with circles A and B. It follows from the solution to problem 256 that any circle which has a common radical axis with A and B will intersect circles σ and σ_1 at a constant angle.

Next, using the result of **311**, we choose, from among the circles with a common radical axis with A and B, two circles A' and B' which are both tangent to σ_1 (or to σ_1'). We will assume for now that these two circles exist. Since we can think of tangent circles as circles which intersect at angles of 0° or 180°, the considerations above show that circles A', B' will also be tangent to circle σ. Moreover, circle A'

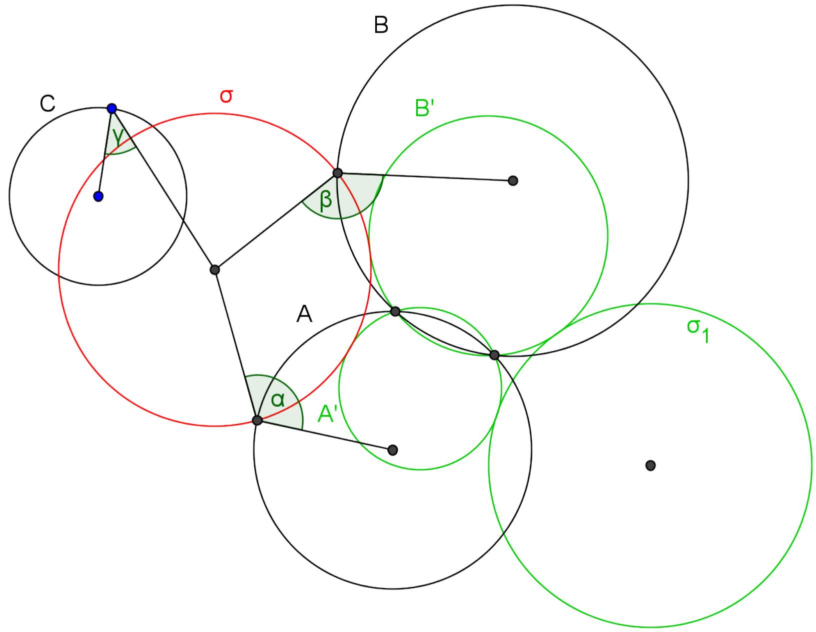

Figure t402bii

will be tangent to σ and σ_1 in the same way (both externally or both internally), since the circles meet at angles which are strictly equal (or supplementary). The same is true of circle B'.

Thus the required circle σ must be tangent to each of circles A', B' (and we can tell whether this tangency is internal or external), and must intersect circle C at an angle strictly equal to γ.

We now construct any circle σ_2 which intersects C at an angle γ, and is tangent to circle A' in the same manner (internally or externally) that A' is tangent to circle σ. (Or, we construct circle σ_2', which intersects C at an angle $180° - \gamma$ and is tangent to circle A' in the opposite manner from the way A' is tangent to σ). Lemma 1 shows us how to do this construction: the fixed circles are σ_2, A' (or σ_2', A'), and the given angles are γ, $0°$ (or their supplements).

The result of exercise 256 assures us that any circle that has a common radical axis with C and A' will intersect circles σ, σ_2 at equal angles (or will intersect circles σ, σ_2' at supplementary angles).

We know that among the circles having a common radical axis with A' and C, there is one circle (namely A') which is tangent to circle σ_2 (or to σ_2'). Therefore (following the construction of **311**) among these same circles, there is in general a second circle C' which is tangent to circle σ_2 (or to σ_2'). Furthermore, circles σ, σ_2 are tangent to C' in the same manner (or tangent to circles σ, σ_2' in different manners).

It follows that the required circles σ must be tangent to three circles A', B', C' which we know how to construct, and that the manner of their tangency (internal or external) is completely determined.

We have been looking for circle σ which intersects the given circles at angles strictly equal to $\alpha,\ \beta,\ \gamma$. Using analogous arguments, it is not hard to see that a circle σ' which intersects the three given circles at angles strictly equal to $180^\circ - \alpha,\ 180^\circ - \beta,\ 180^\circ - \gamma$ must be tangent to the same circles $A',\ B',\ C'$ as circle σ. However, the two circles $\sigma,\ \sigma'$ must have different manners of tangency with each of circles $A',\ B',\ C'$. This follows from the fact that the construction of circle σ' can be accomplished using the same auxiliary circles σ_1 or σ_1' that we used above, and also the same auxiliary circles σ_2 or σ_2' that we used above. However, circles $\sigma_1,\ \sigma_1'$ exchange roles, as do circles $\sigma_2,\ \sigma_2'$.

We can also prove the converse: that any circle tangent to $A',\ B',\ C'$ in the manner indicated (internally or externally) will intersect the given circles at angles equal to $\alpha,\ \beta,\ \gamma$ or $180^\circ - \alpha,\ 180^\circ - \beta,\ 180^\circ - \gamma$. Indeed, suppose some circle σ is tangent to circles $A',\ B'$ in the same manner that $A',\ B'$ are tangent to σ_1, and that σ is tangent to C' in the same manner as it is tangent to σ_2. Circles $\sigma,\ \sigma_1$ are tangent to A' in the same manner; that is, they intersect A' at the same angle (equal to 0° or 180°). Similarly, these circles $\sigma,\ \sigma_1$ also intersect B' at the same angle (0° or 180°). Therefore any circle which has a common radical axis with A' and B' will intersect σ and σ_1 at exactly the same angle. In particular, this is true of circle A. But circle σ_1, by construction, intersects A at an angle α. It follows that circle σ also intersects A at angle α. The same reasoning shows that σ intersects B at angle β.

Also, circles $\sigma,\ \sigma_2$ are both tangent to A' in the same manner, because $\sigma,\ \sigma_1$ are tangent to A' in the same manner (by definition of circle σ_1), and $\sigma_1,\ \sigma_2$ are also tangent to A' in the same manner (by definition of circle σ_2). For analogous reasons, circles $\sigma,\ \sigma_2$ are tangent to C' in the same manner. Thus any circle having a common radical axis with A' and C', and in particular circle C, will intersect σ and σ_2 in angles which are strictly equal. But circle σ_2, by construction, intersects C at an angle strictly equal to γ. Therefore circle σ also intersects C at an angle strictly equal to γ.

In just the same way, we can show that any circle σ' which is tangent to circles $A',\ B'$ in the opposite manner from its tangency to σ_1, and which is also tangent to C' in the opposite manner from its tangency to σ_2 will intersect circles $A,\ B$, and c at angles strictly equal to $180^\circ - \alpha,\ 180^\circ - \beta,\ 180^\circ - \gamma$.

Thus the required circles belonging to the first (i) of the four families listed above, are tangent to the three circles $A',\ B',\ C'$. Also, they comprise one of four pairs of circles which are tangent to $A',\ B',\ C'$, in the sense of the solution to exercise 267. Indeed, it follows from the argument above that if one of the required circles belonging to this first family is tangent to two of the three circles (say A' and B') in the same manner, than any circle belonging to the first family is tangent to them in the same manner.

So far, we have assumed that among the circles having a common radical axis with A and B, there exist two circles $A',\ B'$ which are tangent to the auxiliary circle σ_1 (or σ_1'). This will always be the case, if circles A and B have no points in common. Indeed, if we construct circles $A',\ B'$ as in **311**, we will find two solutions for this case, because the radical center α of circles $A,\ B,\ \sigma_1$ (or $A',\ B',\ \sigma_1'$) which we used there, will lie outside the three circles. But in fact circles $A',\ B'$ can exist even when circles A and B intersect (see figure 402bii).

But suppose there is no circle which has a common radical axis with $A,\ B$ and tangent to the required circle, and suppose this is true also for the pairs of circles $B,\ C$ and $C,\ A$. Then the argument above is no longer valid. This situation can occur only when each pair of the three given circles has points in common. One example of such a situation is shown in figure t402biii.

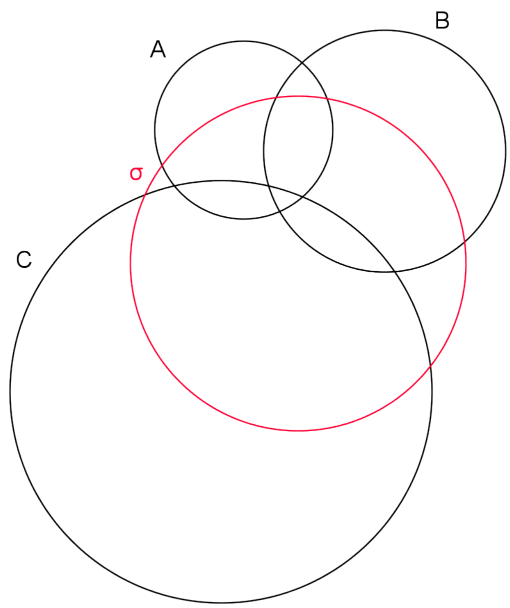

FIGURE t402biii

For a solution to the problem in this situation we can argue differently. We construct auxiliary circle σ_1 (or σ_1') as before. Then we use lemma 2 above to find a circle B_0 having a common radical axis with $A,\ B$ and orthogonal to σ_1 (or σ_1'). The required circle σ (or σ') must also be orthogonal to B_0 , since circles $\sigma,\ \sigma_1$ (or $\sigma,\ \sigma_1'$) intersect B_0 at equal angles. Analogously, if we construct a new auxiliary circle σ_2,which intersects A and B at angles equal to α and γ, we can find a circle C_0 which has a common radical axis with A and C and which is orthogonal to circle σ_2. Circle C_0 will also be orthogonal to the required circle σ.

Thus the required circle σ (or σ') must intersect circle A at an angle strictly equal to α (or $180° - \alpha$), and must be orthogonal to circles $B_0,\ C_0$. Conversely, we can show that any circle which satisfies the conditions just described will also satisfy the conditions of the original problem. Thus we are led to exercise 259.

Note that this second type of argument will also work in those cases where the first argument will not; that is, when the three given circles intersect in pairs. Indeed, through every pair of intersections, for example the intersections of circles A and B, we can draw a circle orthogonal to the known circle σ_1 (see the note in the solution to exercise 258).

However, this second argument can fail, in some of those cases where the first argument holds. Indeed, if two circles (say A and B) do not intersect, then there may not exist a circle having a common radical axis with the two, and orthogonal to, say σ_1. And there are still more cases to consider, if pairs of the given circles intersect.

We have here examined the two cases covered by family (i) of circles as defined above. The discussion for the other three families of circles would proceed analogously, by substituting one of the angles α, β, γ with its supplement. We would need to define auxiliary circles analogous to A', B', C', C_0, B_0. In general, these circles will be different from those constructed in the solution above.

Each of the four families of circles will include at most two circles satisfying the conditions of the problem. Thus there can be as many as eight solutions to the problem.

Problem 403. Given three circles, find a fourth whose common tangents with the first three have given lengths. (Reduce this to the preceding problem by drawing, through each point of contact of these common tangents, a circle concentric with the corresponding given circle.)

Lemma. Given a circle A (with center A) and a segment a, find all those circles whose common tangents with A have length a.

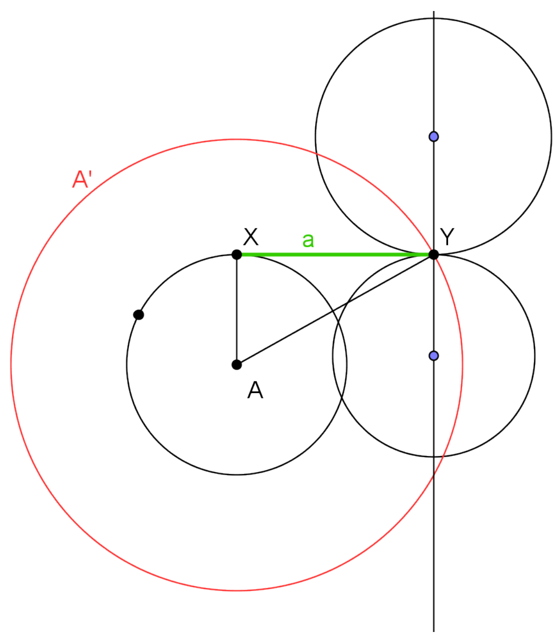

FIGURE t403

Solution to Lemma. We choose some point X on circle A, draw a tangent to A at X, and lay off a segment XY equal to a along this tangent (in either direction). We then draw circle A' with radius AY, and line ℓ through y perpendicular to XY. Clearly any circle with its center on ℓ will have XY as its common tangent with circle A, so these circles will among those we are required to find. And, since

$AX \parallel \ell$, any circle B with its center on ℓ will form an angle equal to $\widehat{XAY}$ with circle A'.

However, note (figure t403) that if the center of the circle is on the same side of XY as point A, then XY is a common *external* tangent, and circles A, B will meet at an angle *strictly* equal to $\widehat{XAY}$. But if the center of the circle is on the opposite side of XY as point A, then XY is a common *internal* tangent, and circles A, B will meet at an angle *strictly* equal to $180° - \widehat{XAY}$.

Any other circle with a common tangent with A of length a can be obtained by starting with other points of circle A playing the role of X. Alternatively, we can rotate figure t403a around point A to get all these circles.

Solution to exercise 403. Suppose the original circles are A, B, C, and the given lengths are α, β, γ. Let the required circle be called σ. Then our lemma tells us that σ must intersect a certain circle A' (fully determined by A and α) at a particular angle. Similarly, σ must meet a certain circle B', fully determined by B and β, at another angle, determined by the same given data, and σ must meet a third determined circle C' at a determined angle.

Thus we are led to the situation of exercise 402b.

Problem 404. We are given a circle, two points A, A' on this circle, and a line D. Show that this line contains points I, I' with the following property: if P, P' denote the intersections of D with the lines joining A, A' with a variable point M on the circle, the product $IP \cdot I'P'$ remains constant, that is, independent of the position of M.

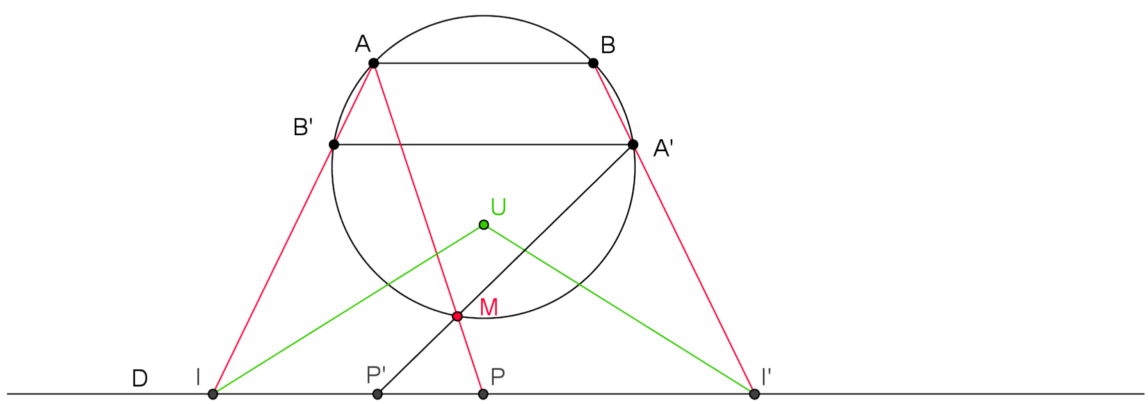

FIGURE t404

Solution. We draw chords $AB,\ A'B'$ through $A,\ A'$ respectively, both parallel to line D. Let $I,\ I'$ be the intersections of $AB', A'B$ with line D. We will show that I, I' are the required points.

Indeed, we have $\widehat{AIP} = \widehat{AB'A'}$; $\widehat{A'I'P'} = \widehat{BA'B'}$ (from parallel lines), and $\widehat{AB'A'} = \widehat{BA'B'}$, since $ABA'B'$ is an isosceles trapezoid. Hence $\widehat{AIP} = \widehat{A'I'P'}$. And $\widehat{IAP} = \widehat{B'A'M}$ (they both intercept arc $\widehat{B'M}$ on the given circle), and $\widehat{B'A'M} = \widehat{A'P'I'}$ (again, from parallel lines). Hence $\widehat{IAP} = \widehat{A'P'I'}$, so triangles $AIP,\ P'I'A'$ are similar. It follows that $IP \cdot I'P' = IA \cdot I'A'$, which is independent of point M.

Note. The main difficulty here is to locate the required points $I,\ I'$.

The proof above must be changed slightly if point M is on minor arc $\widehat{BB'}$, a case not shown in the diagram. But the result still holds.

Special cases occur when M falls on B or B', so that AM or $A'M$ is parallel to D. As M approaches B, point P recedes to infinity, and P' approaches I'. Similarly, as M approaches B', point P approaches I.

Problem 405. In the preceding problem, assume that the line D does not intersect the circle. Show that on each side of the line there is a point from which the segment PP' subtends a constant angle (Exercise 278).

Solution. We use figure t404, which shows a case in which line D does not intersect the given circle.

In exercise 2 we defined the limit point of two circles. We can analogously define the limit point of a given circle C and a line D. We just choose any circle C' for which D is the radical axis of $C,\ C'$. We define the limit points of circle C and line D as the limit points of circles $C,\ C'$. (For example, we can take C' to be the circle symmetric to C in line D.)

This definition makes sense if we consider C' as a member of a family of circles, all of whom have a common radical axis D with C. As the radius of such a circle increases, the circle approaches line D as a limit.

We can use a slight variation of the argument in exercise 152 to prove that any circle with its center on D and orthogonal to C passes through two fixed points, and that these fixed points are the limit points of circle C and line D, as defined above.

We proceed to the proof of the statement in the problem. Let U be one of the limit points of the given circle and line D. As in the solution to exercise 152, we obtain $IU^2 = I'U^2 = IA \cdot IB'$. From the result of exercise 404, this gives us $IU^2 = I'U^2 = IP \cdot I'P'$. By symmetry, we have $\widehat{UIP} = \widehat{UI'P'}$, and these two pieces of information show (**118**, second case) that triangles $UIP,\ P'I'U$ are similar. The similarity of these triangles gives us $\widehat{IUP} = \widehat{I'P'U}$, so $\widehat{PUP'} = 180^\circ - \widehat{IPU} - \widehat{I'P'U} = 180^\circ - \widehat{IPU} - \widehat{IUP} = \widehat{UIP}$, which is constant.

This argument furnishes the point U at which segment PP' subtends a constant angle. The other limit point of D and the given circle lies on the other side of line D, and must have the same property.

Problem 406. We are given circles S, Σ without common points, with centers O, ω, and radii R, ρ, and we consider the circles C which are tangent to S and orthogonal to Σ.

1°. Show that all of these circles are tangent to a second fixed circle;

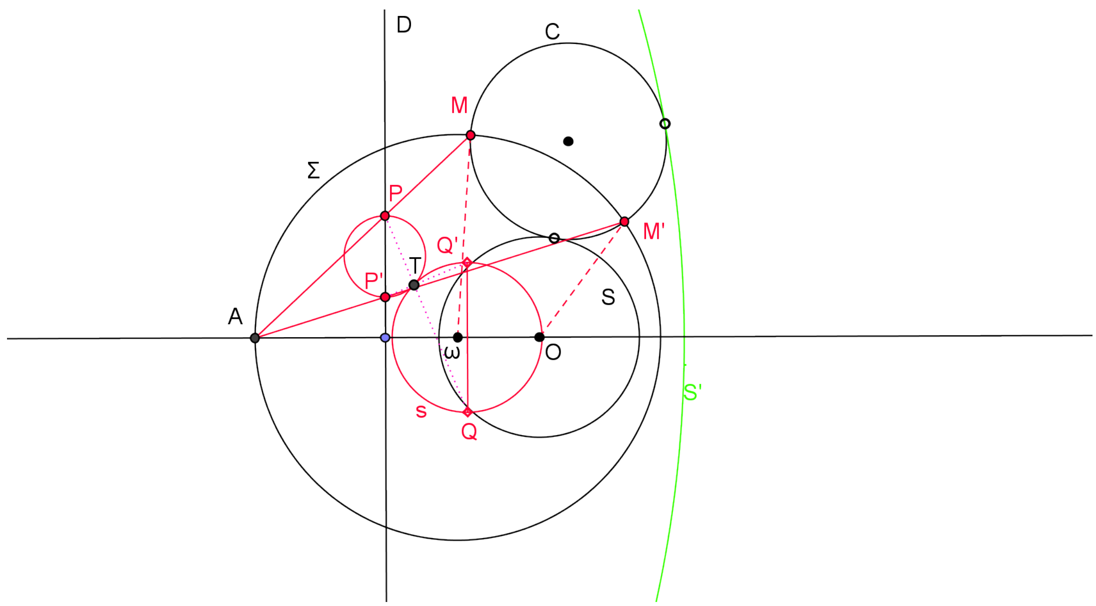

FIGURE t406a

Solution. Suppose C (*fig.* t406a) is any circle tangent to S and orthogonal to Σ. Consider an inversion in circle Σ. This inversion takes C onto itself (**227b**), and S onto another circle S' which must also be tangent to C (**219**, corollary).

2°. Denote by $M,\ M'$ the common points of C with Σ and, through a fixed point A on $O\omega$, draw parallel lines to the bisectors of the angles $\widehat{O\omega M},\ \widehat{O\omega M'}$, to their intersection points $P,\ P'$ with a fixed line D, perpendicular to $O\omega$. Show that there exist two points $X,\ X'$ such that the lines $XP,\ X'P'$ are always perpendicular;

Solution. We first examine a very special case, when the point A mentioned in the problem statement is one of the intersections of line $O\omega$ with circle Σ. We draw $AM,\ AM'$, and note that $\widehat{O\omega M} = 2\widehat{OAM}$ and $\widehat{O\omega M'} = 2\widehat{OAM'}$. Thus $AM,\ AM'$ are, for this special case, the lines through A parallel to the bisectors of $\widehat{O\omega M},\ \widehat{O\omega M'}$. Put another way, for this position of A, the two lines described in the problem statement pass through points $M,\ M'$.

We now consider an inversion about point A which takes circle Σ onto the given line D. (Note that the power of this inversion is the distance from A to the intersection of D and Σ). This inversion takes points $M,\ M'$ onto points $P,\ P'$. The image of circle C is a circle through P and P' which is orthogonal to line D; that is, circle C inverts into the circle c with diameter PP'. Circle S inverts into some circle s. Since C is tangent to S, circle c must be tangent to circle s at a point T. This point T is a center of similarity for circles $c,\ s$, and so if we extend lines $PT,\ P'T$ past T, they will intersect s at the endpoints $Q,\ Q'$ of a diameter parallel to line D. Because we can describe QQ' with respect to line D and circle s, and without regard to the particular circle C we chose, points $Q,\ Q'$ do not depend on C, and so play the role of $X,\ X'$ as required in the problem.

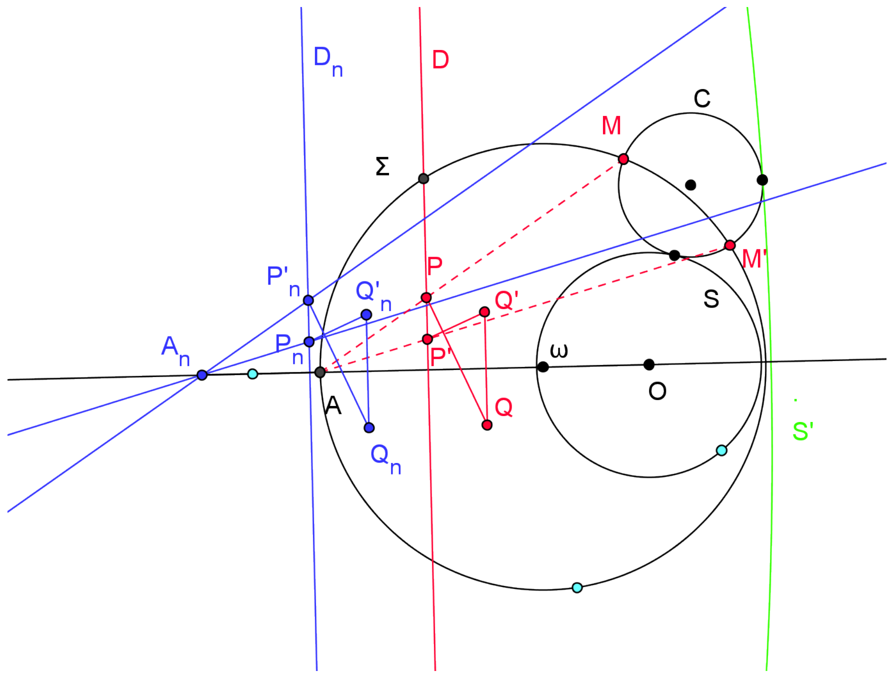

FIGURE t406b

Thus we have solved the problem for a point A which is the intersection of line $O\omega$ and circle Σ, and for any line D perpendicular to $O\omega$. We now translate point A parallel to $O\omega$ by any distance at all to get point A_n, and apply the same translation to line D to get D_n. If we carry out the same construction described in the problem statement, we get two new lines through A_n, parallel respectively to $AM,\ AM'$. Their intersections with D_n are simply the images of $P,\ P'$ under the same translation. Thus if we translate $Q,\ Q'$ to get points $Q_n,\ Q'_n$, these new points will also have the property that $Q_nP_n \perp Q'_nP'_n$, independent of the position fo circle C. Thus $Q_n,\ Q'_n$ fulfill the requirements of the problem statement.

To solve the problem for any point A_n on line $O\omega$, and any line D_n perpendicular to $O\omega$, we first translate A_n to coincide with A (the intersection of Σ and $O\omega$. The same translation will take line D_n onto some line D, perpendicular to $O\omega$. We then solve the problem for $A,\ D$, and use the inverse of the translation we had applied to get a solution for $A_n,\ D_n$.

3°**.** Show that there exist two points at which the segment PP' subtends a constant angle (preceding exercise).

Solution. We will use the results of exercises 404, 405 to show that the limit points of circle s and line D satisfy the conditions of the problem.

First note that if we had known where points $Q,\ Q'$ lie (for any circle C), then we could determine from them points $P,\ P'$ by using the construction described in the statement of exercise 404. That is, $P,\ P'$ are the intersections with D of the lines joining fixed points $Q,\ Q'$ to a variable point T on circle s.

Now by assumption, circles Σ and S do not intersect. Therefore circle s and line D, their images under a certain inversion, also do not intersect . So we can apply the result of exercise 405 to $s,\ D$, and the limit points of this circle and line satisfy the conditions of the problem.

4°**.** Consider a position C_1 of the circle C, intersecting Σ in $M,\ M'$, then a second position C_2 intersecting Σ in M' and a third point M'', then C_3 intersecting Σ in M'', M''', etc. Find a condition for the circle C_{n+1} to coincide with C_1. (If d is the distance $O\omega$, the right triangle whose hypotenuse is $d^2-R^2-\rho^2$ (or $R^2+\rho^2-d^2$) and a leg equal to $2R\rho$, must have an acute angle equal to half the central angle of a regular (convex or star) polygon whose number of sides is n or a divisor of n.) If the circles S, Σ have a common point,the limiting position of points $M,\ M',\ M'',\ M'''$ will be their points of intersection.

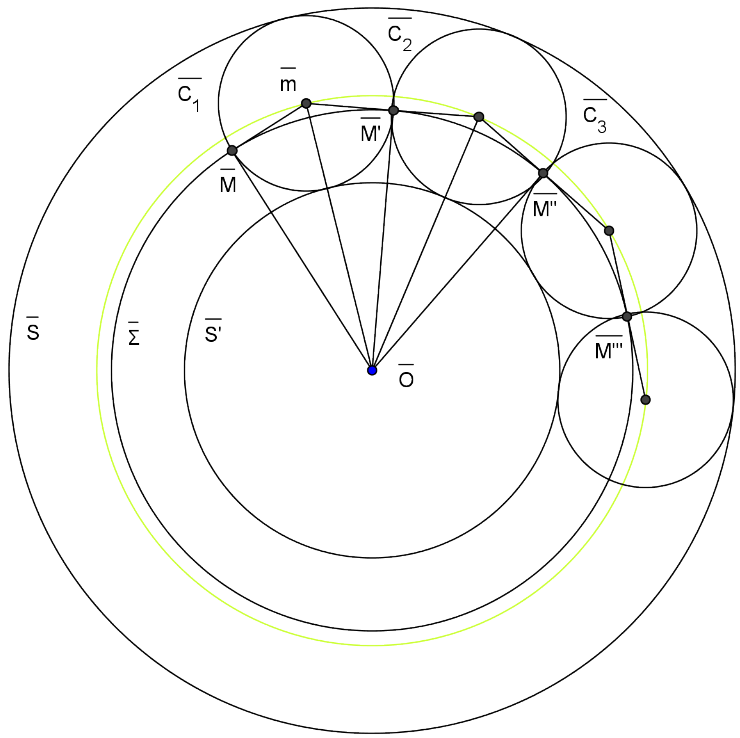

Figure t406c

Solution. We assume that C_{n+1} coincides with C_1 for some n, and continue to assume that circles $S,\ \Sigma$ do not intersect. Under these conditions, the result of exercise 248 assures us that there is an inversion taking S, Σ onto two concentric circles. Figure t406c shows the result of this inversion. The image of S is labeled $\overline{S}$, the image of Σ is labeled $\overline{\Sigma}$, and so on.

We saw, in part (a) of this exercise, that if we invert S in circle Σ, we obtain a circle S' which is tangent to all the circles C_i. It follows from exercise 250 that the image $\overline{S'}$ of S' in figure t406c will be a circle inverse to $\overline{S}$ in circle $\overline{\Sigma}$. Thus $\overline{S'}$ will be concentric with $\overline{S}$ and $\overline{\Sigma}$. And since inversions preserve tangency, (**219**, corollary) $\overline{S'}$ will be tangent to all the circles $\overline{C_i}$. From this it follows that the circles $\overline{C_i}$, which are tangent to two concentric circles, must all have equal radii. And certainly each $\overline{C}_i$ is tangent to $\overline{C_{i+1}}$.

Let $\overline{O}$ be the common center of $\overline{S}$, $\overline{\Sigma}$, $\overline{S'}$, let $\overline{M}$, $\overline{M'}$ be the intersections of $\overline{C_1}$ with circle $\overline{\Sigma}$, and let $\overline{m}$ be the center of $\overline{C_1}$. Angle $\widehat{\overline{MOM'}}$ is equal to the corresponding angles subtended by each circle $\overline{C_i}$. Since $\overline{C}_1$, $\overline{C}_{n+1}$ coincide, this means that n copies of $\widehat{\overline{MOM'}}$ must add up to some integer multiple of $360°$, and so segment $\overline{MM'}$ is the side of one of the polygons described in the problem statement, $\widehat{\overline{MOM'}}$ is one of its central angles, and $\widehat{\overline{MOm'}}$ is half its central angle.

Now suppose $\overline{R}$ is the radius of circle $\overline{S}$, and $\overline{\rho}$ is the radius of circle $\overline{\Sigma}$. We can compute the radius of $\overline{S'}$, the inverse of $\overline{S'}$ with respect to $\overline{\Sigma}$, by looking at the points of tangency of $\overline{C_1}$ to $\overline{S}$ and $\overline{S'}$. These points are diametrically opposite on $\overline{C_1}$, and are endpoints of radii of $\overline{S}$, $\overline{S'}$. Since they are also inverses of each other, we have $\overline{R'} \cdot \overline{R} = \overline{\rho^2}$, where $\overline{R'}$ is the radius of circle $\overline{S'}$. Hence $\overline{R'} = \frac{\overline{\rho^2}}{\overline{R}}$. We can compute $\overline{Om}$ as the average of the radii of $\overline{S}$, $\overline{S'}$, so that $\overline{Om} = \frac{1}{2}\left(\overline{R} + \frac{\overline{\rho^2}}{\overline{R}}\right)$.

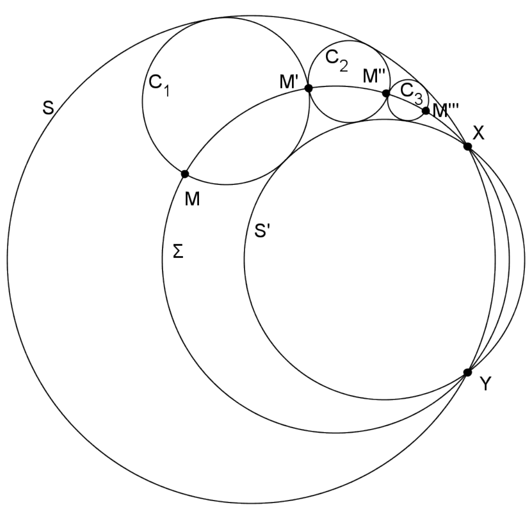

FIGURE t406d

The rest of the computation required by the problem can be done in several ways. One way is to note that triangle $\overline{OMm}$ is right-angled at $\overline{M}$, and we have just computed the lengths of a leg ($\overline{OM}$) and a hypotenuse ($\overline{Om}$). If we multiply these quantities by $2\overline{R}$, we get the length of a leg and the hypotenuse of a triangle similar to $\overline{OMm}$. A leg of this triangle (corresponding to $\overline{OM}$ will have length $2\overline{R\rho}$, and its hypotenuse will have length $\overline{R}^2 + \overline{\rho}^2$. If $C_1,\ C_{n+1}$ coincide (as we have been assuming), an acute angle α of this triangle is half the central angle of a regular polygon such as mentioned in the diagram.

Now we apply the final result of exercise 396. The statement there implies that the quantity $\frac{d^2-r^2-r'^2}{rr'}$ must be invariant under the transformation taking figure t406b onto figure t406c (where d is the distance between any two circles in one of the diagrams, and $r,\ r'$ are the radii of the two circles). We apply this to figure t406c and circles $\overline{S},\ \overline{S'}$. Here, $d = 0$, so the invariant quantity is $\frac{\overline{R}^2+\overline{\rho}^2}{2\overline{R\rho}}$. For figure t406b, and circles $S,\ S'$, the invariant quantity is $\pm\frac{d^2-R^2-\rho^2}{2R\rho}$, where $d = O\omega$.

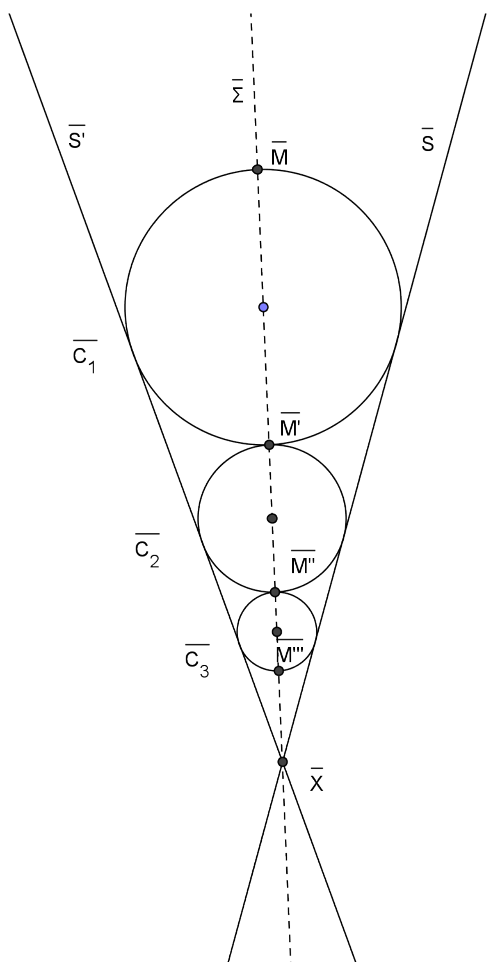

FIGURE t406e

Thus we have, from the invariance described in exercise 396, $\frac{\overline{R}^2+\overline{\rho}^2}{2\overline{R}\rho}=\pm\frac{d^2-R^2-\rho^2}{2R\rho}$. This means that the angle α mentioned above is the same as an angle of a right triangle with hypotenuse $\pm d^2-R^2-\rho^2$ and a leg equal to $2R\rho$. This is exactly the condition described in the problem statement. And since the argument can be reversed, this condition is sufficient as well as necessary.

We now examine the situation in which circles $S,\ \sigma$ have a point in common. This case is significantly easier. In figure t406d, circles $S,\ \Sigma$ intersect at points $X,\ Y$. As before, the circles C_i are all tangent to circle S', the inversion of S in Σ. In this, case circle S' will also pass through $X,\ Y$ (since these points are on the circle of inversion).

We invert figure 406d in any circle centered at point Y, with the result shown in figure 406e. The three circles $S,\ S',\ \Sigma$ invert into three lines through $\overline{X}$ (the image of point X). Since circles $S,\ S'$ make equal angles with Σ, the image $\overline{\Sigma}$ bisects the angle between the lines $\overline{S},\ \overline{S}'$. Thus circles C_i invert into circles $\overline{C_i}$, with centers on line $\overline{\Sigma}$ (the image of Σ) and tangent to lines $\overline{S},\ \overline{S'}$ (the images respectively of $S,\ S'$. The circles $\overline{C_i}$ are also tangent in pairs, at points $\overline{M},\ \overline{M'},\ \overline{M''}$, and so on.

If we consider the homothecy with center $\overline{X}$ which takes point $\overline{M}$ onto $\overline{M'}$, we have $\overline{XM} : \overline{XM'} = \overline{XM'} : \overline{XM''} = \overline{XM''} : \overline{XM''} = \dots$. From this it is clear that the sequence of points $\overline{M},\ \overline{M'},\ \overline{M''}\dots$ tends towards $\overline{X}$, so the s that the sequence $M,\ M',\ M''\dots$ tends towards X as a limit.

Problem 407. The perpendicular from the intersection of the diagonals of a cyclic quadrilateral to the line which joins this point to the center of the circumscribed circle is divided into equal parts by the opposite sides of the quadrilateral. (Apply the remark in no. **211**.)

Solution. Figure t407 shows inscribed quadrilateral with diagonals $AC,\ BD$. Point O is the circumcenter, and opposite sides $AD,\ BC$ intersect at F. Line KL is perpendicular to OE, and we must show that $ME = NE$. The result is unexpectedly difficult to achieve. Our argument is mostly algebraic, and rests heavily on Menelaus' Theorem (**192**).

Our computations, as is usual when using this theorem, will be done using directed line segments. We first apply Menelaus' theorem to triangle FMN, with transversal AEC:

(1) $$\frac{AF}{AM}\cdot\frac{EM}{EN}\cdot\frac{CN}{CF}=1.$$

Next we apply it to the same triangle (FMN), but with transversal BED:

(2) $$\frac{DF}{DM}\cdot\frac{EM}{EN}\cdot\frac{BN}{BF}=1.$$

From **131** we have $AF\cdot DF = BF\cdot CF$, The plan of the following computation is to isolate the expression $\frac{EM}{EN}$ and show that its value is equal to 1. That will solve the problem.

To that end, we multiplying (1) and (2) together (we have rearranged and grouped various terms):

(3) $$\left(\frac{EM}{EN}\right)^2\cdot\frac{AF\cdot DF}{BF\cdot CF}\cdot\frac{BN\cdot CN}{AM\cdot DM}=1.$$

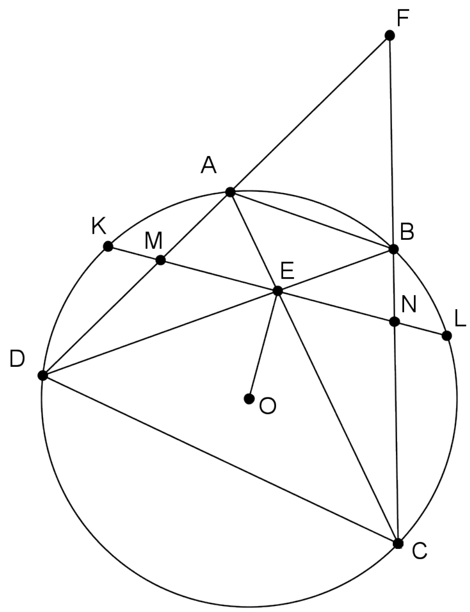

Figure t407

Now $AF \cdot DF = BF \cdot CF$, since both are the power of point F with respect to the circumcircle. So we can ignore this fraction. We examine the product $BN \cdot CN$ in some detail. We have $BN \cdot CN = NK \cdot NL$, since both are the power of N with respect to the circle. And since OE is the perpendicular bisector of segment KL, we have $EK = EL$. Thus $BN \cdot CN = NK \cdot NL = (EK + NE)(EL - NE) = (EK + NE)(EK - NE) = EK^2 - NE^2$.

We perform a similar transformation with the product $AM \cdot DM$. We have: $AM \cdot DM = MK \cdot ML = (EK - ME)(EL + ME) = (EK - ME)(EK + ME) = EK^2 - ME^2$. Substituting these values for the products $BN \cdot CN$, $AM \cdot DM$ into (3), we find

$$\left(\frac{EM}{EN}\right)^2 = \frac{EK^2 - EM^2}{EK^2 - EN^2}.$$

Using properties of proportions (see note), we find that this last equation is equivalent to $\left(\frac{EM}{EN}\right)^2 = \left(\frac{EK}{EK}\right)^2 = 1$, which proves the assertion of the problem.

Note. The last transformation uses the fact that if $a : b = c : d$, then $a : b = (a + c) : (b + d)$. Students not familiar with this property of proportions can be asked to give a quick algebraic proof.

Problem 408. Given two circles C, C', and two lines which intersect them, the circle which passes (Exercise 107b) through the intersections of the chords of the arcs intercepted on C with the chords of the arcs intercepted on C' has the same radical axis as C, C' (use Exercise 149).

Solution. We use the notation of exercise 107b, on which this problem builds. In figure t408a, the two given circles are C, C', the two given lines are AB and CD.

Circle C cuts off chords AB, CD on the given lines, while circle C' cuts off chords $A'B'$, $C'D'$. Line $A'C'$ intersects lines AC, BD at points P and R respectively. Line $B'D'$ intersects lines AC, BD at points Q, S respectively.

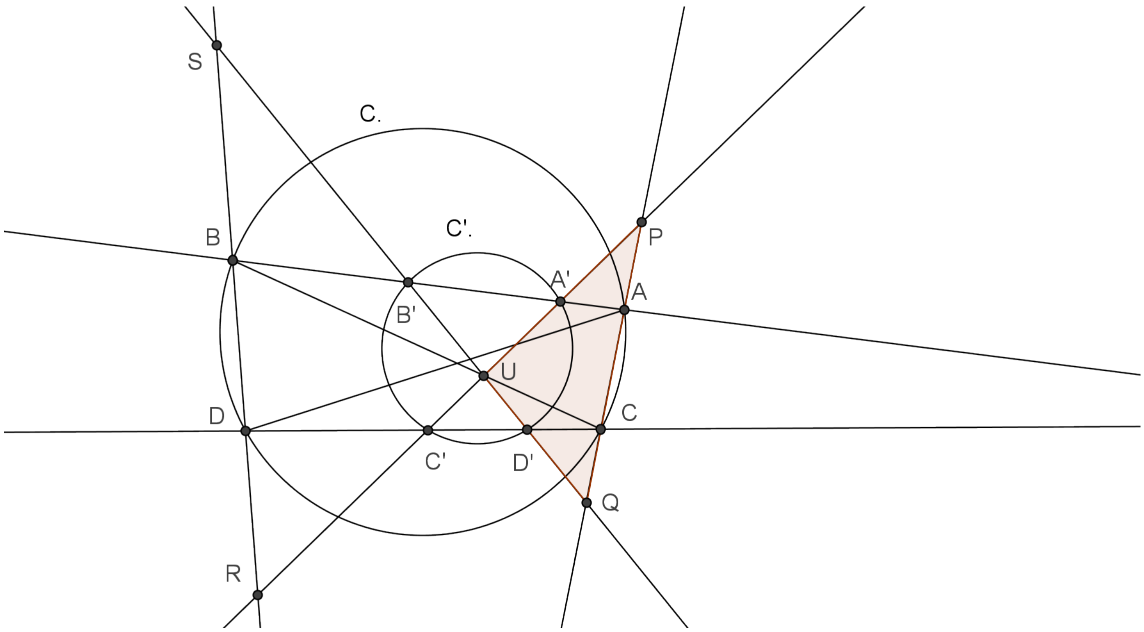

FIGURE t408a

The plan of our proof is to use the result of exercise 149, so we need to compute the powers of points P, Q, R, S with respect to the two given circles. This is not easy.

Let U be the intersection of lines PR, QS. We apply Menelaus' Theorem (**192**) twice to triangle PQU. First, from transversal AB, we have:

(1) $$\frac{AP}{AQ} \cdot \frac{B'Q}{B'U} \cdot \frac{A'U}{A'P} = 1.$$

Next, from transversal CD, we have:

(2) $$\frac{CP}{CQ} \cdot \frac{D'Q}{D'U} \cdot \frac{C'U}{C'P} = 1.$$

Note that

(3) $$A'U \cdot C'U = B'U \cdot D'U,$$

since both products are equal to the power of point U with respect to circle C'.

If we multiply (1) and (2) together, the use (3) to cancel terms, we arrive at:

(4) $$\frac{PA}{PA'} \cdot \frac{PC}{PC'} = \frac{QA}{QB'} \cdot \frac{QC}{QD'}.$$

(Our reversal of the directions of each line segment doesn't affect the equation.)

We now repeat the computation, starting with triangle RSU, and using transversals AB, CD, to get:

$$(5) \qquad \frac{RB}{RA'} \cdot \frac{RD}{RC'} = \frac{SB}{SB'} \cdot \frac{SD}{SD'}.$$

Finally, we note that triangles PAA', SDD' are similar (as we did in the solution to exercise 107a). Indeed, $\widehat{PAA'} = 180° - \widehat{BAC} = \widehat{BDC} = \widehat{SDD'}$ (from cyclic quadrilateral $BACD$), and $\widehat{PA'A} = \widehat{B'A'C'} = \widehat{B'D'C'} = \widehat{SD'D}$ (two of these angle intercept the same arc $\widehat{B'C'}$ on circle C'.). An analogous examination of angles also shows that triangles PCC', SBB' are similar.

From these similar triangles we have $\frac{PA}{PA'} = \frac{SD}{SD'}$; $\frac{PC}{PC'} = \frac{SB}{SB'}$. Multiplying these two equations, we have:

$$(6). \qquad \frac{PA \cdot PC}{PA' \cdot PC'} = \frac{SB \cdot SD}{SB' \cdot SD'}$$

Equations (4), (5), and (6) show that the ratio of the powers of each of the points P, Q, R, S with respect to circles $ABCD$, $A'B'C'D'$ are equal. The result of exercise 149 then implies that points P, Q, R, S lie on a circle which has a common radical axis with the two given circles.

Note. The special case in which the two given lines coincide is of interest, and is simpler. In this case we have the following result (*fig.* t408b):

Give two circles C, C' and a line m intersecting both. The tangents to circle C at its points of intersection with m meet the tangents to circle C' at its points of intersection with m in four points P, Q, R, S. These four points lie on a circle having a common radical axis with C and C'.

Indeed (*fig.* t408b) if V is the intersection of lines PS, QR, we can apply Melenaus' Theorem (**192**) to triangle PQV and transversal AA' to get:

$$\frac{AP}{AQ} \cdot \frac{A'Q}{A'V} \cdot \frac{B'V}{B'P} = 1,$$

or

$$\frac{AP^2}{AQ^2} \cdot \frac{A'Q^2}{A'V^2} \cdot \frac{B'V^2}{B'P} = 1.$$

But $A'V^2 = B'V^2$, so we have $\frac{AP^2}{B'P^2} = \frac{AQ^2}{A'Q^2}$. Thus the ratio of the power of point P and Q with respect to circles C, C' are equal. Analogously, we can show that the same is true for any pair of the points P, Q, R, S, and the result follows as before from exercise 149.

This special case will be used in the solution to exercise 411.

Problem 409. We are given two concentric circles S, C and a third circle C_1. The locus of the centers of the circles orthogonal to C, and such that their radical axis with C_1 is tangent to S, is a circle S_1, concentric with C_1. Conversely, the locus of the centers of the circles orthogonal to C_1, and such that their radical axis with C is tangent to S_1, is the circle S.

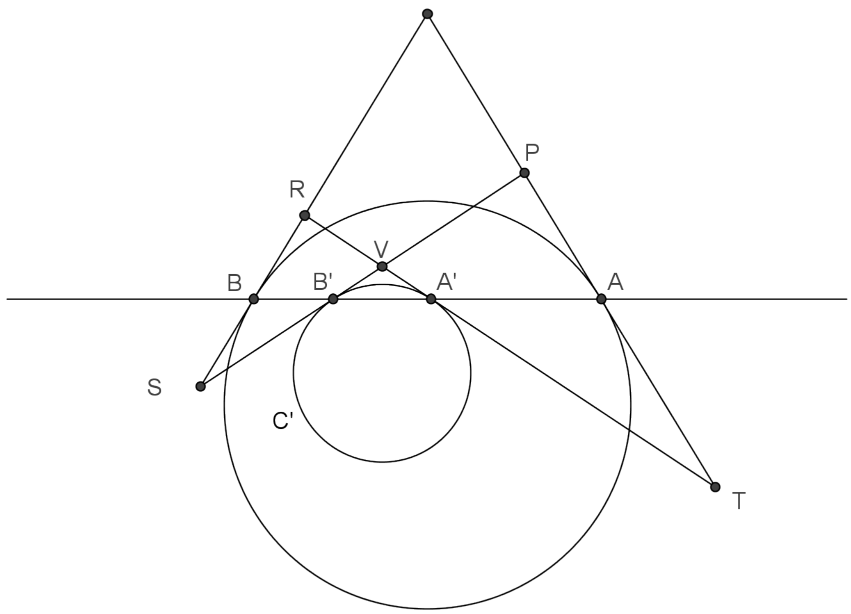

FIGURE t408b

Solution. Suppose we are given (*fig.* t409) circles C, S with a common center O, and radii r, R respectively. Let the third circle C_1 have center O_1 and radius r_1.

Suppose Σ, is a circle centered at some point ω, which is orthogonal to C and such that its radical axis with C_1 is tangent to circle S.

The power of O with respect to Σ is just r^2 (**135**). As figure t409 shows, the tangent from O to circle C_1 is a leg of a right triangle whose other leg is r_1 and whose hypotenuse is OO_1. Hence the power of O with respect to C_1 is $OO_1^2 - r_1^2$.

By **136**, note 3, the difference between the powers of O with respect to circles Σ, C_1 equal to twice the product of the distance $O_1\omega$ and the distance of point O from D. That is:

$$|r^2 - (OO_1^2 - r_1^2)| = 2R \cdot O_1\omega,$$

or

$$2R \cdot O_1\omega = |r^2 + r_1^2 - OO_1^2|. \tag{1}$$

This equation implies that the distance $O_1\omega$ is constant (because all the other quantities in the equation are constant). Thus point ω must lie on a circle with this radius, concentric to C_1.

Conversely, if we choose point ω as the center of a circle orthogonal to C, and satisfying equation (1), then the distance from O to the radical axis S will be equal to R. It follows that the locus of ω is a circle S_1 with center O_1.

Problem 410. Consider the family of those circles which have centers on a given circle C, whose radius has a given ratio to the distance of this center to a given point A in the plane (or, more generally, to the tangent from this point to

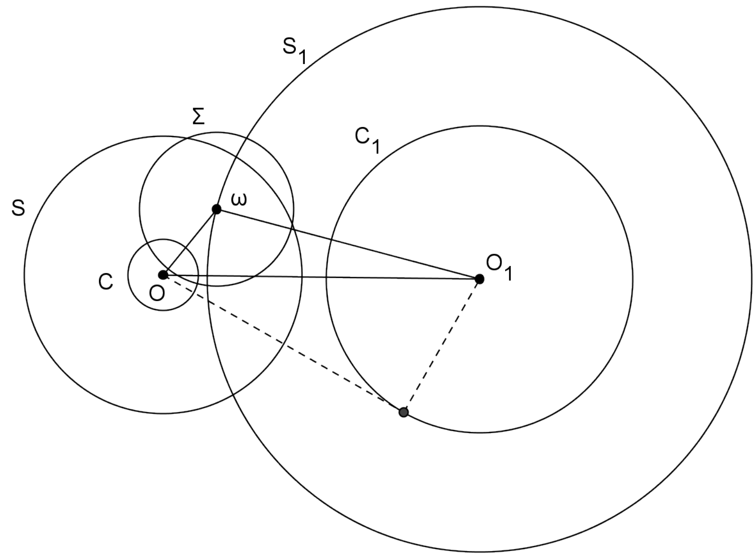

Figure t409

another fixed circle). Show that there exists a point P which has the same power relative to all of these circles. The radical axis of each of these circles with the circle C is tangent to a fixed circle with center P.

Solution. We first treat the case where the quantity referred to is the distance to a fixed point A. Suppose (*fig.* t410) the given circle is C, its center O, and let k be the given ratio.

Suppose M is any point on circle C, and we draw a circle σ centered at M with radius $k \cdot MA$. We seek a point P whose power with respect to σ is constant as M move around circle C. In particular, if we reflect M in line OA, P must have the same power with respect to the reflection of σ as it has to σ. This implies that P must lie on OA (which is the radical axis of σ and its reflection).

The result of exercise 218 (which refers to **127**) then gives us:

$$(1) \qquad MP^2 = MO^2 \cdot \frac{PA}{OA} + MA^2 \cdot \frac{OP}{OA} - OP \cdot PA.$$

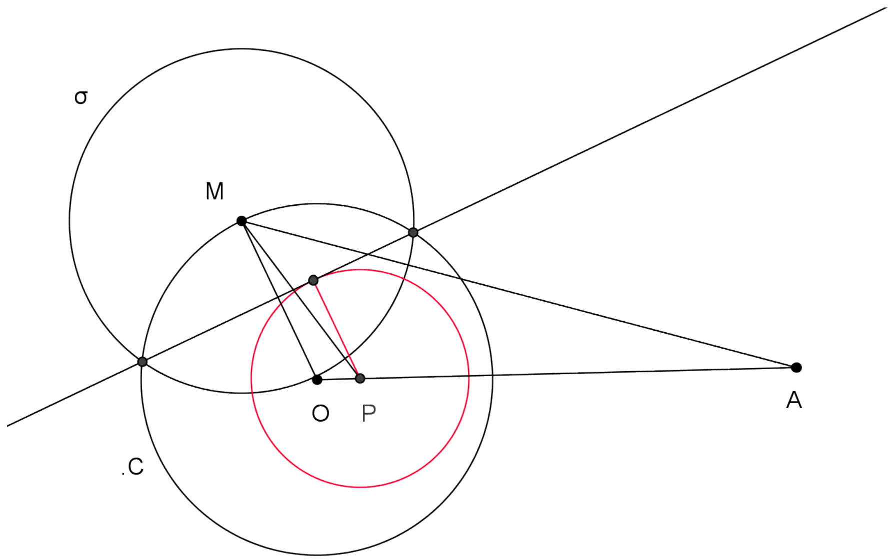

FIGURE t410

It is not difficult to see that the power of P with respect to σ is equal to $MP^2 - k^2 \cdot MA^2$ (this is true whether P is inside, on, or outside σ). Using (1), we can express this power as

$$MO^2 \cdot \frac{PA}{OA} + MA^2 \cdot \left(\frac{OP}{OA} - k^2\right) - OP \cdot PA.$$

For the position of P which we seek, this expression must not depend on the position of M on the circle. We can arrange for this to happen if we can eliminate the quantity MA (other quantities which appear in this equation do not vary with M), and we can eliminate MA by setting $\frac{OP}{OA} - k^2$ equal to zero, or choosing P so that $OP = k^2 \cdot OA$.

We now consider the radical axis of σ and C. In light of **136**, remark 3, we can compute the distance from P to this radical axis by dividing the difference between the powers of P with respect to the circles by twice the distance between their centers. But both these quantities remain constant as M moves around circle C. In other words, this radical axis remains tangent to a fixed circle centered at P with a fixed radius given by **136**, note 3.

Finally, we consider the case where point A is replaced by a circle centered at A with a fixed radius r. Then the length of a tangent from M to this circle is given by $\sqrt{MA^2 - r^2}$, and the computation above still holds, with the quantity $k \cdot MA$ replaced by $k \cdot \sqrt{MA^2 - r^2}$. In particular, the relation $OP = k^2 \cdot OA$ still holds.

Note. Students can fill in the details omitted in the last paragraph, and also check that the argument holds even if point A lies inside circle C.

Problem 411. Through an arbitrary point of a circle C, draw tangents to a circle C'. Show that the line joining the new intersections of these tangents with the circle C is tangent to a fixed circle (this reduces to the preceding exercise). This circle has a common same radical axis with C, C'. Calculate the radius of this new circle, and the distance from its center to the center of C, knowing the radii of the given circles, and the distance between their centers. Deduce from here a solution of Exercise 377.

Solution. Suppose (*fig.* t411a) points O, O' are the centers of circles C, C' respectively, and suppose r, r' are their radii. Let L be the given point on circle C, and let the tangents from L to circle C' intersect C again at M, N. Finally, let line LO' intersect circle C' at point P. Note that LO' bisects angle $\widehat{MLN}$, so arcs $\widehat{PN}$, $\widehat{PM}$ are equal, and $PM = PN$. We draw $OQ \perp NP$ and $O'n \perp LM$, and note that n is the point of contact of tangent LM to circle C'.

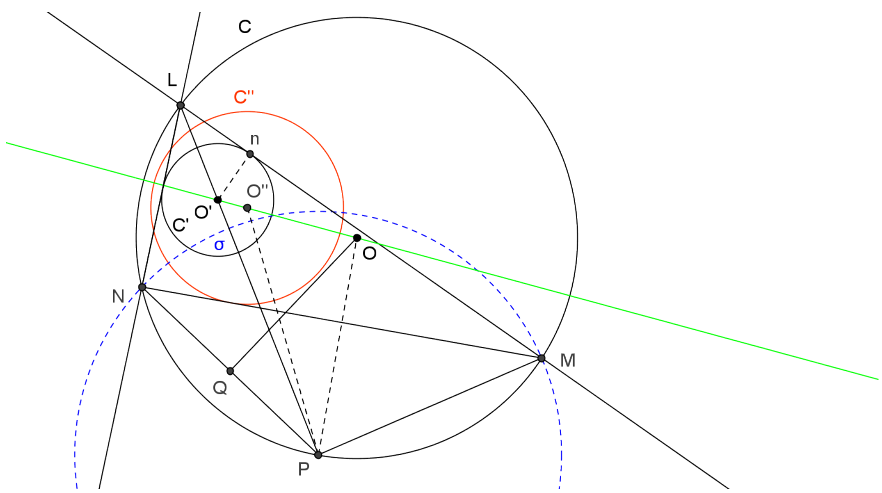

FIGURE t411a

We first show that triangles POQ, $O'Ln$ are similar. Indeed, they are both right triangles, and $\widehat{MLP} = \frac{1}{2} \cdot \widehat{PON}$, since the former is an inscribed angle and the latter is a central angle intercepting arc $\widehat{PN}$. And certainly $\frac{1}{2} \cdot \widehat{PON} = \widehat{POQ}$, so $\widehat{MLP} = \widehat{POQ}$.

From these similar triangles we have $\frac{1}{2} \cdot NP : r = r' : LO'$, so $NP = \frac{2rr'}{LO'}$. Now from **134**, the power of O' with respect to circle C is $r^2 - OO'^2$. But this power is also equal to the product $LO' \cdot PO'$. It follows that

$$\frac{NP}{PO'} = \frac{2rr'}{LO' \cdot PO'} = \frac{2rr'}{r^2 - OO'^2} \tag{1}$$

Following the hint in the problem, we would like to recreate the situation in exercise 410. If we draw a circle σ with center P and radius $PM = PN$, it will do this for us. Indeed, $k = PN : PO'$ does not depend on the point of choice L. The radical axis of C and σ is simply line MN, so it follows from that exercise that

MN is tangent to some fixed circle C'', regardless of the position of point L. The center O'' of circle C'' lies on line OO'.

Now we consider two points $L,\ L'$ on circle C, both external to circle C' (the statement of the problem implies that such points exist). These determine triangles $LMN,\ L'M'N'$ as described in the problem statement (see figure t411b). Let $m,\ n,\ m',\ n'$ be the points of contact of tangents $LN,\ LM,\ L'N',\ L'M'$ respectively with circle C'.

Let line nn' intersects $LL',\ MM'$ at points $p,\ q$ respectively. We will prove that there exists a circle Γ which is tangent to lines $LL',\ MM'$ at points $p,\ q$ respectively. Indeed, we have $\widehat{pLn} = \widehat{L'LM} = \frac{1}{2}\ \widehat{M'L}$, and $\widehat{qM'n} = \widehat{MM'L'} = \frac{1}{2}\ \widehat{M'l}$, so $\widehat{pLn} = \widehat{qM'n'}$.

Now if T is the intersection of ML and $M'L'$, then $Tn = Tn'$, so $\widehat{Tnn'} = \widehat{Tn'n}$ (that is, triangle Tnn' isosceles), which implies that the angles vertical to them are also equal, or $\widehat{pnL} = \widehat{qn'M'}$. Thus triangles $Lnp,\ M'n'q$ agree in two pairs of angles, so their third pair of angles are also equal, or $\widehat{Lpn} = \widehat{M'qn'}$.

The equality of these angles implies the existence of our circle Γ. Indeed, lines $LL',\ MM'$ (but not necessarily those line segments) are symmetric in the perpendicular bisector of pq. Hence a perpendicular at p to LL' will intersect a perpendicular at q to MM' in a point on this perpendicular bisector, which is the center of circle Γ (tangent to LL' at p and to MM' at q).

We now apply the remark from the note to exercise 408 to circles $C',\ \Gamma$ and line nn'. The tangents to C' at $n,\ n'$ intersect the tangents to Γ at $p,\ q$ in points $L,\ L',\ M,\ M'$, all on circle C. Therefore, circles $C,\ C',\ \Gamma$ have a common radical axis.

To summarize: we have shown the existence of a circle Γ which is tangent to $LL',\ MM'$ at their points of intersection $p,\ q$ with line nn', and which has a common radical axis with circles $C,\ C'$. In particular, the center of Γ lies on line OO'.

In just the same way, we can show that there exists a circle Γ' which is tangent to $LL',\ NN'$ at their points of intersection with line mm', and which has a common radical axis with circles $C,\ C'$. In particular, the center of Γ' lies on line OO'.

We now apply the second proposition of exercise 239 to quadrilateral $mm'nn'$. It follows from this proposition that lines $mm',\ nn'$ intersect at point p on line LL'. But this means that circles $\Gamma,\ \Gamma'$ coincide, since they are both tangent to LL' at point p and their centers both lie on line OO'.

That is, circle Γ has a common radical axis with $C,\ C'$, and is tangent to lines $LL',\ MM',\ NN'$ at points $p,\ q,\ s$, (where s is the intersection of mm' and NN') which are also the intersections of these lines with lines $mm',\ nn'$.

Segment qs joins the points of contact of two tangents to circle Γ, so line qs makes equal angles with MM' and NN'. And $\widehat{MM'N'} = \widehat{MNN'}$, since they both intercept (major) arc $\widehat{MN'}$ on circle C. Thus in triangles $M'\ell'q,\ N\ell s$, two pairs of angles are equal, and so the third pair of angles must also be equal; that is, $\widehat{M'\ell'q} = \widehat{N\ell s}$. Therefore (for reasons analogous to the argument establishing the existence of circle Γ) there exists a circle $\overline{C}$ which is tangent to MN and $M'N'$ at points ℓ, ℓ' respectively. (Circle $\overline{C}$ is not shown in figure t411a). And we can show that circle $\overline{C}$ has a common radical axis with circles $C,\ \Gamma$, by using just the same argument we used above for circles $C,\ C', \Gamma$.

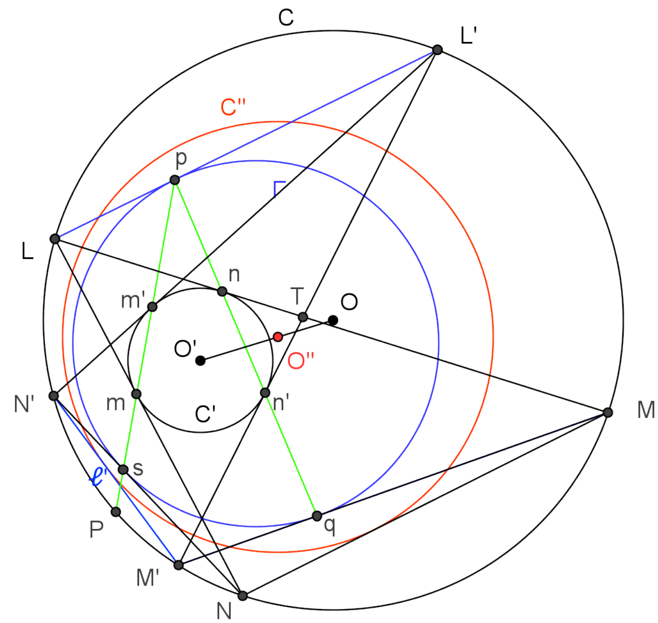

FIGURE t411b

Both circles $\overline{C}$ and C'' are tangent to lines $MN,\ M'N'$, and their centers both lie on line OO'. It is also not hard to see that both their centers lie on the same bisector of one of the angles formed by lines $MN,\ M'N'$. It follows that $\overline{C}$ coincides with C''.

More rigorously, we can note that circle $\overline{C}$ has a common radical axis with C' and Γ, and is tangent to line MN. In **311** it was noted that there are only two such circles, so $\overline{C}$ must be one of these. But if we move L' continuously around the circle, line $M'N'$ also moves continuously, and hence circle $\overline{C}$ moves continuously. But this is not possible if $\overline{C}$ moves at all, because it can only assume one of two positions. So $\overline{C}$ does not move along with $M'N'$; that is, $\overline{C}$ is fixed, and is tangent to any position of $M'N'$, and hence coincides with C''.

Thus circles $C,\ \Gamma$ have a common radical axis with circle C', and also with $\overline{C}$; that is, with C''. It follows that $C,\ C',\ C''$ have a common radical axis.

Now we let $r,\ r'$ be the radii of $C,\ C'$ respectively (*fig.* t411c) and let $d = OO'$. As shown in the solution to exercise 410, the distance from center O'' of circle C'' to the center O of circle C is given by $OO'' = k^2 \cdot OO'$, where (from (1)) $k = \frac{NP}{PO'} = \frac{2rr'}{r^2-d^2}$. Therefore

$$OO'' = \frac{4r^2r'^2d}{(r^2-d^2)^2}. \tag{2}$$

We now compute the radius r'' of circle C''. Let L_0 be the intersection of C with the common centerline of $C,\ C'$. Then $O'L_0 = r + d$. (See figure t411c.) The wording of the problem implies that circle C does not lie inside circle C', so L_0 lies outside circle C'. Let $M_0,\ N_0$ be the second intersections with C of the tangents

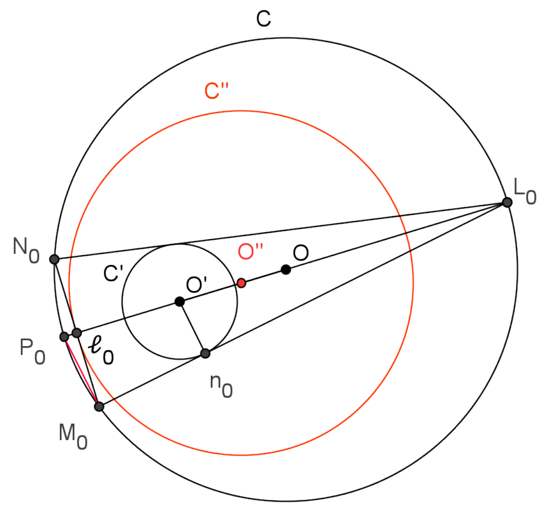

FIGURE t411c

from L_0 to C', and let ℓ_0 be the point of contact of tangent M_0N_0 with circle C''. Let n_0 be the point of contact of tangent L_0M_0 with circle C', and let P_0 be the point diametrically opposite L_0 on circle C.

Triangles $L_0O'n_0$, $L_0P_0M_0$ are similar, so $\frac{M_0P_0}{L_0P_0} = \frac{n_0O'}{L_0O'}$. It follows that $M_0P_0 = \frac{2rr'}{r+d}$. Now in right triangle $L_0P_0M_0$, $M_0\ell_0$ is the altitude to the hypotenuse, so (see **123**) $\ell_0P_0 = \frac{M_0P_0^2}{L_0P_0} = \frac{2rr'^2}{(r+d)^2}$. It follows that

$$r'' = O''\ell_0 = OP_0 - OO'' - \ell_0P_0 = r - \frac{4r^2r'^2d}{(r^2-d^2)^2} - \frac{2rr'^2}{(r+d)^2}$$
$$= r\frac{(r^2-d^2)^2 - 2r'^2(r^2+d^2)}{(r^2-d^2)^2}.$$

Circles C', C'' will coincide (that is, C'' will be an inscribed or escribed circle of triangle LMN) if point O'' coincides with point O'; that is, if $OO'' = d$. This will happen, as we can see from (2) when $d^2 - r^2 = \pm 2rr'$. This is the result of exercise 377.

Problem 412. We are given an angle $\widehat{AOB}$ and a point P.

1°**.** Find a point M on the side OA such that the two circles C, C' tangent to OB and passing through the points M, P intersect at a given angle; .

2°**.** Study the variation of the angle between C and C' as M moves on OA;

3°**.** Let Q, Q' be the points (other than M) where these circles intersect the side OA. Show that circle through P, Q, and Q' is tangent to a fixed line as M moves on OA (this reduces to the preceding exercise).

Solution. We take the original figure, shown as figure t412a, and invert it around pole P by any power. To make things simpler, we choose the power to be OP^2, so that the circle of inversion has radius OP. The result of the inversion is shown in figure t412b, where corresponding elements are labeled with the subscript 1. The points $O,\ P$ are their own images, so we will sometimes refer to O as O_1 and to P as P_1.

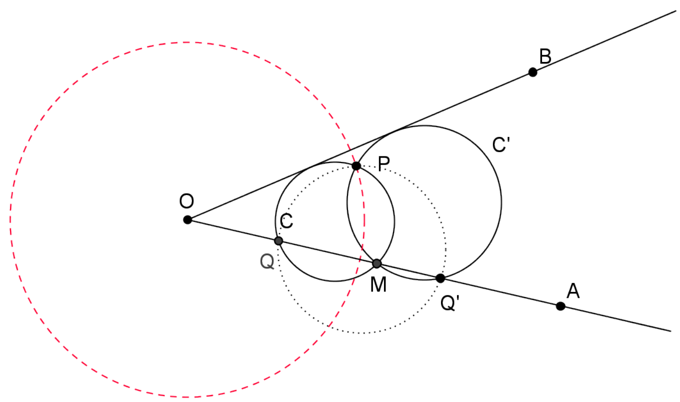

FIGURE t412a

1°. Circles $C,\ C'$ invert into two lines $C_1,\ C_1'$, both tangent to circle $O_1B_1P_1$ and passing through point M_1 on circle $O_1A_1P_1$. The two angles at which circles C and C' meet are equal to the two angles formed by lines $C_1,\ C_1'$. In particular, the angle lying outside both circles, in the region of the plane containing line OB, is equal to the angle in the region of the plane containing circle OB_1P. So we have reduced the problem to that of finding a point on circle OA_1P at which the tangents to circle OB_1P meet at a given angle.

The locus of points whose tangents to a given circle meet at a given angle is itself a circle, concentric to the given circle. Therefore point M_1 is the intersection of circle OA_1P and some circle S_1, the locus of points such that the tangents from those points to circle OB_1P meet at an angle equal to that at which circles $C,\ C'$ meet. Then M is the intersection of line OA with circle S, the inverse image of S_1.

2°**.** As point M_1 moves along arc $\widehat{OA_1P}$, point M moves along side OA of angle $\widehat{OAB}$. the angle between C and C' decreases from 180° to some minimum determined by the greatest distance from M_1 to the center of circle OB_1P, then increases to 180°.

3°**.** Circle PQQ' inverts into line Q_1Q_1', which joins the second points of intersection of tangents from M_1 to circle OB_1P with circle OA_1P. But, by exercise 411, line Q_1Q_1' remains tangent to a certain circle, which has a common radical

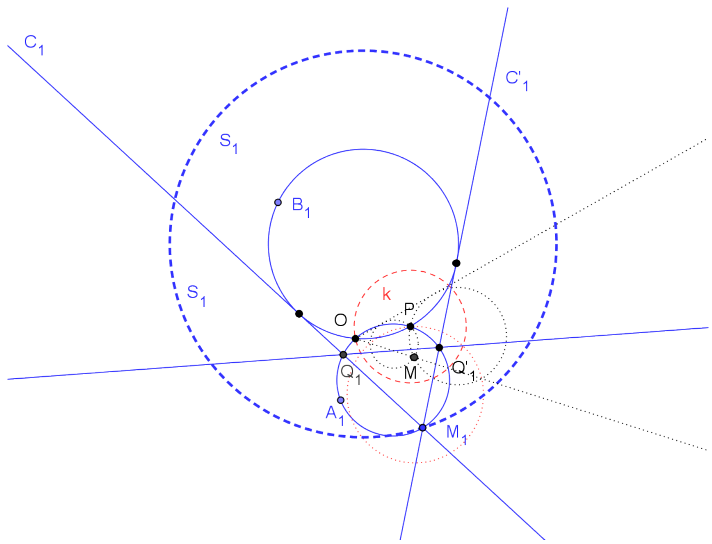

FIGURE t412b

axis with circles OA_1P, OB_1P. Therefore this circle passes through points O and P. Thus, circle PQQ' which is the inversion of line Q_1Q_1', must remain tangent to some line through point O.

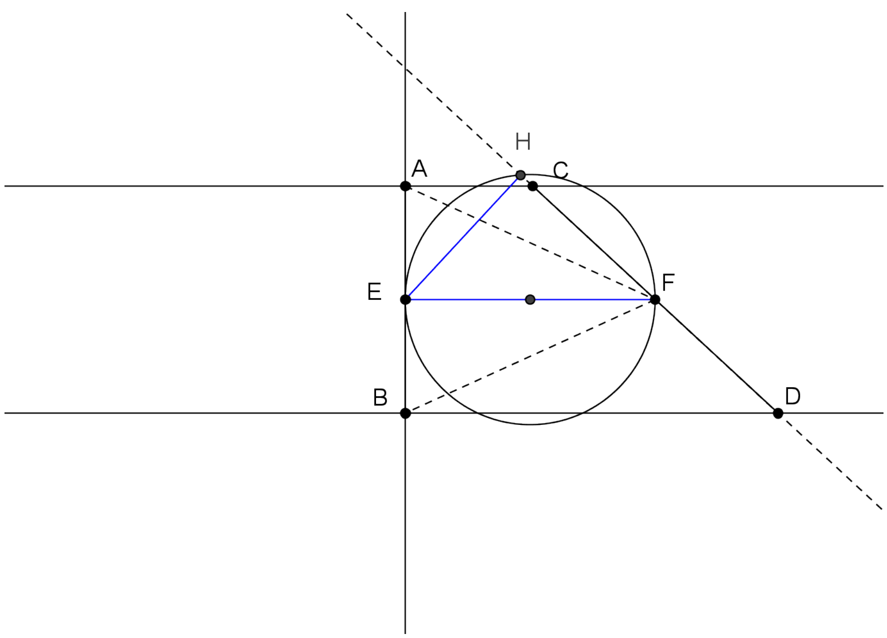

FIGURE t413a

Problem 413. We are given two parallel lines, and a common perpendicular which intersects them in $A,\ B$. Points $C,\ D$ are taken on these lines so that trapezoid $ABCD$ has an area equal to that of a given square. Let H be the projection on CD of the midpoint of AB. Find the locus of H. (One must distinguish two cases, according as the points $C,\ D$ are on the same or opposite sides of the common perpendicular.)

Solution. One key to this problem is the observation that because the area and the height of the trapezoid are both constant, the sum $AC + BD$ of its bases is also constant (**252b**).

1°. Suppose segments $AC,\ BD$ are on the same side of the common perpendicular, as in figure t413a. Then if we join the midpoints $E,\ F$ of $AB,\ CD$ respectively, then $EF = \frac{1}{2}(AC + BD)$, which is also constant. Thus point F is fixed as C and D vary. If we drop perpendicular EH to line CD, then as CD moves, H lies on a circle with (fixed) diameter EF (**78f**). Thus the locus of point H is the part of this circle lying outside triangle AFB.

Note that this solution depends on the length of EF being fixed.

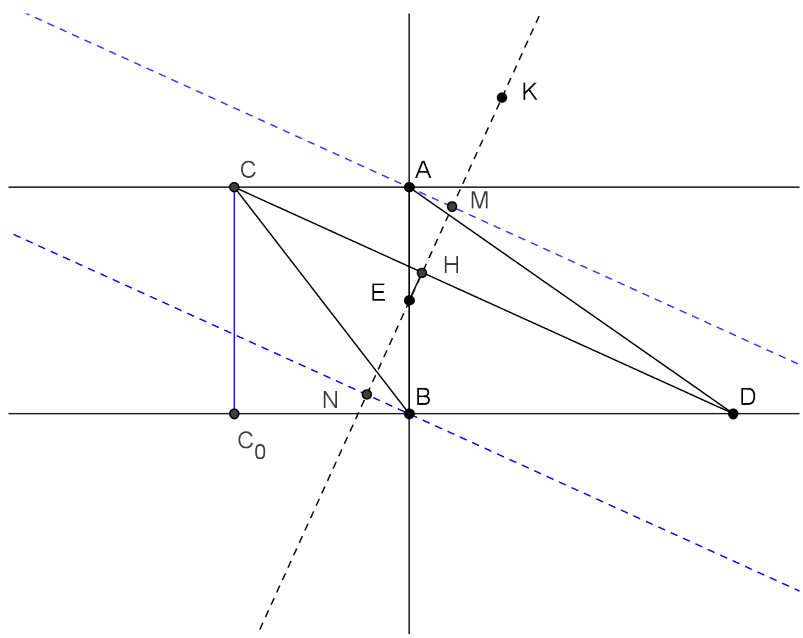

Figure t413b

2°. Suppose segments $AC,\ BD$ are on opposite sides of the common perpendicular, as in figure t413b. Then we can still find a segment of fixed length, if we drop a perpendicular CC_0 from C to line BD. Then $AC = BC_0$, so $C_0B+BD = AC+BD$, which is again constant. Thus as C and D vary, the legs $CC_0,\ C_0D$ of right triangle CC_0D do not vary, so the triangle retains its shape (**24**, 2°). This means that angle $\widehat{CDB}$ has the same measure, and diagonal CD is merely translated parallel to itself. If we drop perpendicular EH onto any position of CD, then the locus of H is a segment of the perpendicular line (EK in figure t413b). Because A is the limiting position of point C, and B is the limiting position of point D, the endpoints of this segment are the intersections M, N of line EK with the parallels to CD through points A and B.

Problem 413b. If four circles are inscribed in the same angle, or in the corresponding vertical angle, and they are also tangent to a fifth circle, then their radii r_1, r_2, r_3, r_4 form a proportion. (One observes that these circles can be arranged in pairs which correspond to each other in the same inversion.)

Lemma. If two tangent circles are inverted around a point P, the mode of tangency (internal or external) is preserved whenever P is outside both circles, and is reversed if P is inside one of the circles.

We leave this proof for the reader. Observe that if P is on one of the circles (neither outside nor inside), then the image of that circle is a line tangent to the image of the other circle, and the mode of tangency is not well-defined. Another special case occurs if P is at the point of tangency of the two original circles.

Solution. Suppose (*fig.* t413bi) that the four circles are C_1, C_2, C_3, C_4, and that they are all tangent to a fifth circle D as well as to two lines m_1, m_2, which intersect at point P. Let k be the power of point P with respect to circle D. We invert around P with power k. (The red circle in figure t413bi is the circle of inversion.) Then clearly lines m_1, m_2 are their own image (**220**, remark), and it is not hard to see that circle D is also its own image.

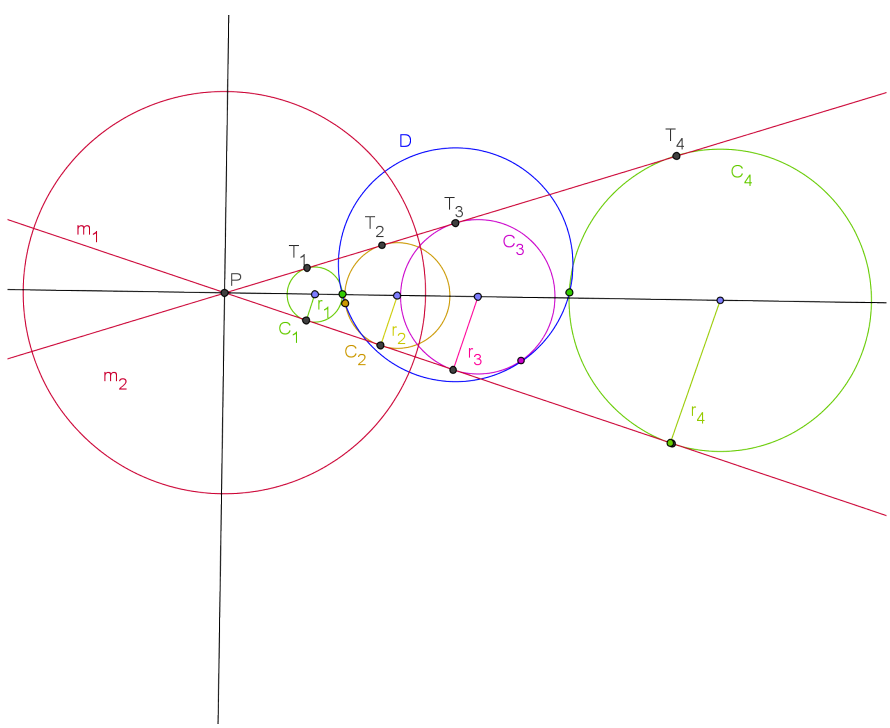

FIGURE t413bi

Then what is the image of circle C_1? It must remain tangent to the two lines, and also to circle D, so it must be one of the other three given circles. In figure 413bi, it is clear from our lemma that C_1 inverts onto C_4, and C_2 inverts onto C_3. This implies that the points of tangency are images: T_1 inverts into T_4 and T_2 into T_3. Then, if $t_1,\ t_2,\ t_3,\ t_4$ denote the segments $PT_1,\ PT_2,\ PT_3,\ PT_4$ respectively, we have

(1) $$t_1t_4 = t_2t_3 = k.$$

Now these circles are all homothetic with respect to point P, so we have $r_1 : r_2 : r_3 : r_4 = t_1 : t_2 : t_3 : t_4$. Using this relationship, we can rewrite (1) as $r_1r_4 = r_2r_3$, so that $r_1 : r_2 = r_3 : r_4$, which is what we wanted to prove.

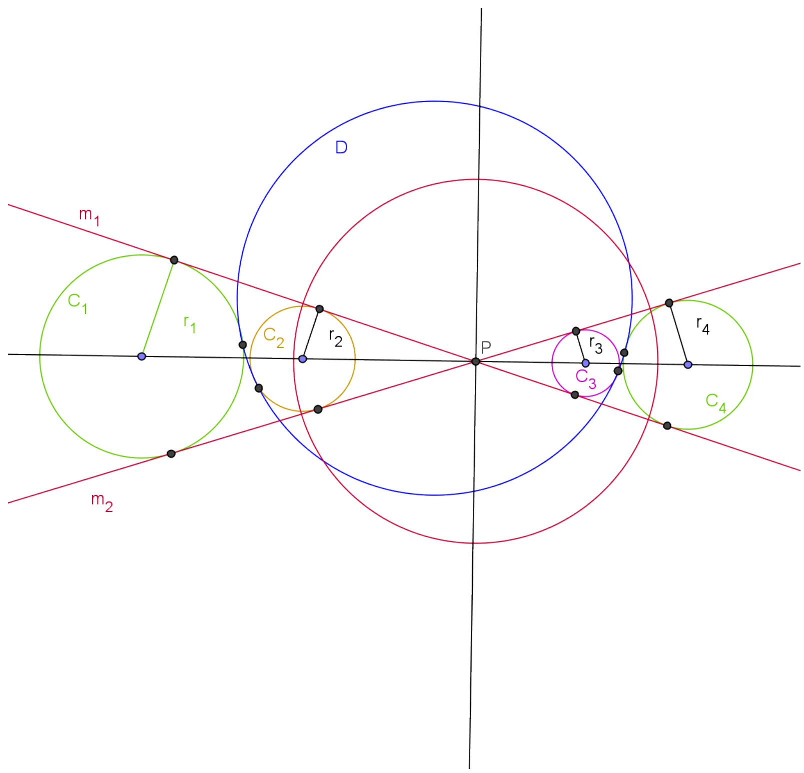

Figure t413bii

Note. Figure t413bii shows a situation where some of the circles are inscribed in the vertically opposite angle, so that their centers are on opposite sides of the bisector of the angle formed by $m_1,\ m_2$. In this case, the proof must be reworded a bit. We can again invert around P so that circle D is its own image, but we must use a negative power of inversion (i.e. inverting in the red circle in figure 413bii, then reflecting in point P).

In this case, our lemma tells us that circle C_1 inverts onto circle C_2, and circle C_3 onto circle C_4. The proof follows as before, but the roles of the circles are changed, and the proportion is now $r_1 : r_4 = r_2 : r_3$. Details are left to the reader.

Problem 414. Another solution to Exercise 329: to draw through a given point inside an angle a secant which forms, with the sides of the angle, a triangle with given area. Construct first the parallelogram with a vertex at the given point, and two sides on the sides of the angle. This parallelogram cuts from the required triangle two partial triangles whose sum is known. Reduce then to the question in Exercise 216.

Solution. Suppose (*fig.* t414) we must draw secant MON through point O inside angle $\widehat{XSY}$, such that (using absolute value for area) $|MNS|$ is equal to some value Δ. We first construct parallelogram $AOBS$. Then, if we drop perpendiculars OP, OQ from O to SY, SX respectively, we have $AM \cdot OP + BN \cdot OQ = 2(\Delta - |AOBS|$.

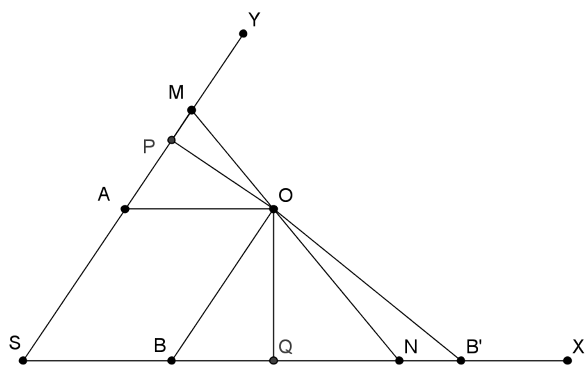

FIGURE t414

We now construct segment BB' such that $|BOB'| = \Delta - |AOBS|$. Then we have $AM \cdot OP + BN \cdot OQ = BB' \cdot OQ$, or (subtracting $2|BON|$ from both sides) $AM \cdot OP = NB' \cdot OQ$.

So we are led to the situation where we are given points A, B', and we must construct a secant MON such that $AM : B'N = OQ : OP$. This is the situation of exercise 216.

Note. Actually, we do not need $AOBS$ to be a parallelogram. It can be any quadrilateral such that $\Delta - 2|AOBS|$ is non-negative, and the argument still holds.

Problem 415. Construct a triangle knowing an angle, the perimeter, and the area (Exercises 90b, 299). Among all triangles with a given angle and given perimeter, which one has largest area?

Solution. Suppose we know the measure of angle $\widehat{A}$ of triangle ABC, as well as its perimeter $2s$. Then, if E_1, F_1 are the points of contact of the escribed circle opposite vertex A (*fig.* t415), then the result of exercise 90b tells us that $AE_1 = AF_1 = s$. This allows us to construct points E_1 and F_1. We can then get excenter I_1 (by erecting perpendiculars to AF_1, AE_1 at F_1, E_1 respectively), and construct the escribed circle itself.

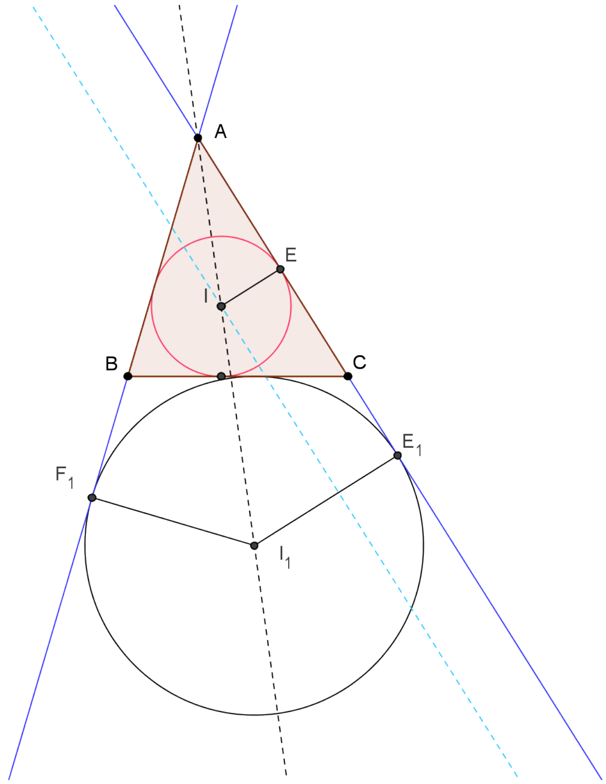

FIGURE t415

We know (**254**) that the inradius r of the triangle is equal to $\frac{K}{s}$, where k is its area. This circumstance allows us to construct the length r. If we then draw a parallel to AE_1 at a distance r, it will intersect the bisector of angle $\widehat{A}$ at the incenter I of triangle ABC. We can then construct the incircle, and side BC is simply the common internal tangent of circles I and I_1. This procedure accomplishes the construction required by the problem.

We next determine the triangle of largest area among those with a given angle and given perimeter. We know that $rs = K$, so if the perimeter is fixed, so is s (the semiperimeter), and to maximize K we must maximize r. It is not hard to see, from the figure, that this will occur when circles I and I_1 are tangent; that is, when triangle ABC is isosceles.

Note. The original problem leaves ambiguous exactly how we are 'given' the area of the triangle. The easiest way to fill in this gap is to assume that a triangle with the same area is given. Then the construction of r involves drawing a triangle with a given base (s) equal in area to a given triangle. Students can solve this sub-problem themselves.

Problem 416. Construct a triangle knowing a side, the perimeter, and the area (construct the figure formed by the inscribed circle and an escribed circle). Among all triangles with a given side and given area, which has the smallest perimeter? Among all triangles with a given side and given perimeter, which has the largest area?

Solution. The solution to this problem is only slightly different from the solution to exercise 415, so we will use the same diagram (*fig.* t415). In that diagram, suppose ABC is the required triangle. We are given $BC = a$, the perimeter of ABC, which we will denote as $2s$, and its area k. Suppose I is the incenter of ABC, r its radius, and E its point of tangency with side AC. Further, let I_1 be the excenter opposite vertex A, let r_1 be the radius of the corresponding escribed circle, and let E_1 be the point of contact of the escribed circle with side AC (extended).

From the result of exercise 90b, we have $AE = s - a$, and $EE_1 = AE_1 - AE = s - (s - a) = a$. From the result of exercise 299, we have:

$$(1) \qquad r = \frac{k}{s}; \qquad r_1 = \frac{k}{s-a}.$$

Thus we can construct point E_1, then segment $EE_1 = a$, then the radii r, r_1 (but see the note to exercise 415), and finally circles I, I_1 (the inscribed and escribed circles of the triangle). The construction is completed by drawing the common internal tangent of the two circles, and another common external tangent.

Next we investigate the possibility of performing the construction. We need circles I, I_1 to be outside each other, or at worst tangent to each other. That is, we need $I_1I^2 = a^2 + (r_1 - r)^2 \geq (r_1 + r)^2$, which simplifies to $a^2 \geq 4rr_1$.

We now determine the triangle with smallest perimeter, among those with a given side and area. For given values of a and k the equations (1) above show that radii r, r_1 both increase as s decreases. The smallest perimeter thus corresponds to the extreme situation where $a^2 = 4rr_1$; that is, when $II_1 = r + r_1$. This means that triangle ABC will be isosceles.

Finally, we look at the question of finding the triangle with largest area, if a side and the perimeter are fixed. We again use (1) to note that if a and s are fixed, then r and r_1 increase as k increases. So the largest value of k corresponds to the same situation, when $a^2 = 4rr_1$, which is again when ABC is isosceles.

Notes (i). We are not given r, r_1: students can use (1) to rephrase this condition for existence of triangle ABC in terms of the quantities which are given.

(ii). We can also find the triangle of largest area of those with a given perimeter and given side by using the results of **251**. The formula derived there can be written as $k^2 = \frac{1}{4}s(s-a)[a^2 - (b-c)^2]$. Thus for fixed s and a, the area will increase as $|b - c|$ decreases. Thus the area will be largest when $b = c$.

(iii). We can use another argument to find the triangle of smallest perimeter, when the area k and a side a are fixed. Suppose Δ is an isosceles triangle with area k, perimeter $2s$, and base a. Suppose Δ' is any non-isosceles triangle with area k, perimeter $2s'$, and base a. We will show that $s < s'$.

Indeed, we can construct isosceles triangle Δ_1 with base a and perimeter $2s'$, and let k_1 be its area. We then compare triangles Δ' and Δ_1. They have the same base, and the same perimeter $2s'$, so from result (ii) above we know that $k_1 > k$.

Let h be the common altitude to side a of $\Delta,\ \Delta'$, and let h_1 be the corresponding altitude of Δ_1. Then $h_1 > h$, and comparing triangles Δ, Δ_1, it follows that $s' > s$. (We use the fact that the larger projection corresponds to the larger declination, so that in isosceles triangles with equal bases the larger altitude corresponds to the larger leg.)

Problem 417. Construct a quadrilateral knowing its four sides and its area. (Let $ABCD$ be the required quadrilateral, so that $AB = a$, $BC = b$, $CD = c$, $DA = d$. Let ABC_1 be a triangle exterior to this quadrilateral, equivalent to ADC, and with an angle $\widehat{C_1BA} = \widehat{ADC}$. We can find each of the following quantities: (i) side BC_1; (ii) the difference of the squares of AC and AC_1; (iii) the projection of CC_1 on AB. Finally (iv) with the help of the known area we will know the projection of CC_1 on a perpendicular to AB. Once the segment $AB = a$ is placed anywhere, this will allow us to construct a segment equal and parallel to CC_1, and therefore to complete the required construction.) The problem, if possible, generally has two solutions. For these two quadrilaterals, the triangle considered in Exercise 270b has the same shape, and consequently (by Part 5 of that exercise), these quadrilaterals can be considered inverse to each other. Given a quadrilateral, construct another, not equal to the first, but with equal corresponding sides and equal area. Among all the quadrilaterals with given sides, the largest in area is cyclic.

Solution. We first separate the various statements in the given problem. Using the notation of the problem statement:

(i) Find side BC_1.

(ii) Find the difference of the squares of AC and AC_1.

(iii) Find the projection of CC_1 on AB.

(iv) Find the projection of CC_1 onto a line perpendicular to AB.

(v) Complete the required construction.

(vi) The problem, if possible, generally has two solutions. For these two quadrilaterals, the triangle considered in Exercise 270b has the same shape, and consequently (by Part 5 of that exercise), these quadrilaterals can be considered inverse to each other.

(vii) Given a quadrilateral, construct another, not congruent to the first, but with equal corresponding sides and equal area.

(viii)Among all the quadrilaterals with given sides, the largest is the cyclic one.

Lemma. In any quadrilateral, the difference between the sums of the squares of pairs of opposite sides is equal to twice the product of a diagonal and the projection on it of the other diagonal.

Proof of Lemma. The statement of the lemma assumes that we are subtracting the smaller product from the larger. In figure t417a, in which $\widehat{AEB}$ is obtuse and $\widehat{BEC}$ is acute, this means that we subtract $BC^2 + AD^2$ from $AB^2 + CD^2$.

Let L, M be the projections of vertices B, D of quadrilateral $ABCD$ onto diagonal BD. We must prove that

$$(AB^2 + CD^2) - (BC^2 + AD^2) = 2AC \cdot ML. \tag{1}$$

We have (**126**):

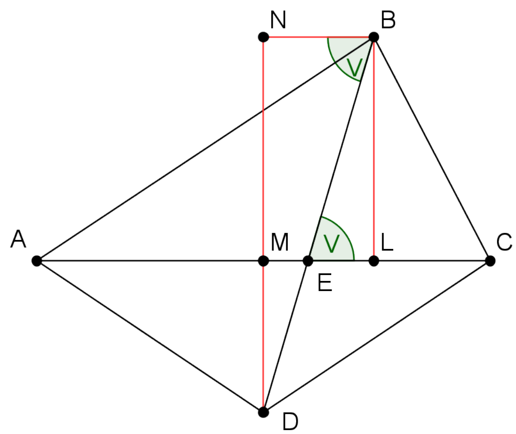

FIGURE t417a

$$AB^2 = AE^2 + BE^2 + 2AE \cdot EL;$$

$$BC^2 = BE^2 + EC^2 - 2EC \cdot EL;$$

$$CD^2 = CE^2 + DE^2 + 2EC \cdot ME;$$

$$AD^2 = AE^2 + DE^2 - 2AE \cdot ME.$$

It follows that $(AB^2 + CD^2) - (BC^2 + AD^2) = 2AE \cdot EL + 2EC \cdot ME + 2EC \cdot EL + 2AE \cdot ME = 2(AE + EC)(ME + EL) = 2AC \cdot ML$. This proves our lemma.

(i). Figure t417b shows the construction of point C_1 as described in the problem statement. Using absolute value for area, this means that we have $|ADC| = |ABC_1|$ and $\widehat{ABC_1} = \widehat{ADC}$. By **256** we have:

$$AD \cdot CD = AB \cdot C_1B, \tag{2}$$

If we let $C_1B = m$, we can write this as

$$m = \frac{cd}{a}. \tag{3}$$

This gives us the length of side BC_1.

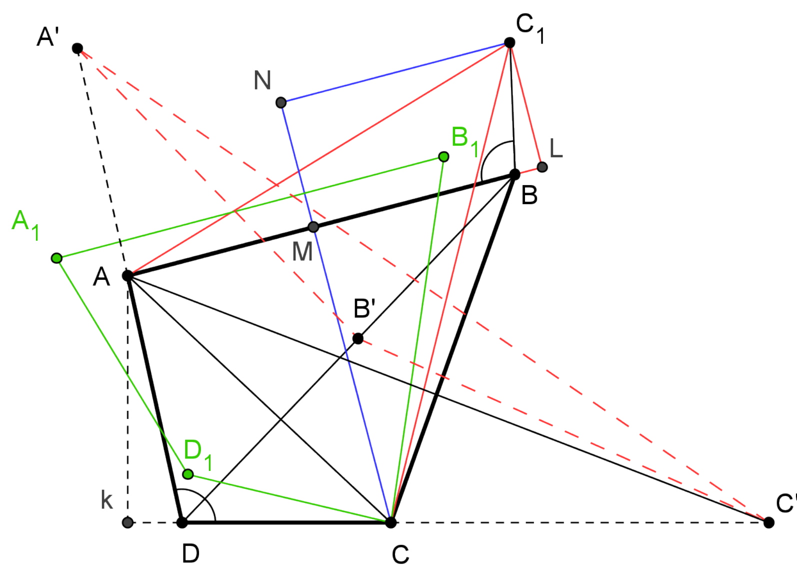

FIGURE t417b

(ii). Let $K,\ L$ be the feet of altitudes $AK,\ C_1L$ of triangles $ACD,\ ABC_1$. Then we have

$$AC^2 = CD^2 + DA^2 \pm 2CD \cdot DK; \tag{4}$$

$$AC_1^2 = AB^2 + BC_1^2 \pm 2AB \cdot BL. \tag{5}$$

Since $\widehat{ADC} = \widehat{ABC_1}$, it follows that the signs noted as ambiguous above are either both positive or both negative. Now triangle ADK, C_1BL are similar (they have equal corresponding angles), so $DK : BL = AD : C_1B$. We can rewrite (2) as $AD : C_1B = AB : CD$, and thus $DK : BL = AB : CD$, or

$$CD \cdot DK = AB \cdot BL. \tag{6}$$

Using (4), (5), and (6), we have:

$$AC_1^2 - AC^2 = AB^2 + BC_1^2 - CD^2 - DA^2 = a^2 + m^2 - c^2 - d^2. \tag{7}$$

This gives us the difference of the squares of AC and AC_1.

(iii). We use (1), applied to quadrilateral AC_1BC to compute the length of projection LM of C_1C onto line AB. We have:

$$(AC_1^2 + BC^2) - (BC_1^2 + AC^2) = 2AB \cdot LM.$$

Then (7) gives us

$$2AB \cdot LM = (AC_1^2 - AC^2) + (BC^2 - BC_1^2) = (a^2 + m^2 - c^2 - d^2) + (b^2 - m^2).$$

It follows that

$$LM = \frac{a^2 + b^2 - c^2 - d^2}{2a} \tag{8}$$

This is the result required by (iii)

(iv). We draw $CM \perp AB$, and let CN be the projection of CC_1 onto line AB. Then we have (again using absolute value for area) $|ABCD| = |ABC| + |ACD| = |ABC| + |ABC_1| = \frac{1}{2}AB(CM + C_1L) = \frac{1}{2}AB \cdot CN$. Thus:

$$CN = \frac{2|ABCD|}{a}. \tag{9}$$

This is the result of (iv)

(v). We know the length of $C_1N = LM$ from (8), and the length of CN from (9), so we can construct segment CC_1 as the hypotenuse of a right triangle with these two legs. Then we use (3) to construct BC_1. Now we know all three sides of triangle BCC_1, so we can construct it (**24**, 3°). We next locate vertex A by drawing $BA \parallel C_1N$ through B, and laying off segment $BA = a$, which is given. Finally, we locate the position of vertex D by constructing triangle ACD knowing its sides.

(vi). If the construction is possible, then it will generally have two solutions. Indeed, having drawn segment CC_1, we can construct triangle BCC_1 (from its three sides) in two different ways: the locations of point B will be symmetric in line CC_1. These two positions are labeled B, B_1 in figure 417b, and the resulting quadrilaterals (both of which include vertex C) are labeled $ABCD$, $A_1B_1CD_1$.

To investigate the situation of exercise 270b, we invert points A, B, C in point D using any power of inversion to obtain points A', B', C'. Then, from **217**, quadrilaterals $AA'BB'$, $BB'CC'$ are cyclic, so we have $\widehat{A'B'B} = \widehat{BAA'}$ and $\widehat{BB'C'} = \widehat{BCC'}$. It follows that:

$$\begin{aligned} \widehat{A'B'C} &= \widehat{A'B'B} + \widehat{BB'C} = 360^\circ - \widehat{BAD} - \widehat{BCD} = \\ &= \widehat{ADC} + \widehat{ABC} = \widehat{C_1BA} + \widehat{ABC} = \widehat{C_1BC}. \end{aligned} \tag{10}$$

As in the solution to exercise 270 (part 1°), we have:

$$A'B' : B'C' = (AB \cdot CD) : (BC \cdot AD) = ac : bd. \tag{11}$$

In the same way, we can invert A_1, B_1, C in D_1 (using the same power of inversion) to obtain points A_1', B_1', C_1'. Arguing analogously, we will find:

$$\widehat{A_1'B_1'C_1'} = \widehat{C_1B_1C}; \tag{12}$$

$$A_1'B_1' : B_1'C_1' = ac : bd. \tag{13}$$

An examination of (10), (11), (12), and (13) leads to the conclusion that the triangles obtained by the construction of exercise 270 from quadrilaterals $ABCD$, $A_1B_1CD_1$ have the same shape, as asserted by (vi).

(vii, solution 1). Suppose we are given quadrilateral $ABCD$. We must construct a second quadrilateral, not congruent to the first, but with the same sides and area. We can do this by following the steps above: construct point C_1, then point B_1, symmetric to B in line CC_1, and complete the construction of quadrilateral $A_1B_1CD_1$ as indicated above. This quadrilateral will have the same area as $ABCD$, and the same sides, but will not be congruent to $ABCD$.

(vii, solution 2). Suppose the given quadrilateral is $ABCD$, and the required quadrilateral is $A_0B_0C_0D_0$. We will find an inversion that takes the given quadrilateral onto the required one, by determining its pole and its power. If O is the pole, and k the power, we must have $A_0B_0 = \frac{k \cdot AB}{OA \cdot OB} = AB$ (**218**), with similar relationships for the other corresponding vertices. It follows that $k = OA \cdot OB = OB \cdot OC = OC \cdot OD = OD \cdot OA$, so that we have $OA = OC$ and $OB = OD$. Also, $OA_0 = \frac{k}{OA} = \frac{OA \cdot OB}{OA} = OB$, and similarly $OA_0 = OC_0 = OB = OD$ and $OB_0 = OD_0 = OA = OC$.

From this inversion we have the following construction. We take for point O the intersection of the perpendicular bisectors of the diagonals of $ABCD$. Then $OA = OC$ and $OB = OD$. On rays OA, OC we lay off segments OA_0, OC_0, both equal to OB. Then on rays OB, OD we lay off segments OB_0, OD_0, both equal to OA (*fig.* t417c). It is clear that (using absolute value for area) $|OAB| = |OB_0A_0|; |OBC| = |OC_0B_0|$, and so on, so that $|ABCD| = |A_0B_0C_0D_0|$. This observation provides a second solution for (vii).

Note. If quadrilateral $ABCD$ turns out to be cyclic, then points B, B_1 coincide (on segment CC_1), and there is only one quadrilateral satisfying the conditions of the problem.

(viii). From (9) we have $|ABCD| = \frac{1}{2}a \cdot CN$. If sides a, b, c, d of $ABCD$ are fixed, the largest value of its area thus corresponds to the largest value of CN. But in right triangle C_1NC, side $C_1N = LM$ is determined by (8) from the lengths of the sides. So the length of leg CN depends only on the length of hypotenuse CC_1. But as B varies in position, side BC is fixed, and (3) shows that the length of BC_1 is also fixed. So, applying **26** to triangle CC_1B, $CC_1 \leq CB + BC_1$. Since the segments on the right of this inequality have fixed length, CC_1 is largest when C, C_1, B are collinear. This means that $\widehat{CBC_1} = \widehat{B} + \widehat{D} = 180°$, and quadrilateral $ABCD$ with the largest possible area is cyclic.

Problem 417b. Given a quadrilateral with sides a, b, c, d, diagonals e, f, and area S, we have

$$4e^2f^2 = (a^2 + c^2 - b^2 - d^2)^2 + 16S^2.$$

The angle V of the diagonals is given by

$$\tan V = \frac{4S}{a^2 + c^2 - b^2 - d^2}.$$

Deduce from this a solution of the preceding exercise. (Having fixed the position of one side, each of the remaining two vertices will be the intersection of two circles.)

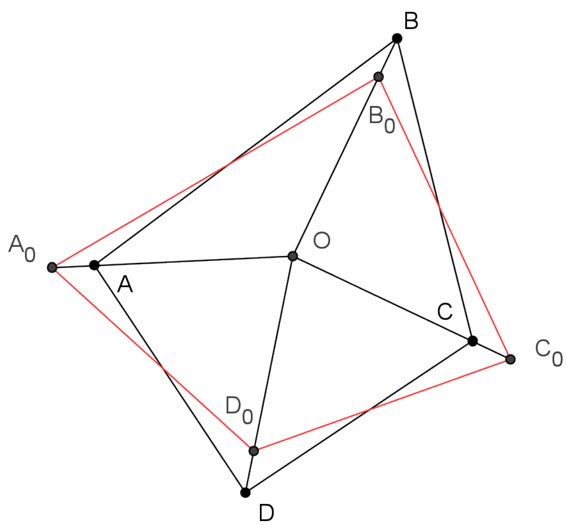

FIGURE t417c

Solution. We use figure t417a, letting $AB = a,\ BC = b,\ CD = c,\ DA = d,\ AC = e,\ BD = f$. Let E be the intersection of the diagonals, and let $L,\ M$ be the projections of vertices $B,\ D$ on line AC. Finally, let N be the projection of vertex B on line DM.

From right triangle BDN we have $f^2 = BN^2 + DN^2 = ML^2 + DN^2$. We multiply the left side by $4e^2$, and the right side by $4AC^2$, which are two equal expressions, to get:

$$4e^2 f^2 = (2AC \cdot ML)^2 + (2AC \cdot DN)^2. \tag{1a}$$

We work on the right hand side of this equation term by term. From equation (1) in the solution to exercise 417, we have $2AC \cdot ML = AB^2 + CD^2 - BC^2 - AD^2 = a^2 + c^2 - b^2 - d^2$. Also, $2AC \cdot DN = 2AC \cdot DM + 2AC \cdot BL = 4|ACD| + 4|ABC| = 4S$ (using absolute value to denote areas). Substituting these expressions into (1a), we get:

$$4e^2 f^2 = (a^2 + c^2 - b^2 - d^2)^2 + 16S^2, \tag{2a}$$

as required.

We next compute $\tan \widehat{V}$, where $\widehat{V}$ is the (acute) angle between the two diagonals of the quadrilateral. We have: $tan\ \widehat{V} = tan\ \widehat{DBN} = \frac{DN}{NB} = \frac{DN}{ML} = \frac{2AC \cdot DN}{2AC \cdot ML}$. Replacing the numerator and denominator of this fraction by the expressions derived above, we have:

$$tan\ \widehat{V} = \frac{4S}{a^2 + c^2 - b^2 - d^2},$$

as required.

We now address the task of exercise 417: to construct a quadrilateral, knowing its four sides and its area. In figure t417bi, point P is chosen so that triangle ABP is similar to triangle ADC. Then we have: $BP = \frac{ac}{d}$; $AC : AP = AD : AB = d : a$. Also, $\widehat{CAD} = \widehat{BAP}$, which implies that $\widehat{CAP} = \widehat{DAB}$. Therefore triangles ACP, ADB are also similar (**118**, 2°), and $CP = \frac{ef}{d}$.

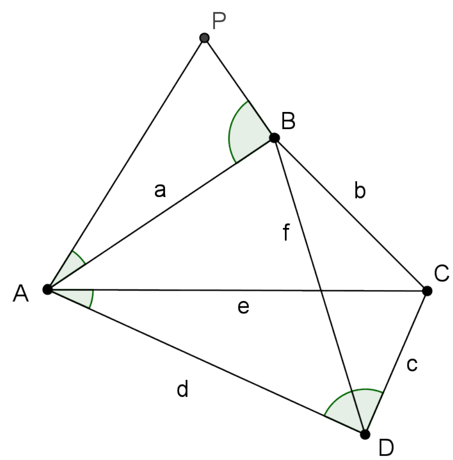

FIGURE t417bi

We now can compute CP from the given sides and area of the quadrilateral. Indeed, if we divide both members of (2a) by $4d^2$, we find:

$$\left(\frac{ef}{d}\right)^2 = \frac{1}{4}\left(\frac{a^2}{d} + \frac{c^2}{d} - \frac{b^2}{d} - d\right)^2 + \left(\frac{2S}{d}\right)^2. \tag{3a}$$

Let us fix the position of side BC (whose length is given) on the plane. Then from our discussion, we know the distances from point P to B and to C: $BP = \frac{ac}{d}$; $CP = \frac{ef}{d}$, and the last is given by (3a). These two pieces of information determine point P. We can determine point A by finding the intersection of two circles: (i) the circle which is the locus of points A such that $AC : AP = d : a$ (**116**), and (ii) the circle with center B and radius a. Finally, we can locate vertex D because we know its distance c from point C and its distance d from point A. This allows us to construct the required quadrilateral.

Problem 418. Construct a cyclic quadrilateral knowing its sides.

Solution. Suppose (*fig.* t418) $ABCD$ is the required quadrilateral, and let P be a point on the plane such that triangle ABP is similar to triangle ADC.

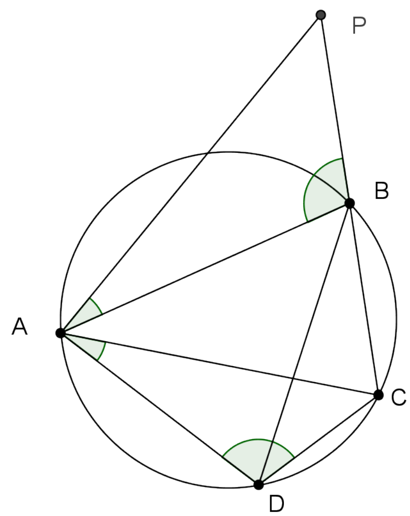

Figure t418

First note that points $P,\ B,\ C$ are collinear, because $\widehat{ABP}+\widehat{ABC}=\widehat{ADC}+\widehat{ABC}=180°$. Also, from similar triangles $ABP,\ ADC$, we have $BP=\frac{AB\cdot CD}{AD}$. If we fix side BC, this gives us the length of BP in terms of the given sides, and thus allows us to locate point P.

From the same similar triangles, we have $AC:AP=AD:AB$. Thus point A lies on the intersection of two circles: (i) the locus of points whose distances to C and P are in the ratio $AD:AB$ (**116**), and (ii) a circle centered at B with radius AB (which is given). Finally, we can locate vertex D from its given distances $CD,\ AD$ from vertices C and A.

Problem 418b. Among all polygons with the same number of sides and the same perimeter, the largest is the regular polygon. (Assuming that a polygon of maximum area exists, we can use the preceding exercises and Exercise 331 to show that this polygon must be regular.) The result can be restated as follows: If S is the area of a polygon, and p its perimeter, the ratio $\frac{S}{p^2}$ is larger for a regular polygon than for an irregular polygon with the same number of sides.

Solution. Suppose $ABCDE...$ is the polygon referred to, with a given number of sides and a given perimeter, and with the largest possible area. (We assume that such a polygon exists.) We have solved this problem completely for triangles in exercise 416, so we can assume that our polygon has at least four sides.

Take four consecutive vertices $A,\ B,\ C,\ D$ of our polygon. If these four vertices did not lie on the same circle, then we could increase the area of our polygon by replacing quadrilateral $ABCD$ with cyclic quadrilateral $AB'C'D$ (keeping vertices A and D fixed), by the last result of exercise 417. We can repeat this argument for any four consecutive vertices of the polygon, so the polygon of largest area must be cyclic.

Now suppose that two adjacent sides of the polygon, say AB and BC, are not equal. Then we can replace triangle ABC with an isosceles triangle with the same perimeter, but having a larger area (by the result of exercise 416).

Since the quadrilateral of largest area is cyclic, and has equal sides, it must be regular.

We can phrase this result in terms of the ratio $S : p^2$. If we compare a regular polygon with a non-regular polygon with the same number of sides and the same perimeter, this ratio will be larger for the regular polygon, since its area will be larger. And since all regular polygons with the same number of sides are similar, this ratio will not depend on the size of the regular polygon.

Note. In more advanced work, we can obtain this result even without assuming that the polgyon of largest area exists.

Problem 419. Among all the closed curves of same length, the circle is the one whose interior has the largest area. (Consider the ratio $\frac{S}{p^2}$ for a polygon inscribed in a circle and a polygon inscribed in a curve of the same length, the number of sides being the same in the two cases, and let the number of sides increase indefinitely.)

Solution. We consider a circle with circumference C, and a closed curve with the same length. In the circle, we inscribe a regular polygon with n sides, and let $S_n,\ P_n$ denote its area and perimeter respectively. In the closed curve we inscribe another polygon with n sides, and let $s_n,\ p_n$ denote its area and perimeter. The result of exercise 418b tells us that

$$\frac{S_n}{P_n^2} \geq \frac{s_n}{p_n^2} \tag{1}$$

for any n.

Now let n increase without bound. Then we can choose the vertices of the polygon inscribed in the closed curve in such a way that the length of each of the sides approaches 0. Thus P_n approaches C, and p_n approaches c. Also, S_n approaches S and s_n approaches s, where S and s denote the area bounded by the circle and the other closed curve, respectively. From (1), it follows that $\frac{S}{C^2} \geq \frac{s}{c^2}$, so that $S \geq s$.

Thus, of all closed curves with a given length, none can enclose a larger area than that of the circle with the given length as circumference.

Note. It remains to prove that no closed curve other than the circle can also enclose the largest possible area, of all curves with a given length. Indeed, it is possible, in light of the inequality $S \geq s$ that there is some other closed curve, not a circle, with length C and area S (that is, $s = S$ for this curve). In fact, a more advanced investigation would show that there is no such curve.

Problem 419b. Let O be the intersection of the diagonals of a quadrilateral $ABCD$, and let O_1, O_2, O_3, O_4 be the centers of the circles OAB, OBC, OCD, ODA; these four centers are the vertices of a parallelogram P.

1°**.** When this parallelogram is known, then the area and the diagonals of the quadrilateral are determined;

2°. If points $O_1,\ O_2,\ O_3,\ O_4$ are fixed, and point O moves along a line Δ, the vertices of the quadrilateral move about the sides of a parallelogram P'. Study the changes in this parallelogram as the line Δ varies. Find the positions of Δ for which its area is maximum;

3°. Construct a quadrilateral $ABCD$ knowing two of its angles and the parallelogram P; or, knowing P and the ratios $\frac{AB}{AD}, \frac{CB}{CD}$. Discuss.

Solution. We separate the various assertions of the problem:

(0) points $OAB,\ OBC,\ OCD,\ ODA$ are the vertices of a parallelogram P;

(1) when parallelogram P is known, then the area and the lengths of the diagonals of the quadrilateral are determined;

(2a) if points $O_1,\ O_2,\ O_3,\ O_4$ are fixed, and point O moves along a line Δ, the vertices of the quadrilateral move about the sides of a parallelogram P';

(2b) study the changes in this parallelogram as the line Δ varies.

(2c) find the positions of Δ for which its area is maximum;

(3a) Construct a quadrilateral $ABCD$ knowing two of its angles and the associated parallelogram P.

(3b) Construct a quadrilateral $ABCD$ knowing P and the ratios $\frac{AB}{AD}, \frac{CB}{CD}$.

We give solutions to these problems separately and in order.

(0). In figure t419bi, line O_1O_4 is the perpendicular bisector of AO (**68**), and in particular $O_1O_4 \perp AO$. Similarly, $O_2O_3 \perp OC$. But OC and OA lie along the same line, so $O_1O_4 \parallel O_2O_3$. In the same way, we can show that $O_1O_2 \parallel O_3O_4$, so that $O_1O_2O_3O_4$ is a parallelogram.

(1). First we show that each diagonal of quadrilateral $ABCD$ is twice one of the altitudes of P. Indeed, point A is the reflection of O in line O_1O_4 (**68**), and point C is the reflection of O in O_2O_3. Thus the portion of segment AC that lies inside P is half the length of AC, and this portion is also equal to an altitude of P. Similarly, BD is half the other altitude of P. This observation shows that if we know P, then the lengths of the diagonals of $ABCD$ are determined.

Now we determine the area $|ABCD|$, given P. We construct $BH \perp AC$ and $DH \parallel AC$ (that is, these two lines determine the position of point H). Then BH is equal in length to the sum of the altitude to AC in triangle ABC and the altitude to AC in triangle ADC. It follows (using absolute value for area) that $|ABCD| = \frac{1}{2}BH \cdot AC$. We know that P determines AC, so we must now show that P also determines BH.

To this end, we draw altitude O_4K of P. Then $O_4K \parallel BD$ (they are both perpendicular to O_1O_2, and $O_1O_4 \parallel BH$ (they are both perpendicular to AC), so $\widehat{O_1O_4K} = \widehat{DBH}$ (**43**), and right triangles $O_1O_4K,\ DBH$ are similar. Thus $BH = \frac{O_4K \cdot BD}{O_4O_1}$, and since each segment in this last fraction is determined by P, so is BH, and therefore so is $|ABCD|$.

Note. We have actually proved a bit more about the diagonals of $ABCD$. Not only their lengths, but their directions are determined by P. For example, since O is the reflection of A in O_1O_4, we know that line AO, along which diagonal AC lies, is perpendicular to side O_1O_4 of P. Likewise, $BD \perp O_1O_2$. We will use this fact in our discussion of statement (3a).

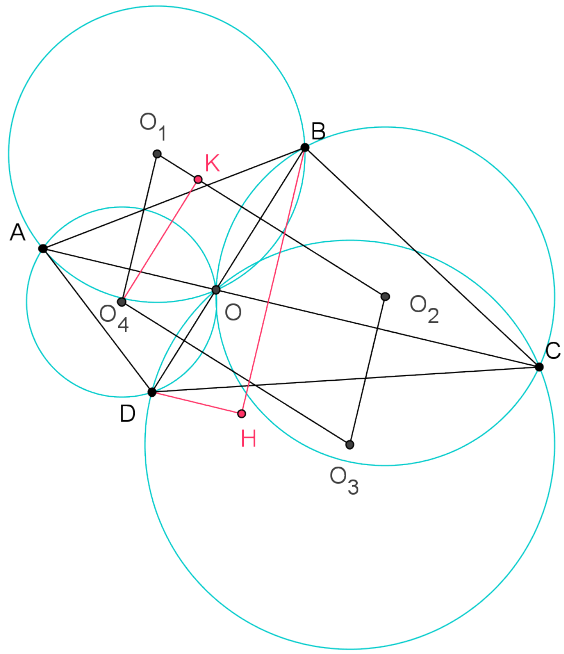

FIGURE t419bi

2a. The key observation here is that points A and O are symmetric with respect to line O_1O_4. Thus if O_1 and O_4 remain fixed, and O moves along a line Δ, then A will describe a line a, symmetric to Δ in O_1O_4. Analogously, C will describe a line b, symmetric to Δ in O_2O_3. Now if we reflect line a in O_1O_4, we get Δ, and if we reflect Δ in O_2O_3 we get line c. Hence c is the image of a in the composition of reflections in two parallel lines. Such a composition is a translation (**102b**), so $a \parallel c$.

In the same way, we can show that B and D move along two parallel lines b and d, as O moves along Δ. Thus $ABCD$ moves along a parallelogram P'.

Note. In fact, as O slides along Δ, the vertices of $ABCD$ are not confined to the sides of P', but vary along the lines containing those sides. A dynamic sketch, starting with fixed points $O_1,\ O_2,\ O_3,\ O_4$ will confirm this.

2b. We can make Δ coincide with any given line by applying to it a translation and a rotation (in either order). We study the effects of each separately. Our discussion is illustrated by figure t419bii.

First we move Δ so that it remains parallel to its former position. In this case it is clear that lines $a,\ b,\ c,\ d$ retain their direction. Now c is obtained from a by a translation in a direction perpendicular to that of O_1O_4 and O_2O_3, and through a distance equal to twice the distance between these two lines. But this distance does not vary with Δ, so as Δ is translated parallel to itself, a and c remain the

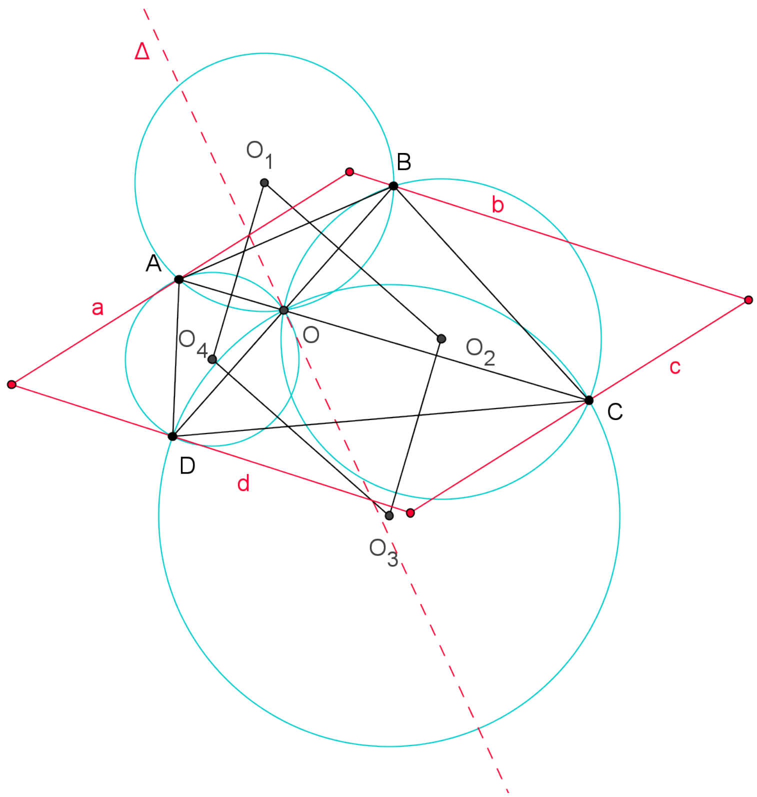

Figure t419bii

same distance apart. In the same way, we can show that b and d remain the same distance apart. It follows that parallelogram P' is simply translated.

Now suppose we rotate Δ about a fixed point through some angle. Then lines a and b will be changed by the same rotation, and so the angle between them will not vary. The same is true of lines b and c, of lines c and d, and of lines d and a. It follows that if we rotate Δ about a fixed point, parallelogram P' will change into another parallelogram with the same angles.

Lemma for part (2c). Given parallelogram $\alpha\beta\gamma\delta$, construct a parallelogram with given angles, whose sides pass through the vertices of $\alpha\beta\gamma\delta$, and which has the largest possible area.

Solution. Suppose that $TUVW$ is a parallelogram whose sides pass through the vertices of $\alpha\beta\gamma\delta$ (*fig.* t419biii). We note in passing that the choice of any one of its vertices, say U, determines the others.

Now α and γ are symmetric with respect to point M, so it is not hard to see (using absolute value for area) that $|\alpha\gamma VU| = \frac{1}{2}|UVWT|$. From the result of exercise 297, it follows that $|\alpha\gamma UV| = \alpha\gamma \cdot SL$, where SL is the perpendicular from the midpoint S of UV to line $\alpha\gamma$. Thus $|UVWT| = 2 \cdot \alpha\gamma \cdot SL$, and we need to maximize SL.

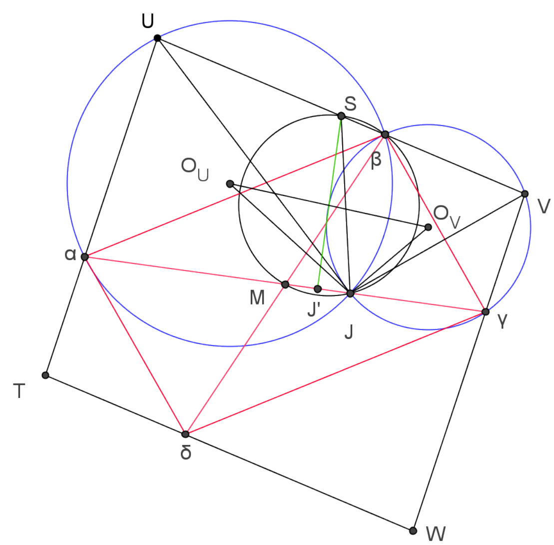

FIGURE t419biii

Angle U is fixed, so vertex U lies on an arc of a circle through α and β. Let the center of this circle be O_U. Likewise, V moves along a circle through β and γ, whose center we will call O_V. We will show that S moves along a circle as well. Let J be the second point of intersection of circles O_U, O_V. Note that $\widehat{\alpha J \beta}$, $\widehat{\gamma J \beta}$ are supplementary (they are each supplementary to the angles $\widehat{U}$, $\widehat{V}$, which are certainly supplementary. So J lies on diagonal $\alpha\gamma$ of $\alpha\beta\gamma\delta$.

By the result of exercise 69, angle $\widehat{UJV} = \widehat{O_U J O_V}$, which does not vary with U. It is also not hard to see that $\widehat{JO_U V} = \frac{1}{2}\,\widehat{J\beta}$ (where the arc is taken in circle O_U), which also does not vary with U. Hence triangle UJV retains its shape as U moves along its circle. Since S is always the midpoint of UV, triangle JUS also retains its shape as U varies. That is, the ratio $JS : JU$ remains constant. This means that S is obtained from U by a rotation about J through a fixed angle $\widehat{JU\beta}$ followed by a dilation centered at J with ratio $JS : JU$. So the locus of S is a circle Σ, obtained from O_U by this combination of rotation and dilation.

But in fact we can characterize Σ more easily by looking at limiting positions of U and V. There is one position of U where S must coincide with β: the corresponding position of V is obtained by reflecting circle O_U in β and finding the intersection of the image with circle O_V. And when U and V coincide at point J, this is also a limiting position of S. Thus Σ passes through β and J. Finally, it is not hard to see that there are two positions for U, V where parallelogram $TUVW$ degenerates to a segment lying along the diagonals of $\alpha\beta\gamma\delta$. For these positions, S coincides with point M, the intersection of the diagonals of $\alpha\beta\gamma\delta$. Thus Σ is the circle passing through β, J, and M.

Now we can find the parallelogram of maximal area, by locating the point on Σ furthest away from line $\alpha\gamma$. This is the intersection of the perpendicular bisector of JM with circle Σ. This concludes the discussion of our lemma.

Solution for part (2c). Since a translation of Δ will not affect the area of P', we look for the largest area of P' that we can obtain by rotating Δ about some point. Without loss of generality, we can take this point to be M, the center of P (*fig.* t419biv).

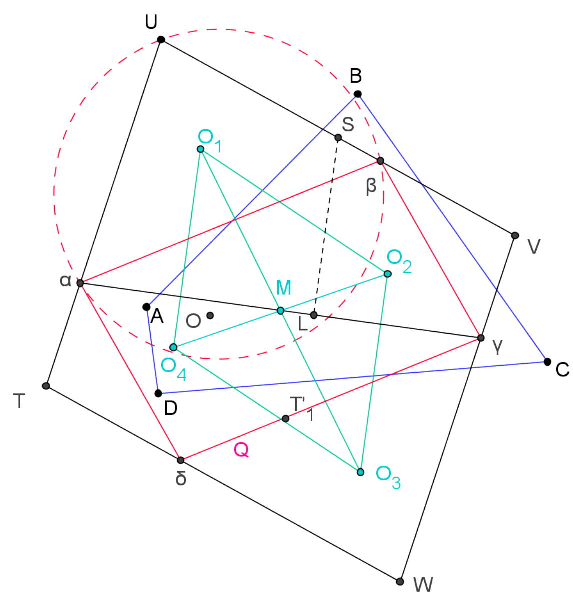

FIGURE t419biv

Now Δ is a line rotating about M, and a is the reflection of this line in the fixed line O_1O_4. Thus if α is the reflection of M in O_1O_4, then a must always pass through point α; that is, a rotates about point α. Similarly, b, c, d rotate about points β, γ, δ, the reflections of M in the other sides of P. Opposite sides of quadrilateral $\alpha\beta\gamma\delta$ are reflections of M in the parallel sides of P, so $\alpha\beta\gamma\delta$ must be a parallelogram, which we will call Q. The intersections of a, b, c, and d determine P', whose area we must maximize. Note that we have already proved that the angles of this parallelogram P' do not vary with Δ.

That is, we have reduced the problem to the situation of our lemma. Using the discussion there, we can determine the position of U for which the area of P' is largest, and then determine from it the required position of line Δ, since Δ is symmetric to a (that is, to line $U\alpha$) in line O_1O_4.

3a. There are two cases to consider: we could be given two adjacent angles of the quadrilateral, or two opposite angles.

First suppose we are given two opposite angles, say $\widehat{A}$ and $\widehat{C}$. We have seen, in (1), that knowing P gives us the lengths and directions of the diagonals of $ABCD$.

Thus we can lay of a segment with the length and direction of BD anywhere on the plane. Vertex A is located on an arc of a circle through B and D, and vertex

C is located on another such arc. So we are led to the situation of exercise 75, and the solution to that exercise completes this solution.

Now suppose we are given parallelogram P and angles A and B. The length and direction of diagonals AC, BD are again determined by P. Thus we can construct the parallelogram whose vertices are the midpoints of the sides of the required quadrilateral: its sides are equal to half of the diagonals, and are parallel to them. Then vertex A lies on an arc of a circle through two of the vertices of the parallelogram, and vertex B lies on another arc of a circle through two of its vertices.

So we need to draw a line through a vertex of this parallelogram such that the segments of it contained between the circles of these two arcs is bisected by the chosen vertex. This task is essentially completed in the solution to exercise 165. We reflect one of the arcs in the vertex mentioned, and find the intersection of the reflected arc with the second arc. This will locate one of the endpoints of the required segment.

3b. Suppose we are given the ratios $AB : AD$ and $CB : CD$ (in figure t419bi). Then we can determine the length and direction of diagonal BD as before. Since we know the ratio of the distances from A to B and from A to D, it follows that A must lie on a circle (**116**). Similarly C must lie on a different circle, and we again are in the situation of exercise 75.

Problem 420. The radii of the circles circumscribing (Exercise 66) the quadrilaterals determined by the bisectors of the interior (or exterior) angles of a quadrilateral are in the ratio $\frac{a+c-b-d}{a+c+b+d}$, where $a,\ b,\ c,\ d$ are the sides of the given quadrilateral, taken in their natural order.

Solution. Suppose the given quadrilateral is $MNPQ$, and suppose the quadrilaterals formed by its internal and external angle bisectors are $ABCD$ and $A'B'C'D'$ respectively (*fig.* t420).

Lemma: Quadrilaterals $ABCD$, $A'B'C'D'$ have the same interior angles.

Proof: Since AM, $A'M$ bisect $\widehat{QMT}$, $\widehat{QMN}$ respectively, and since the last two angles are supplementary, we have $A'M \perp AM$. Likewise, $A'Q \perp AQ$, so quadrilateral $AQA'M$ is cyclic. This means that angles $\widehat{QAM}$, $\widehat{QA'M}$ are supplementary (**80**), and therefore $\widehat{QAM}$, $\widehat{B'A'D'}$ are supplementary. But we know, from exercise 66, that quadrilateral $A'B'C'D'$ is also cyclic, so $\widehat{B'A'D'}$ supplements $\widehat{B'C'D'}$, and $\widehat{QAM} = \widehat{B'C'D'}$. We can show that the other three angles are equal in pairs in just the same way.

Note. Students can be reminded that this does not make the quadrilaterals similar. Any two rectangles have pairs of equal angles, but may not be similar.

Note also that the angles are not in 'corresponding' positions in figure t420: angles $\widehat{A}$, $\widehat{A'}$ are supplementary, not equal.

We turn now to the assertion itself.

We know from exercise 66 that $ABCD$ is cyclic. And it is clear from the construction of $ABCD$ by intersecting exterior angle bisectors, that $MNPQ$ is the quadrilateral investigated in exercise 362b: it has the minimal perimeter of any quadrilateral inscribed in $ABCD$, and in particular, its sides make equal angles with those of $ABCD$. So we can use various relations derived in the solution to

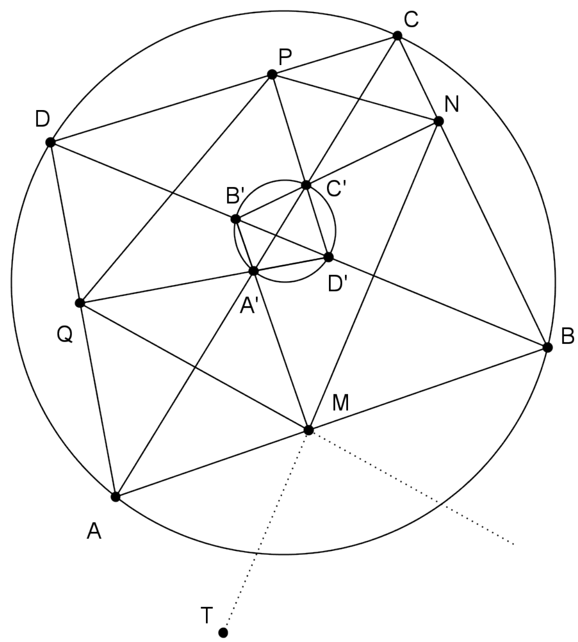

Figure t420

exercise 362b. (We note that these relationships are far from obvious: the solution to that exercise involved some difficult computation.)

We have, from that discussion:

$$R \cdot (MN + NP + PQ + QM) = AB \cdot CD + AD \cdot BC, \tag{1}$$

in which R is the radius of the circle circumscribing quadrilateral $ABCD$.

Also drawing on the arguments in the solution to exercise 362b, we have (from equation (2) and others following it, in that solution):

$$2R' \cdot MN = B'M \cdot C'D' + B'N \cdot D'A'$$

$$2R' \cdot NP = C'N \cdot D'A' + C'P \cdot A'B'$$

$$2R' \cdot PQ = D'P \cdot A'B' + D'Q \cdot B'C'$$

$$2R' \cdot QM = A'Q \cdot B'C' + A'M \cdot C'D',$$

where R' is the circumradius of quadrilateral $A'B'C'D'$.

From these relationships, and Ptolemy's theorem **237** we have

$2R'(MN - NP + PQ - QM) = A'B'(D'P - C'P) + B'C'(D'Q - A'Q) + C'D'(B'M - A'M) + D'A'(B'N - C'N) = 2(A'B' \cdot C'D' + B'C' \cdot D'A') = 2A'C' \cdot B'D'$

Combining this with (1) gives us

$$(1) \qquad \frac{R(MN+NP+PQ+QM)}{R'(MN-NP+PQ-QM)} = \frac{AC\cdot BD}{A'C'\cdot B'D'}$$

We now relate the product on the right hand side of this equation to $R,\ R'$. From the note in **251** we have (using absolute value for area),

$$4R\cdot|ABD| = AB\cdot BD\cdot DA;$$
$$4R'\cdot|A'B'D'| = A'B'\cdot B'D'\cdot D'A',$$

so that

$$(2) \qquad \frac{R}{R'}\cdot\frac{|ABD|}{|A'B'D'|} = \frac{AB\cdot BD\cdot AD}{A'B'\cdot B'D'\cdot D'A'}.$$

Then **256**, together with our lemma, gives us $\frac{|ABD|}{|A'B'D'|} = \frac{AB\cdot AD}{A'B'\cdot A'D'}$. Comparing this with (2), we see that

$$\frac{R}{R'}\cdot\frac{AB\cdot AD}{A'B'\cdot A'D'} = \frac{AB\cdot BD\cdot AD}{A'B'\cdot B'D'\cdot D'A'},$$

or

$$\frac{R}{R'} = \frac{BD}{B'D'}.$$

Reasoning analogously from triangles $ACD,\ A'C'D'$, we find that $\frac{R}{R'} = \frac{AC}{A'C'}$. Combining these results with the result in (1), we have:

$$\frac{R(MN+NP+PQ+QM)}{R'(MN-NP+PQ-QM)} = \frac{R^2}{R'^2}.$$

and a bit of algebra shows that this last equation is equivalent to the announced result.

Problem 420b. The opposite sides of a cyclic quadrilateral are extended to their intersections $E,\ F$, and we draw the bisectors of the angles they formed.

1°**.** Show that these bisectors intersect on the line joining the midpoints of the diagonals of the quadrilateral, and divide this segment into a ratio equal to the ratio of the diagonals.

Solution. Suppose opposite sides $AB,\ CD$ of cyclic quadrilateral intersect at point E (*fig.* t420bi), and sides $AD,\ BC$ intersect at F. Let $M,\ N$ be the respective midpoints of diagonals $AC,\ BD$. Let $EQ,\ FS$ be the bisectors of angles $\widehat{AED},\ \widehat{AFB}$, as in the diagram. We will show that EQ and FS divide segment MN in the same ratio, which will prove that they intersect at a point on the segment.

To this end, note that triangles $AEC,\ DEB$ are similar: they share angle $\widehat{AED}$, and $\widehat{EAC} = \widehat{EDB}$ because they both intercept arc $\widehat{BC}$ on the circumscribing circle. Now $EM,\ EN$ are corresponding medians in these similar triangles, so we have:

$$(1) \qquad EM : EN = AC : BD,$$

and

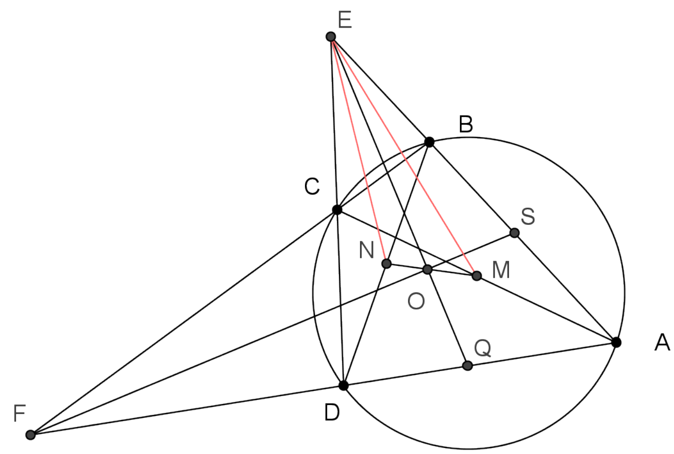

FIGURE t420bi

(2) $$\widehat{AEM} = \widehat{DEN}.$$

(This last from the same similar triangles.) It follows from (2) that EQ is also the bisector of angle $\widehat{MEN}$, so from (**115**) we have:

(3) $$EM : EN = MO : ON.$$

From (1) and (3) we have:

(4) $$MO : ON = AC : BD,$$

so that EQ, the bisector of angle $\widehat{AED}$, divides segment MN in the ratio of the diagonals of the quadrilateral.

In just the same way, we can show that the bisector FS of angle $\widehat{DFC}$ divides MN in the same ratio. It follows that these two angle bisectors intersect at point O on MN, dividing it into the ratio indicated.

2°**.** These lines also bisect the angles which this segment subtends at E and F.

Solution. The result was achieved in solving part 1°.

3°**.** These bisectors intersect the sides of the quadrilateral in four points (other than E, F) which are the vertices of a rhombus. The sides of the rhombus are parallel to the diagonals of the quadrilateral, and their length is the fourth proportional to the lengths of these diagonals, and their sum.

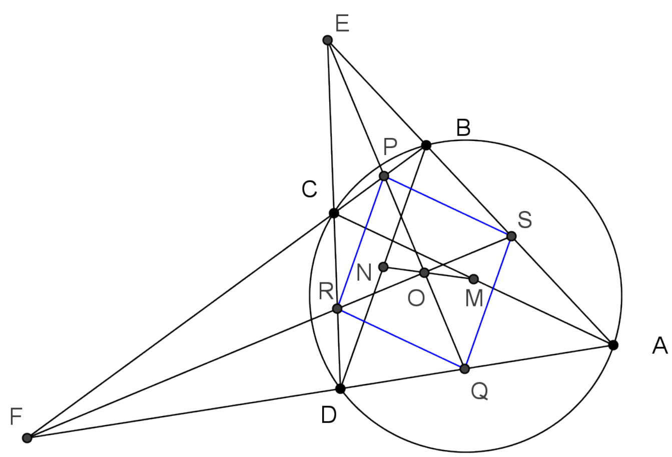

Figure t420bii

Solution. Suppose (*fig.* t420bii) the bisector of angle $\widehat{AED}$ intersects BC, AD in points P, Q respectively, and the bisector of angle $\widehat{DFC}$ intersects AB, CD in points R, S respectively. We again use similar triangles AEC, DEB, to find that $EC : EB = AC : BD$. Now EP is an angle bisector in triangle EBC, so $EC : EB = CP : PB$. It follows that

$$CP : PB = AC : BD. \tag{5}$$

In the same way, we can show that $BS : SA = DB : AC$. Thus in triangle ABC, line PS divides BC, BA in the same ratio. It follows (**114**) that $PS \parallel AC$.

We can use the proportions already noted to express the length of PS in terms of the diagonals of the quadrilateral. Indeed, from (5), we have $\frac{BP}{CP} = \frac{BD}{AC}$, or, using **106**, $\frac{BP}{BP+CP} = \frac{BD}{BD+AC}$. But, from **114** $\frac{PS}{AC} = \frac{BP}{BC} = \frac{BP}{BP+CP} = \frac{BD}{BD+AC}$, so that

$$PS = \frac{AC \cdot BD}{AC + BD}.$$

Analogously, we can express the lengths of segments SQ, QR, RP in terms of the lengths of the diagonals, and we will get just the same expression. It follows that $PQRS$ is a rhombus whose sides are parallel to the diagonals of the quadrilateral, and have the length indicated in the problem statement.

4°**.** Analogous statements for the bisectors of the supplements of the angles at E and F.

Solution. We here list some statements whose proofs are analogous to the arguments in (1) and (3), and which lead to the required analogous results.

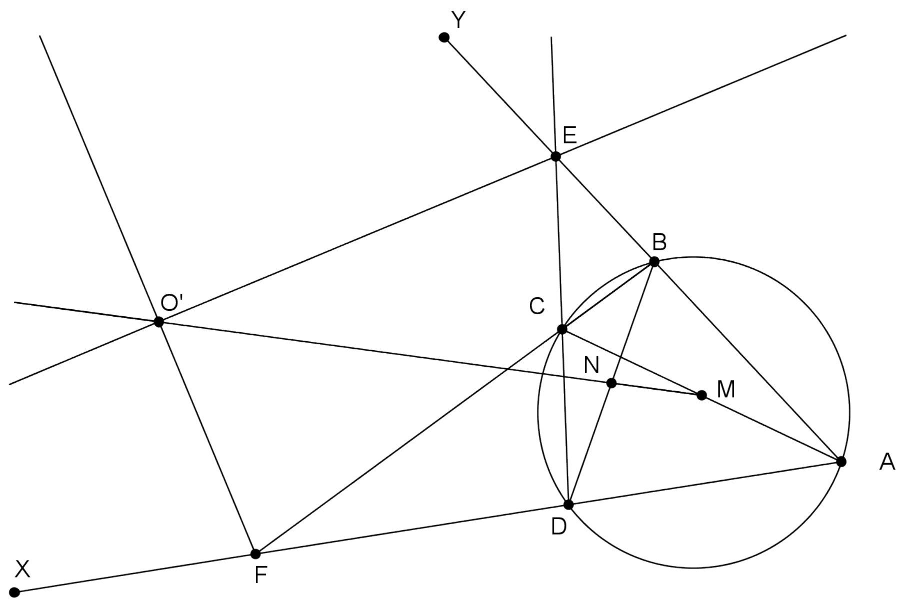

Figure t420biii

(a) Let O' be the point where the bisector of angle $\widehat{DEY}$, supplementary to $\widehat{AED}$ intersects line MN (extended; see figure t420biii). Then $MO' : O'N = EM : EN$ (from **115**; the last two segments are not shown in the figure), and as in (1), $EM : EN = AC : BD$. Hence

$$MO' : O'N = AC : BD \tag{6}$$

.

Analogously, the bisector of $\widehat{CFX}$, supplementary to $\widehat{CFD}$ intersects line MN in a point dividing segment MN externally into the same ratio. Hence the two angle bisectors pass through the same point O', which divides segment MN externally in the ratio of the diagonals.

(b) Line EO' bisects the angle supplementary to $\widehat{MEN}$, and FO' bisects the angle supplementary to $\widehat{MFN}$. For clarity, these angles are not shown in figure t420biii.

(c) Let the bisector of the angle supplementary to $\widehat{AED}$ meet AD (extended) at Q', BC (extended) at P' (*fig.* t420biv). Likewise, let the bisector of the angle supplementary to $\widehat{BFA}$ intersect CD, AB in points S', R'. Then $P'R'Q'S'$ is a rhombus.

To show this, note that EQ' is an exterior angle bisector in triangle EAD, so $Q'D : Q'A = ED : EA$. From similar triangles AEC, BED, we have $ED : EA = BD : AC$. So $Q'D : Q'A = BD : AC$.

And FS' is an exterior angle bisector in triangle FDC, so $S'D : S'C = FD : FC$. From similar triangles BFD, AFC we have $FD : FC = BD : AC$, the same ratio as above. So $Q'D : Q'A = S'D : S'C$, which shows that $Q'S' \parallel AC$. In the same way, we can show that $P'R' \parallel AC$, and that $Q'R' \parallel BD \parallel P'S'$.

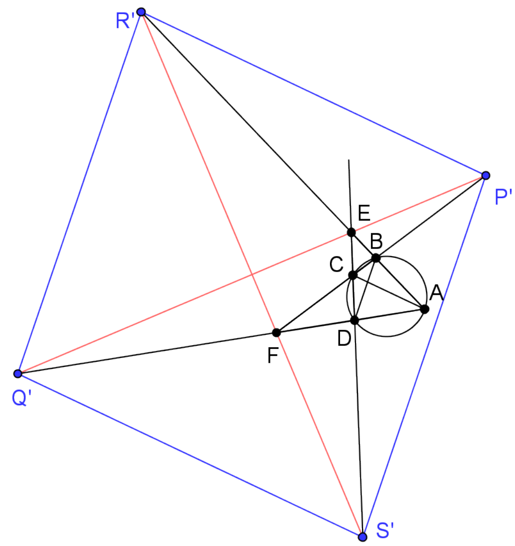

FIGURE t420biv

We can now use the same argument as in (3) to compute the lengths of $P'S'$, $S'Q'$, $Q'R'$, $R'P'$ in terms of the diagonals of the quadrilateral. The computation shows these segments to be equal, so $P'Q'R'S'$ is a rhombus with sides parallel to the diagonals of $ABCD$.

5°. Show that the ratio of EF to the segment joining the midpoints of the diagonals is the same as the ratio of twice the product of these diagonals and the difference of their squares. Calculate EF knowing the sides of the quadrilateral.

Solution. A glance at figure t420bii shows that EO, FO lie along two diagonals of rhombus $PSQR$, so $EO \perp FO$. And since (*fig.*t420bv) EO, EQ' are angle bisectors of two supplementary angles, $EO \perp EQ'$. For the same reason, $FO \perp FO'$, and $FOEO'$ is a rectangle. Thus $EF = OO'$ (**48**).

From (4) we have $\frac{ON}{MN} = \frac{BD}{AC+BD}$, and from (6), $\frac{NO'}{MN} = \frac{BD}{AC-BD}$ (**106**). Thus $\frac{EF}{MN} = \frac{OO'}{MN} = \frac{ON+NO'}{MN} = \frac{ON}{MN} + \frac{NO'}{MN}\frac{BD}{BD+AC} + \frac{BD}{AC-BD} = \frac{2AC \cdot BD}{AC^2-BD^2}$. That is, the ratio of segment EF to MN, the segment joining the midpoints of the diagonals, is equal to the ratio of twice the product of the diagonals to the difference of their squares. This is what the problem requires.

We now calculate the length of EF in terms of the lengths of the sides of the quadrilateral. We can do this directly, from the last relationship we derived. Indeed, **240b** gives expressions for diagonals AC and BD in terms of the sides, and exercise 139 gives us a similar expression for MN.

Alternatively, we can use results from the theory of poles and polars. Note that E and F are conjugate points with respect to the circumcircle, in the sense of **205**: the result of the first paragraph of **211** shows that the polar of E passes through F and vice-versa. Thus, from the first result proved in the solution to exercise 237, the square of EF is equal to the sum of the powers of E and F with respect to the circle, or

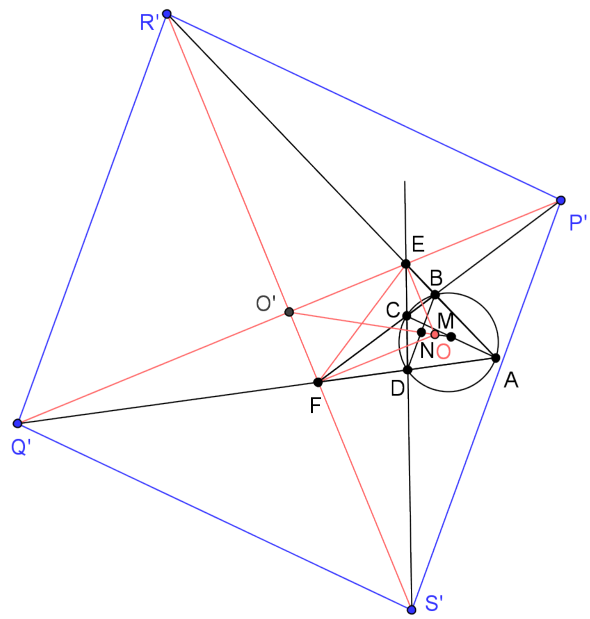

Figure t420bv

$$EF^2 = EA \cdot EB + FA \cdot FD. \tag{7}$$

Let $AB = a$, $BC = b$, $CD = c, DA = d$. Then from similar triangles EAD, ECB we have:

$$EA : EC = ED : EB = AD : BC = d : b.$$

Also,

$$EA - EB = a;$$
$$ED - EC = b.$$

We can eliminate EC and ED from these three equations, to get expressions for EA and EB in terms of the sides:

$$EA = \frac{d(ad + bc)}{d^2 - b^2};$$
$$EB = \frac{b(ab + cd)}{d^2 - b^2}.$$

Multiplying, we have

$$EA \cdot EB = \frac{bd(ab+cd)(ad+bc)}{(b^2-d^2)^2}.$$

In the same way, we can get

$$FA \cdot FD = \frac{ac(ab+cd)(ad+bc)}{(a^2-c^2)^2}.$$

If we substitute these values of $EA \cdot EB$ and $FA \cdot FD$ into equation (6), we find:

$$EF^2 = (ab+cd)(ad+bc)\left(\frac{ac}{(a^2-c^2)^2} + \frac{bd}{(b^2-d^2)^2}\right).$$

Problem 421. Let O be an interior point of triangle $A_1A_2A_3$, and let (k_1'), (k_2'), (k_3') be the circles inscribed in the triangles $A_2A_3O, A_3A_1O, A_1A_2O$.

Note. Throughout the solution of this problem we will use the notation (P) of the original problem for the circle with center at point P, except where this notation is ambiguous.

1°**.** If (k_1) is a circle concentric with (k_1'), one can find (k_2) concentric with (k_2') and (k_3) concentric with (k_3') such that $(k_2), (k_3)$ intersect in a point N_1 on A_1O, $(k_3), (k_1)$ intersect in a point N_2 on A_2O, and $(k_3), (k_1)$ in a point N_2 on A_3O.

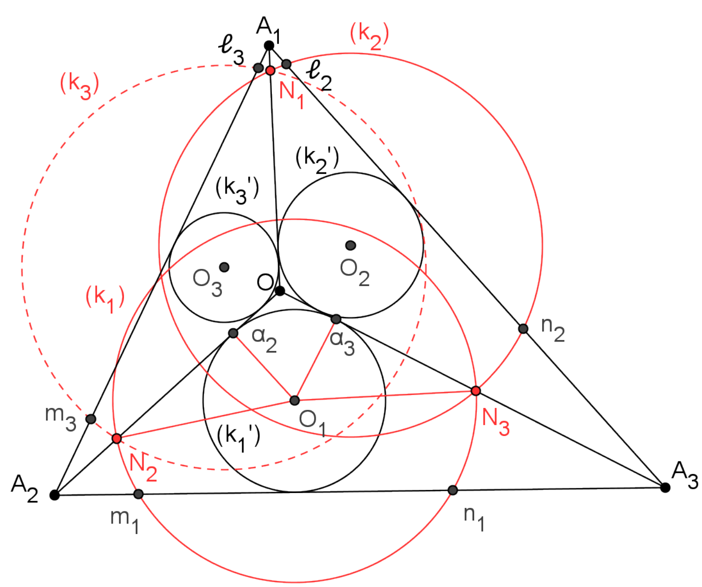

Figure t421a

Solution. Suppose (*fig.* t421a) (k_1) is some circle concentric to (k_1'), and (k_1) intersects $OA_2,\ OA_3$ at $N_2,\ N_3$ respectively. Let O_1 be the common center of $(k_1'),\ (k_1)$, and let $\alpha_2,\ \alpha_3$ be the points of contact of (k_1') with $OA_2,\ OA_3$. Then (**92**)

$$O\alpha_2 = O\alpha_3. \tag{1}$$

Now $O_1\alpha_2 = O_1\alpha_3$ (they are both radii of (k_1')), and $O_1N_2 = O_1N_3$ (they are both radii of (k_1). Hence right triangles $O_1\alpha_2N_2,\ O_1\alpha_3N_3$ are congruent (**34**, case 2), and

$$\alpha_2N_2 = \alpha_3N_3. \tag{2}$$

Adding (1) and (2), we have $ON_2 = ON_3$.

We now construct circle (k_2), concentric to (k_2') and passing through N_3. Let O_2 be the center of (k_2'). Then if N_1 is the intersection of (k_2) with segment OA_1, we can use an argument analogous to the one above to show that $ON_1 = ON_3$. We have solved the problem if we can show that circle (k_3), with center O_3 and radius O_3N_1, passes through N_2.

This is not difficult. Triangles $O_3N_1O,\ O_3N_2O$ are congruent (they are not drawn in figure t421a). Indeed, $ON_1 = ON_3 = ON_2$, they share a common side OO_3, and $\widehat{N_2ON_1}$ is formed by two tangents to (k_3'), and so is bisected by OO_3 (**92**), so the triangles are congruent by SAS (**24**, SAS). This concludes the proof.

2°**.** Circle (k_1) intersects A_2A_3 in two points m_1, n_1 such that $A_2m_1 = A_2N_2$, $A_3n_1 = A_3N_3$. In the same way, (k_2) intersects A_1A_3 in two points ℓ_2, n_2 such that $A_1\ell_2 = A_1N_1,\ A_3n_2 = A_3N_3$, and (k_3) intersects A_1A_2 in two points $\ell_3,\ m_3$ such that $A_1\ell_3 = A_1N_1,\ A_2m_3 = A_2N_2$.

Solution. Let α_1 (*fig.* t421b) be the point of contact of circle (k_1') with A_2A_3. Then $A_2\alpha_1 = A_2\alpha_2$ (**92**, or simply note that the figure formed by concentric circles $(k_1),\ (k_1')$ and two tangents to (k_1') from point A_2 is symmetric in line A_2O_1). From the same symmetry (or using congruent triangles, not shown in the figure), we see that $\alpha_2N_2 = \alpha_1m_1$. Subtracting, we have $A_2m_1 = A_2N_2$.

The proofs of the other results are completely analogous.

3°**.** As the radius of (k_1) varies, the radii of $(k_2), (k_3)$ vary so that the properties in 1° remain true. The points P_1, P_2, P_3, other than N_1, N_2, N_3, where pairs of these circles intersect, move along the common tangents $(t_1), (t_2), (t_3)$ to the pairs of circles $(k_2'), (k_3');\ (k_3'), (k_1');\ (k_1'), (k_2')$ respectively. These three lines are concurrent in a point obtained from O by the construction indicated in Exercise 197, the triangle ABC in that exercise being replaced by the triangle formed by the centers of $(k_1'), (k_2'), (k_3')$.

Solution. The first sentence of the problem statement is simply a statement of fact: the arguments in (1) and (2) do not refer to any properties of the radius of (k_1).

A nice symmetry argument will show that P_2, for instance, lies on the common internal tangent of (k_1') and (k_3') (*fig.* t431c). Indeed, P_2 is the reflection of N_2 in O_1O_3, their common centerline. Now N_2 lies on OA_2, which is a common internal tangent of (k_1') and (k_3'). Hence its reflection P_2 must lie on the other common

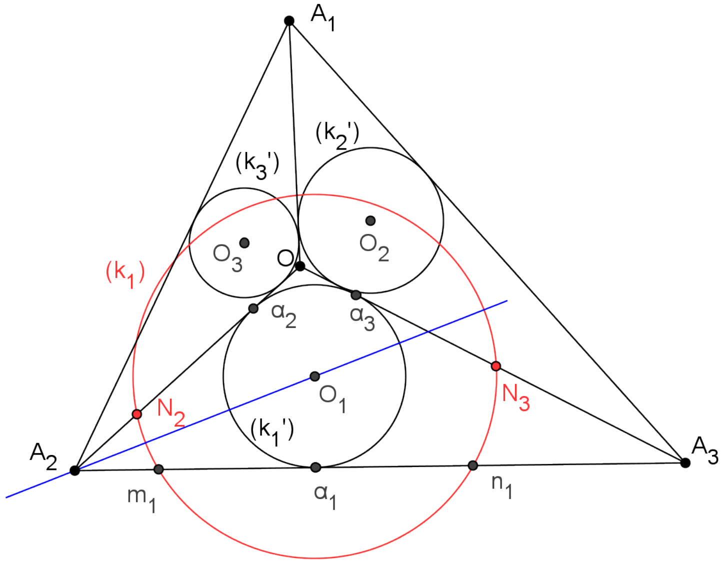

FIGURE t421b

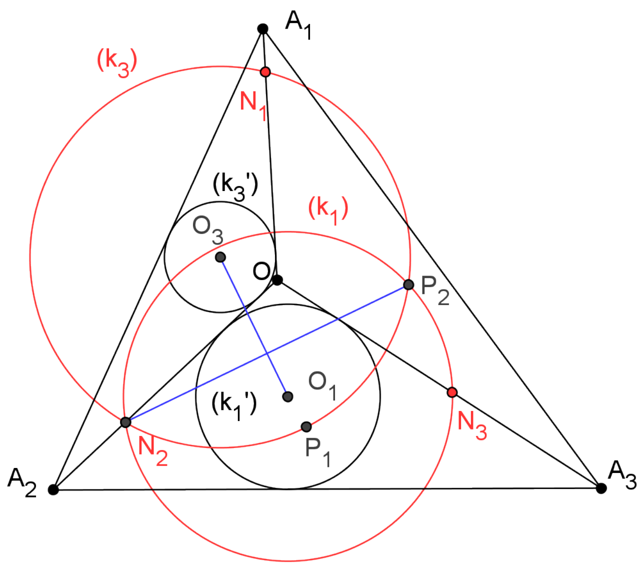

FIGURE t421c

internal tangent of (k_1') and (k_3'), since the common internal tangents of two non-intersecting circles are symmetric in the common centerline. Analogous arguments hold, of course, for P_1 and P_3.

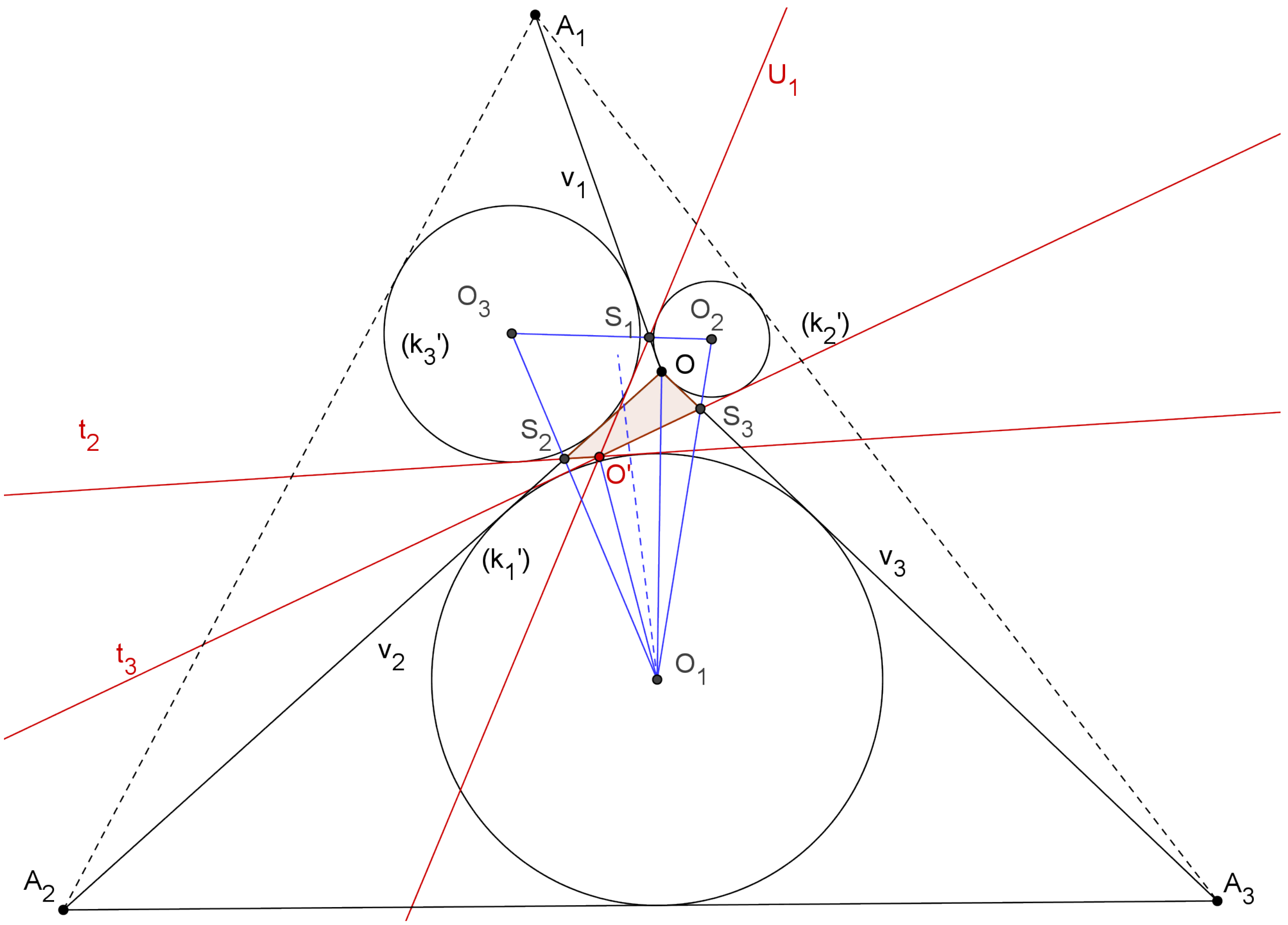

FIGURE t421d

Next we prove the concurrence of lines $t_1,\ t_2,\ t_3$. This is more difficult to show. Figure t421d shows the parts of the diagram essential to the argument. In it, the three original common tangents to the three circles $(k_1'),\ (k_2'),\ (k_3')$, called $ON_1,\ ON_2,\ ON_3$ in previous diagrams, are here labeled $v_1,\ v_2,\ v_3$.

We let O' be the intersection of t_2 and t_3, which are the new common internal tangents to $(k_1'),\ (k_3')$ and $(k_1'),\ (k_2')$ respectively. Let U_1 be the tangent from O' to circle (k_2'). Our strategy will be to show that U_1 is also tangent to (k_3'), so that it is actually none other than line t_1, which must therefore pass through O'.

To this end, we let S_2 be the intersection of t_2 and v_2, let S_3 be the intersection of t_3 and v_3, and let S_1 be the intersection of U_1 and v_1.

We consider quadrilateral $OS_2O'S_3$. It is circumscribed about circle (k_1') in the sense of exercise 87: the lines along which its four sides lie are all tangent to (k_1'). So the result of that exercise applies, or $OS_2 + O'S_2 = OS_3 + O'S_3$. Similarly, quadrilateral $OS_3O'S_1$ is circumscribed about circle (k_2'), so $OS_3+O'S_3 = OS_1 + O'S_1$. From these equations we find that $OS_2 + O'S_2 = OS_1 + O'S_1$. But, again from the result of exercise 87, this means that $OS_1O'S_2$ is circumscribed about a circle, and this circle can only be (k'_3), which we know is tangent to three of the sides of the quadrilateral. (Note that in figure t421d this last quadrilateral turns out to be re-entrant. Students can check that the result of exercise 87 still holds.) That is, since three sides of $OS_1O'S_2$ are tangent to (k'_3), the fourth side must also be tangent to this circle. But this means that U_1 is tangent to (k'_3), and U_1 must coincide with t_1.

We have proved that $t_1,\ t_2,\ t_3$ all pass through the same point O'. It remains to show that O and O' are related as in exercise 197. That is, we must show that if we reflect lines OO_1, OO_2, OO_3 in the bisectors of the angles of triangle $O_1O_2O_3$, we get three lines which are concurrent at O'.

To do this, we consider circle (k'_1) and tangents $v_2,\ t_2$ to it, applying to these the result of exercise 89. This result tells us that the segments intercepted on any other tangents to (k'_1) by these two fixed tangents subtend equal angles at O_1. That is, segments $OS_3,\ O'S_2$ subtend equal angles at O_1. We can express this fact in various ways. Most simply, $\widehat{OO_1S_3} = \widehat{O'O_1S_2}$. Or we can say that lines O_1O and O_1O' make equal angles with sides $O_1O_2,\ O_1O_3$ of triangle $O_1O_2O_3$. Most importantly, because an angle is symmetric in its bisector, we can say that lines $OO_1,\ O'O_1$ make equal angles with the bisector of angle $\widehat{O_2O_1O_3}$ (which is shown as the dotted blue line in figure t421d).

Analogously, we can show that lines $OO_2,\ O'O_2$ make equal angles with the bisector of angle $\widehat{O_1O_2O_3}$, and lines $OO_3,\ O'O_3$ make equal angles with the bisector of angle $\widehat{O_1O_2O_3}$. Thus point O' is the point of concurrence of three lines which are the reflections in the angle bisectors of triangle $O_1O_2O_3$ of three other lines, concurrent at O, and this is the construction described in exercise 197.

4°**.** Quadrilateral $P_2P_3\ell_2\ell_3$ is cyclic (Exercise 345) and the circumscribed circle (x'_1) intersects the sides A_2A_1, A_1A_3 and the lines $(t_2), (t_3)$ at equal angles; likewise, P_3, P_1, m_3, m_1 are on a circle (x'_2) intersecting at equal angles $A_2A_3, A_2A_1, (t_3), (t_1)$, and P_1, P_2, n_1, n_2 on a circle (x'_3) intersecting at equal angles $A_3A_1, A_3A_2, (t_1), (t_2)$. The center of (x'_1) remains fixed as the radii of $(k_1), (k_2), (k_3)$ vary as in (3); the line which joins it with the center of (k'_1) passes through the intersection of $(t_1), (t_2), (t_3)$. Similar statements for the centers of $(x'_2), (x'_3)$. There exists a circle (x_1) tangent to $A_2A_1, A_1A_3,\ (t_2), (t_3)$, a circle (x_2) tangent to $A_2A_3, A_2A_1,\ (t_3), (t_1)$, and a circle (x_3) tangent to $A_3A_1, A_3A_2,\ (t_1), (t_2)$.

Solution. We separate the various statements in this part of the exericise:

4a. Quadrilateral $P_2P_3\ell_2\ell_3$ is cyclic.

Proof: From 2°, we know that points $\ell_2,\ \ell_3, N_1$ are equidistant from A_1 (*fig.* 421e), so let (A_1) be the circle through those three points with center A_1. From 1° we know that $ON_1 = ON_2 = ON_3$, so the circle (O) through $N_1,\ N_2,\ N_3$ has its center at O. Now N_1 is on both circles, and also on their common centerline, so circles (A_1) and (O) must be tangent at N_1.

To apply the result of exercise 345, we must start with a cyclic quadrilateral. Here, it is convenient to think of triangle $N_1N_2N_3$ as a "quadrilateral" with a "double point" at N_1. As a "quadrilateral", it is certainly cyclic (it is circumscribed by circle (O)), and circles $(k_1),\ (k_2),\ (k_3),\ (A_1)$ pass through its vertices. If we take the second intersections of pairs of these circles, we get points P_2 (for N_2), P_3 (for N_3), ℓ_2 (for N_1 as a point on circles (k_2) and (A_1)), and ℓ_3 (for N_1 as a

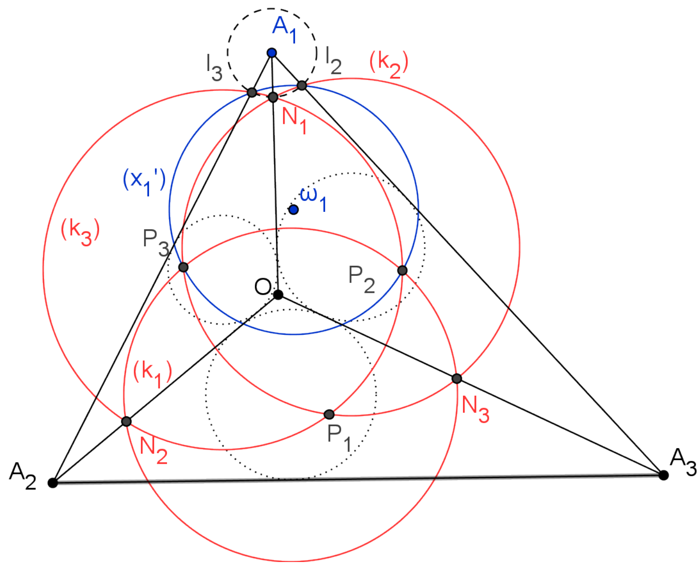

FIGURE t421e

point on circles (k_3) and (A_1)). The result of exercise 345 then guarantees that quadrilateral $P_2P_3\ell_2\ell_3$ is cyclic.

4b. Circle (x_1') intersects sides A_2A_1, A_1A_3 and lines $(t_2), (t_3)$ at equal angles;

Proof: Let ω_1 be the center of circle (x_1'). We know from 1° that $A_1\ell_2 = A_1\ell_3$. If we draw $\omega_1\ell_2,\ \omega_1\ell_3$, we find that triangles $A_1\omega_1\ell_2,\ A_1\omega_1\ell_3$ are congruent (SSS, **24**). Hence $\widehat{\ell_2A_1\omega_1} = \widehat{\ell_3A_1\omega_1}$, so lines $A_1A_2,\ A_1A_3$ are symmetric with respect to $A_1\omega_1$. It follows that angles $\widehat{\omega_1\ell_2A_3} = \widehat{\omega_1\ell_3A_2}$. Finally, it is not hard to see that if the radii to two secants make equal angles at a point of contact, then the same is true of the two tangents at those points of contact. This proves the equality of the first pair of angles mentioned.

Note. Students might see this result more easily by noting that because $A_1\ell_2 = A_1\ell_3$, lines A_1A_2 and A_1A_3 are symmetric with respect to line $A_1\omega_1$.

To prove that circle (x_1') makes equal angles with lines t_2 and t_3, we let $\beta,\ \beta'$ be the points of contact of circle (k_2') with lines A_1A_3 and t_3 respectively. If K is the intersection of lines A_1A_3 and t_3, then $K\beta,\ K\beta'$ are the tangents to circle (k_2') from K, so $K\beta = K\beta'$.

We next show that $\beta\ell_2 = \beta'P_3$. Indeed, from circle (k_2'), we know that $O_2\beta = O_2\beta'$, and from (concentric) circle (k_2) we know that $O_2\ell_2 = O_2P_3$. Also, $\widehat{O_2\beta\ell_2} = \widehat{O_2\beta'P_3} = 90°$ (**58**), so right triangle $O_2\beta\ell_2,\ O_2\beta'P_3$ are congruent, and $\beta\ell_2 = \beta'P_3$.

Now the angles at which (x_1') intersect A_1A_3 and t_3 are equal if angles $\widehat{\omega_1\ell_2K}$, $\widehat{\omega_1P_3K}$ are equal, and this last assertion is easily seen to be true, either by symmetry about line $K\omega_1$ or from congruent triangles $\omega_1\ell_2K,\ \omega_1P_3K$.

Similarly, (x_1') intersects lines A_1A_2 and t_2 at equal angles. Thus the four angles mentioned in the problem statement are all equal.

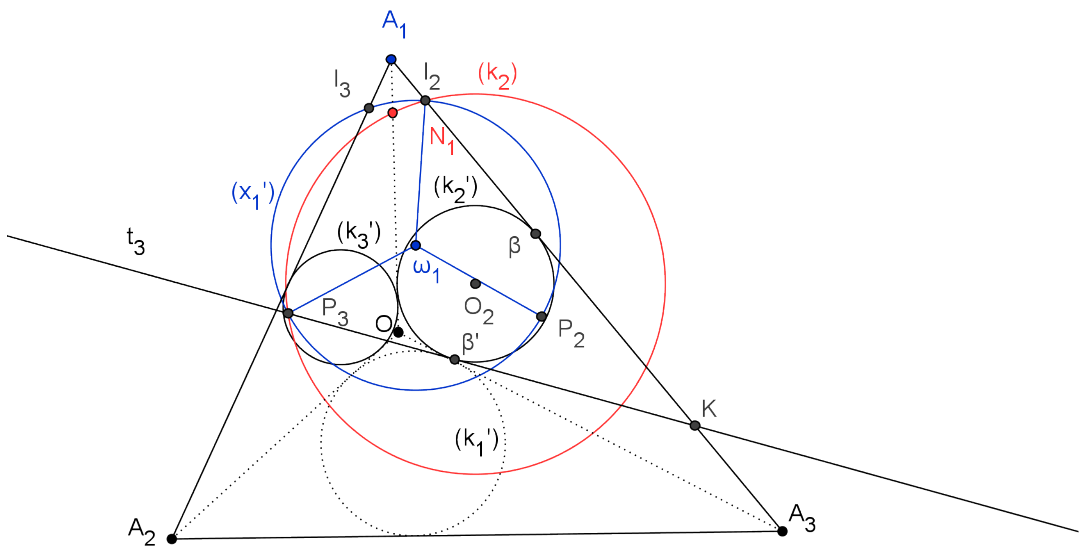

FIGURE t421f

4c. The center ω_1 of x_1' remains fixed as $(k_1),\ (k_2),\ (k_3)$ vary. The line joining ω_1 with the center of (k_1') passes through the intersection of $(t_1),(t_2),(t_3)$. Similar statements for the centers of $(x_2'),(x_3')$.

Lemma: If a circle makes equal angles with two lines, then its center is equidistant from the lines.

Proof of lemma: It is not hard to see that the figure formed by the two lines and the circle is symmetric in the line joining the circle's center to the intersection of the lines. This observation leads to the desired conclusion, or motivates an easy proof by congruent triangles.

Proof of statement 4c: From our lemma and 4b, it follows that ω_1 is equidistant from lines $A_1A_2,\ A_1A_3,\ t_2$ and t_3, and thus lies on the intersection of the bisectors of angle $\widehat{A_2A_1A_3}$ and of the angle between t_2 and t_3; that is, the line $O'O_1$. Since the location of lines $t_2,\ t_3$ do not depend on the radii of $(k_1),\ (k_2),\ (k_3)$, it follows that the position of ω_1 does not depend on the choice of these radii, so that line $O_1\omega_1$ passes through O'.

The proofs of similar statements for the centers of $(x_2'),(x_3')$ are analogous.

4d. There exists a circle (x_1) tangent to $A_1A_2, A_1A_3,\ (t_2),(t_3)$, a circle (x_2) tangent to $A_2A_3, A_2A_1,\ (t_3),(t_1)$, and a circle (x_3) tangent to $A_3A_1, A_3A_2,\ (t_1),(t_2)$.

Proof: We have seen that center ω_1 of circle (x_1') is equidistant from lines $A_2A_3, A_2A_1,\ (t_3),(t_1)$. Thus there is a circle (x_1) centered at ω_1 which is tangent to these four lines.

The result for the other sets of lines is proved analogously.

5°**.** The intersection of $m_1P_3,\ n_1P_2$ is on the radical axis of $(x_2'),(x_3')$.

Proof: Line m_1P_3 (*fig.* t421h) contains the common chord of circles $(k_1),\ (x_2')$, so it is the radical axis of these two circles. Similarly, line n_1P_2 is the radical axis of circles $(k_1),\ (x_3')$. Therefore the intersection M of these two lines is the

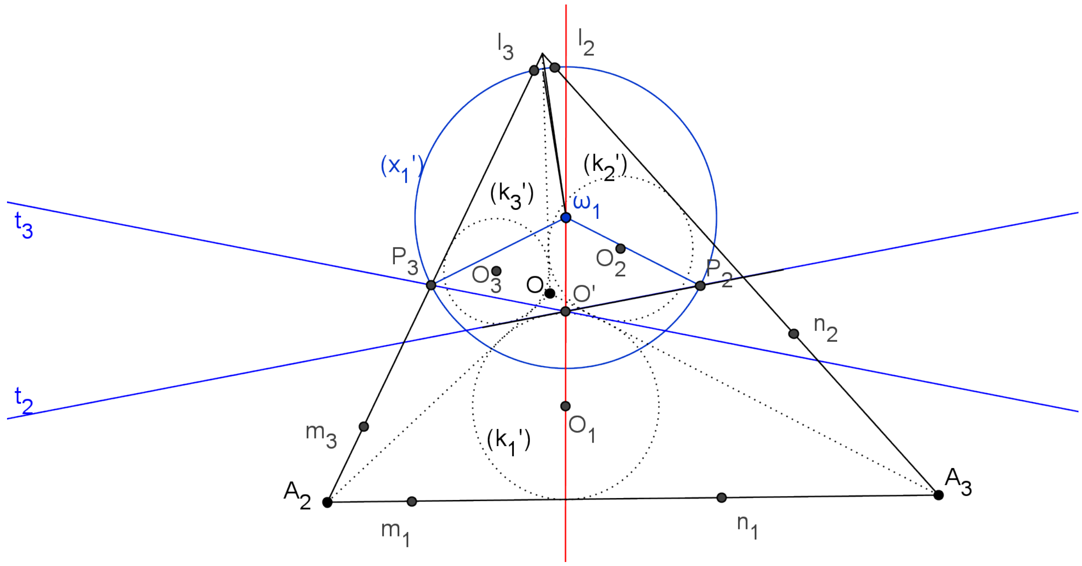

FIGURE t421g

radical center of circles $(k_1),\ (x_2'),\ (x_3')$ (see **139**), and so lies on the radical axis of $(x_2'), (x_3')$. This proves the assertion.

6°**.** As the radii of $(k_1), (k_2), (k_3)$ vary, the intersection of m_1P_3 and n_1P_2 describes the line joining the intersection of $(t_1), (t_2), (t_3)$ with the point of contact of (k_1') with A_2A_3.

Proof: Let α be the point of contact of circle (k_1') with line A_2A_3. Let $T_2,\ T_3$ be the intersections of lines $t_2,\ t_3$ respectively with line A_2A_3, and let M' be the intersection of lines n_1P_2 and $O'\alpha$. We will show that M' coincides with M (the point described in 5°).

We do this by showing that lines $m_1P_3,\ n_1P_2$ divide segment $O'\alpha$ externally in the same ratio, and so their point of intersection is on line $O'\alpha$, and M coincides with M'.

To do this, we first apply Menelaus' Theorem **192** to triangle $\alpha O'T_2$ and transversal n_1P_2, to get

$$\frac{P_2O'}{P_2T_2} \cdot \frac{n_1T_2}{n_1\alpha} \cdot \frac{M'\alpha}{M'O'} = 1.$$

But we can show that $P_2T_2 = n_1T_2$. Indeed, both lines are tangent to circle (k_1'), and so are equidistant from point O_1. Since this point is also the center of circle (k_1), segments $T_2P_2,\ T_2n_1$ are chords of this circle equidistant from its center, and so are equal.

Thus $\frac{M'\alpha}{M'O'} = \frac{n_1\alpha}{P_2O'}$; that is, line n_1M' divides segment $O_1\alpha$ externally in the ratio $\frac{n_1\alpha}{P_2O'}$.

the same theorem to triangle $\alpha O'T_3$, with transversal m_1P_3. Following the same line of reasoning, we find

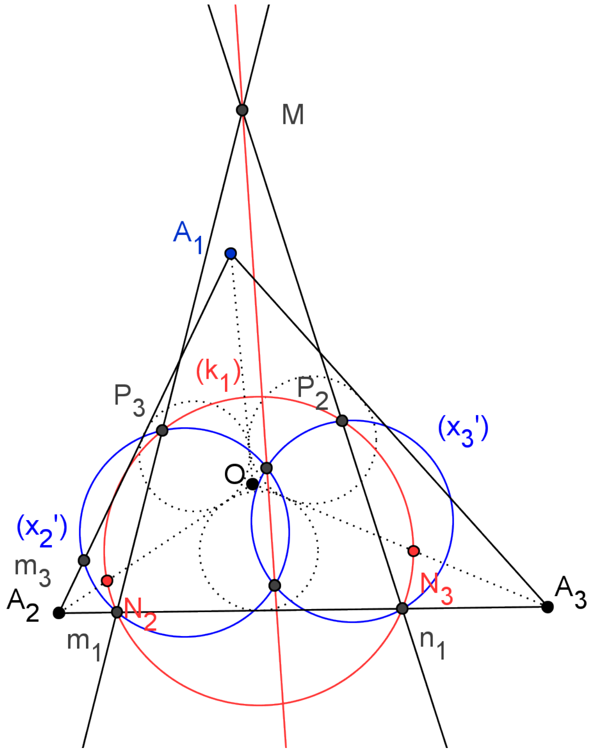

Figure t421h

$$\frac{P_3O'}{P_3T_3} \cdot \frac{m_1T_3}{m_1\alpha} \cdot \frac{M'\alpha}{M'O'} = 1.$$

As before, we can show that $m_1T_3 = P_3T_3$, so we find that $\frac{M'\alpha}{M'O'} = \frac{m_1\alpha}{P_3O'}$. This says that that line m_1P_3 divides segment $\alpha O'$ externally in the ratio $m_1\alpha : P_3O'$.

But we can show that $m_1\alpha = n_1\alpha$. Indeed, triangles $O_1n_1\alpha,\ O_1m_1\alpha$ (not show in figure t421k) are both right-angled at α (since $O_1\alpha \perp m_1n_1$), they have leg $O_1\alpha$ in common, and $O_1m_1 = O_1n_1$, as radii of circle (k_1). This proves that $m_1\alpha = n_1\alpha$.

We can also show that $O'P_2 = O'P_3$, repeating an argument used above. Indeed lines $O'P_2,\ O'P_3$ (that is, $t_2,\ t_3$) are tangent to circle (k_1') and are therefore equidistant from the center O_1 of circle (k_1). Hence, as chords equidistant for the center of (k_1), we have $O'P_2 = O'P_3$.

Thus lines m_1P_3 and n_1P_2 divide segment $\alpha O'$ in the same ratio, and so must intersect at point M' on line $\alpha O'$. But it was shown, in 5°, that their point of intersection was M, as defined in that section. This means that points $M,\ M'$ coincide.

That is, the intersection M of lines m_1P_3 and n_1P_2 lies on the line joining the point of contact α of circle (k_1') with line A_2A_3 to the intersection O' of lines $t_1,\ t_2,\ t_3$, no matter how the radii of circles $(k_1),\ (k_2),\ (k_3)$ are chosen.

7°. A condition for $(x_2), (x_3)$ to be tangent is that line $O_1\alpha$ coincide with line t_1.

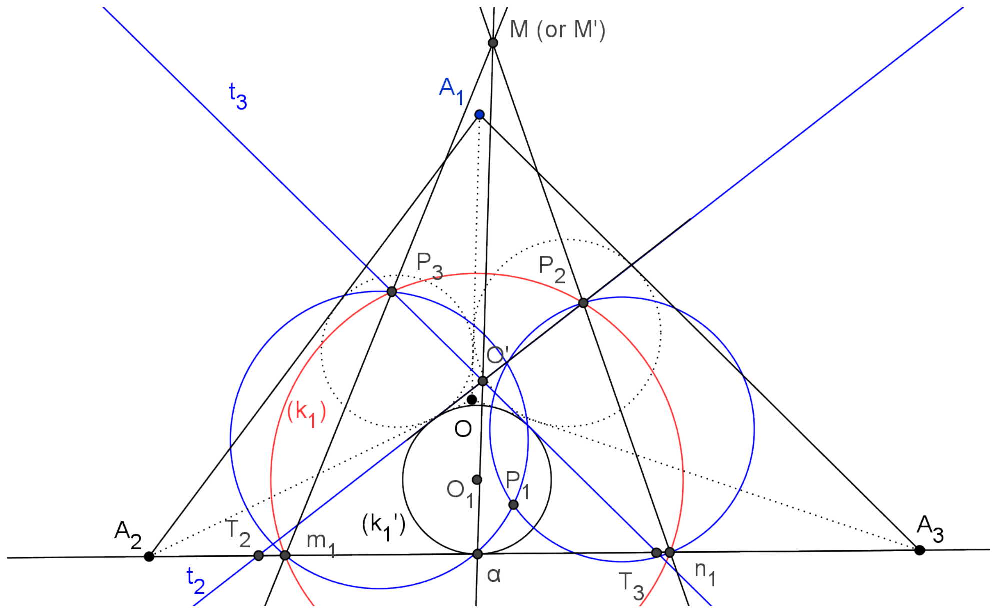

FIGURE t421k

Proof: Suppose circles (x_2), (x_3) are tangent. Then line t_1, which is tangent to both of them, must be their common (internal) tangent.

Since one of the intersection points P_1 of circles $(x_2'($, (x_3') – which are concentric with (x_2), (x_3) – lies on line t_1, it follows that the second point of intersection of the two circles lies on line t_1. Indeed, the common centerline of (x_2'), (x_3') (which is also the common centerline of (x_2), (x_3)) is perpendicular to the common internal tangent t_1. So t_1 is the (unique) perpendicular from P_1 to the common centerline. Thus the common chord of $(x_2'), (x_3')$ lies along t_1, as does their second point of intersection.

Since t_1 is the radical axis of circles (x_2'), (x_3'), it passes through the radical center M of circles (k_1), (x_2'), (x_3'). And since it also passes through point O', it must coincide with $O'M$, which (from 6°) is the same line as $O'\alpha$.

Conversely, if line t_1 coincides with $O'M$, then it passes through the radical center M of circles (k_1), (x_2'), (x_3'). Since this line also passes through one of the points P_1 of intersection of circles (x_2') and (x_3'), it must in fact be their radical axis, and so it passes through their second point of intersection as well. Circles (x_2), (x_3) are therefore tangent to line t_1 on the line of centers of circles (x_2'), (x_3'), and therefeore are tangent to each other.

Problem 421b. Through a given point A in the plane, construct a line on which two given circles C, C' intercept equal chords MN, $M'N'$. (The notation being such that these segments are in the same sense, one should look for the common midpoint of MN' and NM'.) More generally, draw a line through A such that the chords intercepted by C, C' have a given ratio k. (Use Exercise 149.) Is the maximum number of solutions the same for $k \neq 1$ as it is for $k = 1$?

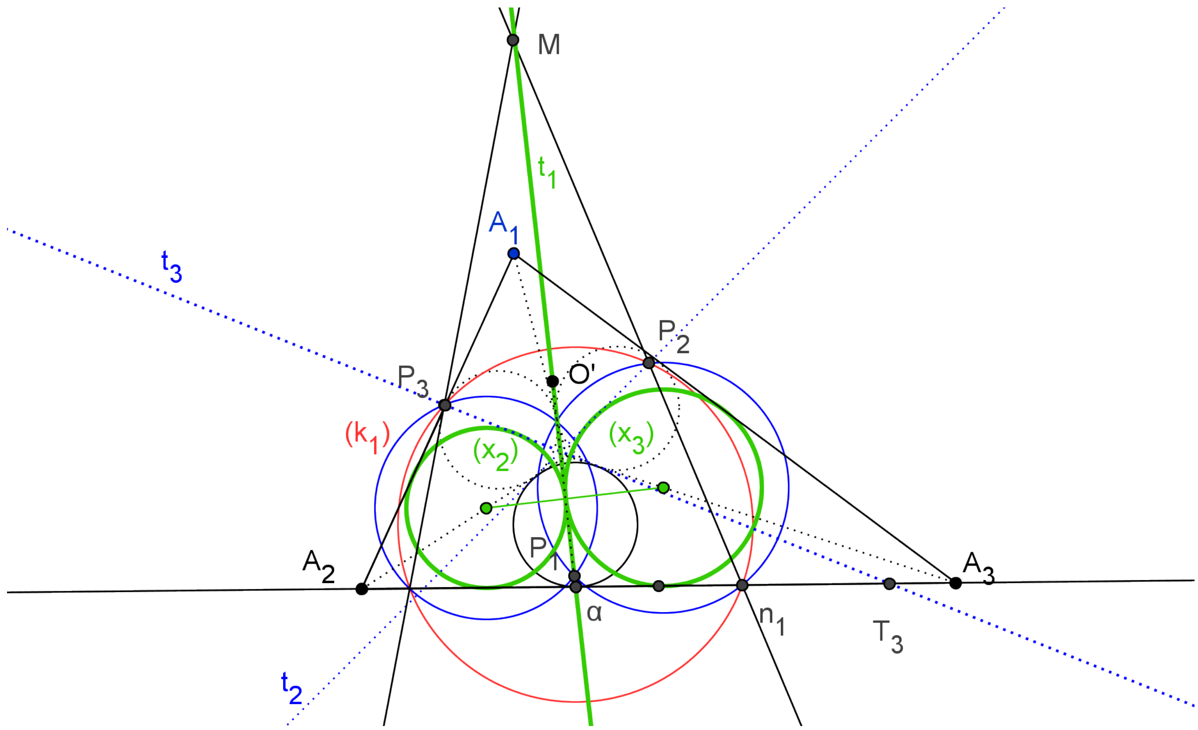

FIGURE t421m

Solution. (This solution is due to Behzad Mehrdad.) Suppose (*fig.* t421bi) the given circles C, C' (we use this notation both for the circles and for their centers) intercept equal chords MN, $M'N'$ along line AS. Following the hint in the problem statement, we first note that if D is the midpoint of segment MN', then it is also the midpoint of segment $M'N$. Next we note that D must be on d, the radical axis of circles C, C'. Indeed, we have $DM' = DN$, $DM = DN'$, so that $DM \cdot DN = DM' \cdot DN'$. That is, the powers of D with respected to the two given circles are equal, and D lies on their radical axis.

Let the midpoints of MN, $M'N'$ be U and V respectively. Then both CU and $C'V$ are perpendicular to AS. If we draw the perpendicular bisector DT of segment MN' (where T is on line CC'), then DT will be parallel to CU and $C'V$, and by **113**, T is the midpoint of CC'. That is, point D must be located at the intersection of the radical axis of circles C, C' with the perpendicular bisector of MN'. The argument is also reversible: if we locate point D as described, then AD will satisfy the conditions of the problem.

Thus we have the following construction. We find the midpoint of T of segment CC', and draw a circle with diameter AT. If the intersection of this circle with the radical axis of circles C, C' is D, then AD is the required line.

The number of solutions depends on the number of intersections of the radical axis with the circle on diameter AT. There can be 0, 1, or 2 solutions.

Notes. Students can investigate when these cases occur, for example, by looking at the relative positions of point A and the common tangents to the given

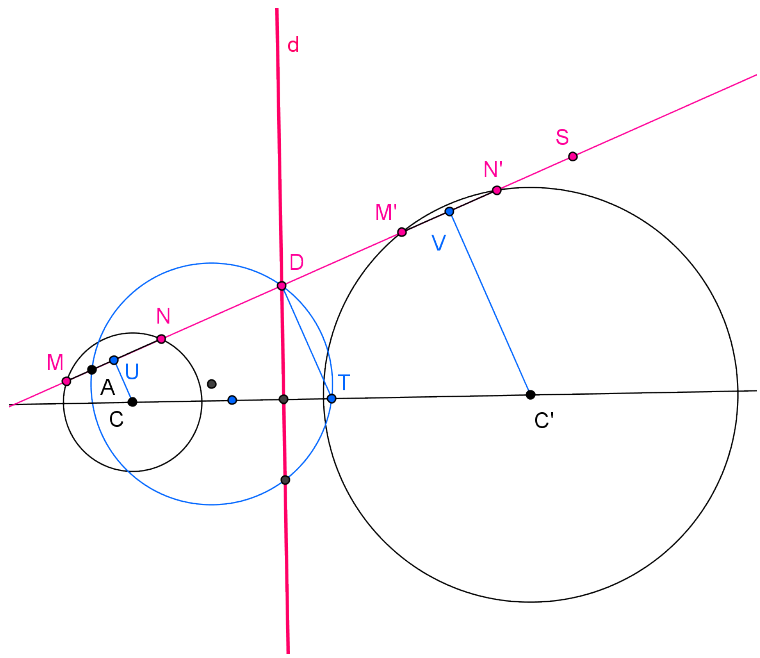

FIGURE t421bi

circles. They can also look at the (easy) special case when A is on the radical axis of the two circles. In that case, they need only reflect one of the given circles in P, and the common chord of the reflected circle and the other circle gives the required line.

We now turn to the generalization, where the chords MN, $M'N'$ are in a given ratio k. The proof given above generalizes immediately. This time, assuming the construction completed, we find point D on segment NM' such that $DN : DM' = k$. Then it is not hard to see that the ratio of the powers of D with respect to circles C, C' is also k, so by the result of exercise 149, D must lie on a circle P with the same radical axis as C and C'. Circle P plays the role of the radical axis in the case $k = 1$.

Next we draw UC, VC' from the centers of the circles perpendicular to MN, $M'N'$ respectively. If the required line is AS, then, again by **113**, a perpendicular from D to CC' will intersect CC' at a point T such that $CT : TC' = k$, a point we can easily construct. This point lies on a circle Q which is the locus of points the ratio of whose distances to C, C' is k. Point D can be either of the intersections of circles P and Q. A reversal of this argument will show that if point D is constructed as described, then AD will satisfy the conditions of the problem.

Problem 422. (Morley's theorem) Divide each angle of a triangle ABC into three equal parts by lines AS, AT ($\widehat{CAS} = \widehat{SAT} = \widehat{TAB}$); BT, BR($\widehat{ABT} = \widehat{TBR} = \widehat{RBC}$); CR, CS ($\widehat{BCR} = \widehat{RCS} = \widehat{SCA}$). The lines through B, C and closest to BC intersect in R; the ones from C, A and closest to CA intersect in S, and the ones from A, B and closest to AB intersect in T. The three points R S, T obtained this way are the vertices of an equilateral triangle. (Extending BT, CS

to their intersection I we form in R with RI and on each side of RI, angles of $30°$ until their intersections in T', S' with BT, CS, respectively. The triangle $RT'S'$ is equilateral: it suffices to show that the lines AT', AS' divide $\widehat{BAC}$ into three equal parts. For this, denote by B', C' the symmetric points of R relative to BT, CS, respectively, and show that $C'T'S'B'$ is a regular broken line (**160**), and that the circumscribed circle passes through A.)

Solution. (This solution is due to Behzad Mehrdad.) The wording of the hint is a bit confusing. What is meant is that we first construct a certain triangle, which turns out to be equilateral, then show that its vertices lie at the intersection points of pairs of angle trisectors.

More specifically (*fig.* t422), we first construct point R, the intersection of the trisectors of angles $\widehat{B}$, $\widehat{C}$ nearest side BC. This will be one vertex of our equilateral triangle. Then we take the other two trisectors of these angles (those that are not nearest side BC), and find their point I of intersection. Finally, we construct angles $\widehat{IRT'}$, $\widehat{IRS'}$, both equal to $30°$, with T' and S' on the BI, CI respectively.

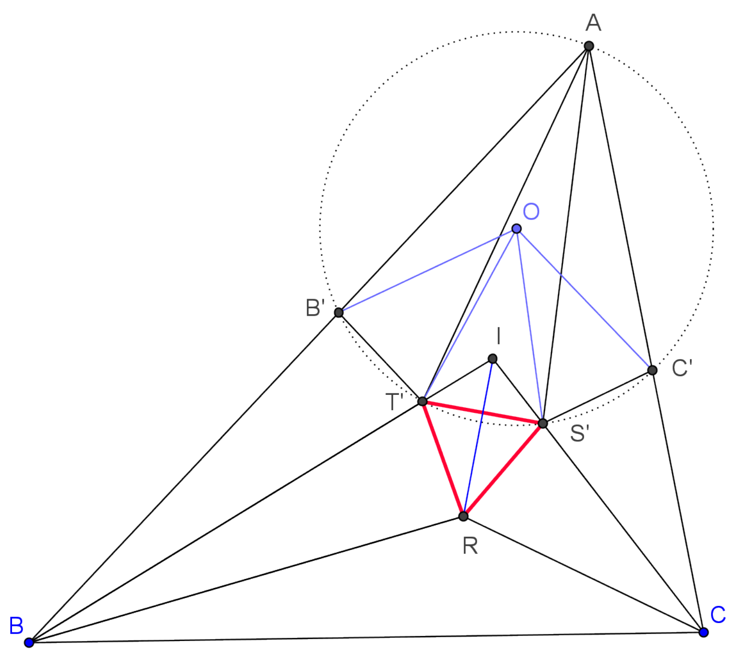

FIGURE t422

We will prove (a) that triangle $RS'T'$ is equilateral, and (b) that points S', T' are actually the points S, T mentioned in the problem statement, by showing that triangle AS', AT' trisect angle $\widehat{A}$.

Proof of (a): In triangle BIC, point R is the intersection of two angle bisectors, so it lies on the third angle bisector as well (**54**). That is, $\widehat{RIT'} = \widehat{RIS'}$. Then triangle RIT', RIS' are congruent (SAS, **24**, case 2), so $RT' = RS'$, and $RS'T'$ is isosceles. Since $\widehat{T'RS'} = 60°$, triangle $RT'S'$ is in fact equilateral.

Proof of (b): We reflect point R in line BI to get point B'. Since BI bisects angle $\widehat{ABR}$, it follows that B' lies on line AB. Similarly, the reflection C' of R in line CI lies on line AC.

Clearly (from the equilateral triangle and properties of reflection), $B'T' = T'R = RS' = S'C'$. Following the hint in the problem, we now show that $\widehat{B'T'S'} = \widehat{C'S'T'}$. We do this by comparing the angles around points S' and T'. Note that triangles $T'IR$, $S'IR$ are congruent (SAS, **24)**, so IR bisects $\widehat{T'IS'}$. And $\widehat{BT'R}$, $\widehat{CS'R}$ are exterior angles for the same two triangles, so they are both sums of equal pairs of remote interior angles and are thus equal. Finally, angles $\widehat{B'T'B}, \widehat{C'S'C}$ are reflections of the last pair of equal angles, and so are equal.

Then we note that since all the other angles around S', T' are equal in pairs, $\widehat{B'T'S'}$ and $\widehat{T'S'C'}$ must also be equal. Thus $B'T'S'C'$ is a 'regular' broken line, as the hint indicates.

Finally, we will show that the circle through $B'T'S'C'$ also passes through A, by computing various angles.

Let $\widehat{B'BT'} = \widehat{T'BR} = \widehat{RBC} = \beta$, $\widehat{C'CS'} = \widehat{S'CR} = \widehat{RCB} = \gamma$. Also let $\widehat{BAC} = 3\alpha$ (note that we must not assume that AS', AT' trisect $\widehat{BAC}$). From triangle ABC, we have $\alpha + \beta + \gamma = 60°$. From triangle BIC, we have $\widehat{T'IS'} = 180° - (2\beta + 2\gamma) = 180° - (120° - 2\alpha) = 60° + 2\alpha$.

Next we note that triangles $RT'I$, $RS'I$ are congruent (SAS, **24**), so $\widehat{T'IR} = \widehat{S'IR} = \frac{1}{2}\widehat{BIC} = 30° + \alpha$. Finally, $RI \perp S'T'$. (It is an angle bisector in isosceles triangle $T'IS'$, and so is also an altitude in that triangle.) Thus $\widehat{S'T'I} = 90° - (30° + \alpha) = 60° - \alpha$, which is also the measure of $\widehat{T'S'I}$.

(We note in passing that many of these angle measure depend on the measure of $\widehat{BAC}$, and not on the other two angles of the original triangle.)

Finally, from triangle RIT', we see that exterior angle $\widehat{BT'R} = \widehat{T'IR} + \widehat{IRT'} = 30° + \alpha + 30° = 60° + \alpha$. By reflection, this is also the measure of $\widehat{B'T'B}$.

We now add up the angles around point T':

$$360° = \widehat{B'T'S'} + \widehat{S'T'R} + \widehat{RT'B} + \widehat{BT'B'} =$$

$$= \widehat{B'TS'} + 60° + 2(60 + \alpha) = \widehat{B'TS'} + 180 + 2\alpha.$$

It follows that $\widehat{B'T'S'} = 180° - 2\alpha$.

We can do the same computation for $\widehat{C'S'T'}$, but in fact we really don't need to. The angles about S' are easily seen to be equal in pairs to the angles about T', so $\widehat{C'S'T'} = \widehat{B'T'S'} = 180° - 2\alpha$ as well.

That is, broken path $B'T'S'C'$ is in fact 'regular': it is composed of equal segments intersecting at equal angles. This means that the vertices lie on a circle. Indeed, if we let the bisectors of $\widehat{C'S'T'}$ and $\widehat{B'T'S'}$ intersect at point O, then triangles $B'OT'$, $T'OS'$, $S'OC'$ are all congruent (by SAS, **24**), and since $TO'S'$ is isosceles, they are all isosceles. Then O is the center of a circle through B', T', S', C'.

Furthermore, by direct computation, $\widehat{OT'S'} = \frac{1}{2}B'T'S' = 90° - \alpha$, so $\widehat{T'OS'} = 180° - 2\widehat{OT'S'} = 180° - 180° + 2\alpha = 2\alpha$. In other words, arc $\widehat{S'T'}$ has a central angle of 2α. So arc $\widehat{B'T'S'C'}$ has a central angle of 6α, and any point at which chord $B'C'$ subtends an angle 3α lies on this circle. Point A is such a point. Then $\widehat{B'AT'}$, $\widehat{T'AS'}$, $\widehat{S'AC'}$ are all inscribed angles subtending equal arcs, the angles are themselves equal, and AT', AS' trisect $\widehat{BAC}$. This completes the proof.