Endnotes for Really Big Numbers:

These notes contain some additional information about my book. One of the things I did before publishing the final copy of the book was go through and justify all the estimates I made. I have included many of the calculations here. These notes (which I'll update periodically) also point out any mistakes that I find. Finally, I have included some commentary here and there.

The pages in the book are not numbered, so I have introduced my own system of numbering. To illustrate my method by way of example, the number **3L** refers to the left half of the 3rd two-page spread and the number **3R** refers to the right half. Sometimes I will add in some additional comments to help the reader locate the page more easily.

1L: This refers to the first of the two thought bubbles in the beginning of the book. I have in mind factorization of primes. The primes are written as numbers and the composites are written as square or cubical grids, with the sides being the prime factors. The question marks around the number 1 are supposed to allude to the fact that children are sometimes puzzled as to whether 1 should be considered prime or composite.

1R: Here are the things shown in the thought bubble.

- A ternary tree.
- A genus two surface
- The Euler product formula for primes.
- The 2-adic integers.
- Billiards in a 100-degree obtuse triangle. This alludes to my theorem that a triangle has a periodic billiard path provided that all its angles are less than 100 degrees.
- The pentagram map acting on a 7-gon. This is a dynamical system I introduced 20 years ago. V. Ovsienko, S. Tabachnikov, and I recently proved that it is completely integrable, and (independently) so did F. Soloviev. The equation $\{E_{7,1}, E_{7,2}\} = 0$ expresses the fact two of the invariants for the pentagram map Poisson commute.

19L: The number of ways to 3-color a 3×3 grid is $3^9 = 19683$. This is pretty close to 20000.

19R: The number of ways to place 3 checkers on a checkerboard is

$$\frac{64 \times 63 \times 62}{3 \times 2 \times 1} = 41664$$

which is pretty close to 40000.

20L: If you want to 3-color a 3×3 grid using all the colors, you have to take the total number of 3 colorings, and subtract off 3 times the number of ways to 2-color a 3×3 grid. This comes to

$$3^9 - 3 \times 2^9 = 18147.$$

21L: How many hairs on the head of a monkey? It was easy to find on the internet that there are about 100000 hairs on a human head, and I figured that the answer would be roughly similar for monkeys. Their hair is coarser, but on the other hand, more of their head it covered with hair.

22L: How many leaves on a typical forest tree? This is kind of a vague question. The website *J. W. Teaford's Wetland Notes* (http://wetlandnotes.com) gives a method for estimating the number of leaves on an oak tree. They come up with the answer that a large oak tree has between 1.25 and 2 million leaves. I had in mind a smaller tree, where the answer would be closer to $100,000$. I could imagine a school project where the students have to formulate this question more precisely and try to answer it.

22R: There are $2^{17} = 131,072$ leaves on a binary tree after 17 years of growth. This is pretty close to $100,000$.

23L: In 114 years, there are

$$114 \times 365 \times 24 = 998,640$$

hours, not accounting for irregularities in the calendar. This is about a million.

24L: The US Census gives the 2012 population of Rhode Island as $1,050,292$. This is very close to the 2010 figure, so probably the population is fairly stable in the short run.

24R: The driving distance from Providence to Chicago is listed as 964 miles, which comes to

$$964 \times 5280 = 5,089,920$$

feet. It seems plausible that the average arm span of a Rhode-Islander is about 5 feet, so about a million people would stretch from Providence to Chicago.

25L: I looked up that satellites fly about 150 miles above the surface of the earth. That comes to $150 \times 5280 = 792,000$ feet, which is fairly close to a million feet.

28L: The circumference of the earth is listed as $24,901$ miles. Fifty times the circumference is

$$50 \times 24,901 \times 5280 = 6,573,864,000$$

feet. If 7 million people lined up, one person per foot, they would between them, they would would perhaps be about a foot of space between them, so 7 million people would wind 50 times around the equator.

28R: The website
http://www.universetoday.com/14824/distance-from-earth-to-mars/
says that the distance from Earth to Mars varies between 41 million kilometers and 401 million kilometers. On a good day, with the distance being minimum, a quarter of the distance from Earth to Mars is

$$.25 \times 41,000,000 \text{ km.} \times 3280.24 \text{ ft./km.} = 3.36 \times 10^{10} \text{ ft.}$$

If 7 billion people stetch out, with a 5 foot span between them, they would reach 3.5×10^{10} feet, which is about the same thing. It seems reasonable, given all the children on the planet, that 5 feet would be the average span of a person in a giant chain of people.

29L: On wikipedia, I learned that the diameter of *very fine sand* is between 62.5 and 125 μm (micrometers) At the same time, the diameter of a basketball is about 24.2 centimeters, or $242,000\mu m$. Let's round this up slightly to $250,000\mu m$. The ratio of diameters is between 2000 and 4000. So, the ratio of volumes is between 2 billion and 64 billion. On the other hand, the sand would not pack in the basketball perfectly, so 20 billion seems like a fair estimate.

29R: The area of New York City as about 302 sq. miles So, the volume of the space above New York City, from ground level to 5 feet, is

$$302 \text{ mi}^2 \times (5280\text{ft/mi})^2 \times 5 \text{ ft} \approx 42,000,000,000\text{ft}^3$$

cubic feet. On the other hand, radius of a basketball is about .4 feet, giving it a volume of about $.25\text{ft}^3$. The volume ratio is 160 billion, so it seems reasonable that 100 billion basketballs would pack this space.

30L: Even the biggest surfboard is less than 3 meters tall and 1 meter wide. I figure that you could bury a man and his surfboard in a casket which was 3 meters by 1 meter by .5 meter casket. The volume of such a casket would be $1.5 \times 10^{18}\mu m^3$. On the other hand, it is reasonable for a grain of very fine sand to have radius about $50\mu m$, giving it a volume of

$$\frac{4\pi}{3}(60)^3 \approx 9 \times 10^5 \mu m^3.$$

The volume ratio is about 1.66 trillion. Given that the sand would not pack perfectly, about a trillion grains of sand would pack the casket.

31: There are $26^9 = 5,249,503,678,976$ different 9-letter combinations. This is between 5 trillion and 6 trillion.

32: The words "axaxismlo" and "mlomlomlo" are references to J. L. Borges story, *The Library of Babylon.* One of the book in Borges' library has the enigmatic words "axaxaxis mlo".

37R: The surface area of the earth is about 5 quadrillion square feet. A personal exercise trampoline has about 4 square feet of surface area, giving an area ratio of just over a quadrillion.

38: A strip of coastline 10 miles long and 300 feet deep and 3 feet high has a volume of

$$52800 \text{ ft} \times 300 \text{ ft} \times 3\text{ft} \times (304800\ \mu\text{m/ft})^3 = 1.34 \times 10^{24} \mu m^3.$$

The volume of a grain of very fine sand is about $9 \times 10^5 \mu m^3$. The volume ratio is $1,5$ quintillion.

41L: There isn't much news in my hometown of Barrington, RI. Two of the biggest stories in the past few years have been the capture of the bearded bandit, a man who robbed two banks in Barrington, and the fierce debates over the installation of a wind turbine in our town.

43L: There are n^{n-2} distinct labelled trees with n vertices. When $n = 20$, this comes to $20^{18} = 262144 \times 10^{18}$ ways.

44R: One proof I know of the fact about trees uses Prufer codes, and can be found in any number of books on graph theory. Douglas West's book on graph theory is one source. This would be quite a difficult problem for a kid.

48R: The volume of the earth is 1.083×10^{12} km^3, or approximately 10^{27}cm^3. A reasonable estimate on the volume of the average person is 20 liters, or 2×10^4cm^3. The volume ratio is 5×10^{22}. So, 6.02×10^{23} people would fill about 15 copies of the earth, accounting for some space between them.

53: The number of atoms in the Earth: This is a pretty rough estimate. I used Avogadro's number for this. The mass of the earth is about 6×10^{24}kg and every 12 grams accounts for about 6×10^{23} atoms. So, the mass of the earth can be divided into 5×10^{26} units of 12 grams each, and this gives

$$(5 \times 10^{26}) \times (6 \times 10^{23}) = 3 \times 10^{50}$$

atoms. This isn't too far off from my estimate of 10^{50}.

54R: 48 factorial is about 1.24×10^{61}, which is certainly more than 10^{50}.

55: It is always possible to connect up any finite number of dots in the plane without crossing, no matter how the dots are arranged. This isn't really a counting problem so much as a topology problem, but I like the problem.

56: I looked up the fact that the observable universe has about 10^{80} atoms in it. The source is the Observable Universe webpage.

57R: There is a mistake on this page. I wrote "quadrillion" but meant to write "quintillion". In a certain sense, there is also a second mistake. Yuguang Wu pointed out that while the observable universe has about 10^{80} atoms in it there is a lot of empty space between the atoms. So, depending on what the word *fill* means, it would be possible to fill the observable universe with more than 10^{80} atoms. Wikipedia lists the radius of the universe as about $.5 \times 10^{27}$ meters, and the atomic radius of an isolated neutral atom as about 10^{-10} meters. This means that the ratio of the volume of an atom to the volume of the observable universe is about 10^{110}. So, if the atoms were packed extremely tightly, you could fit about 10^{110} of them in the observable universe. This is somewhat more than a googol.

68: If you have an alphabet with a 10^{100} letters, and you can write a total of $10^{18} \times 10^{80} = 10^{98}$ letters, than the number of possibilities is

$$\left(10^{100}\right)^{10^{98}} = 10^{100 \times 10^{98}} = 10^{10^2 \times 10^{98}} = 10^{10^{2+98}} = 10^{10^{100}},$$

which is a googol plex.

69: This is the same kind of calculation as the one done on p 68.

75: This is the page about comparing the iterated powers. I'll give a solution at the end of these notes.

81: I wanted to go lightly on the theory in the book, presenting the idea of recursively defined functions in an informal and idiosyncratic way. On the other hand, I wanted to somehow allude to the fact that this is a well-developed subject, with many ideas and references. So, the bookshelf behind the narrator had books about the Ackerman function computability, Goodstein sequences, Turing machines, etc.

84-86: The symbols I am defining, taken together, are more or less equivalent to what is known as the Ackerman function.

Iterated Powers:
Here is a solution to the iterated exponential problem in my book. Let $P(a,n)$ be a tower of n powers of a. So, for instance, $P(5,3) = 5^{5^5}$. I'll derive the following inequalities.

$$P(1000,3) < P(10,4) < P(5,5) < P(2,7). \tag{1}$$

The first two inequalities are pretty crude. The third one seems delicate.

We will use some notation below which is rather tedious to define in complete generality. Here we illustrate what the notation means by example.

$$3,(7),3, \qquad 3,3,(7),3 \qquad 3,3,(7),3,3$$

mean, respectively,

$$3^{(7\times 3)}, \qquad 3^{3^{(7\times 3)}}, \qquad 3^{3^{(7\times 3^3)}}.$$

Lemma 0.1 *If $A > a > 1$ and $A \leq a^{a-1}$, then*

$$(a),A,A,...,A \leq a,(a),A...,A. \tag{2}$$

The two sequences are meant to have the same length.

Proof: We will illustrate the idea of the proof with a concrete example.

$$(a),A,A,A \leq$$
$$(a),a^{a-1},A,A =$$
$$a \times (a^{a-1})^{A^A} =$$
$$a \times (a^{A^A})^{a-1} \leq^*$$
$$(a^{A^A}) \times (a^{A^A})^{a-1} =$$
$$(a^{A^A})^a =$$
$$(a^a)^{A^A} =$$
$$a^{a\times A^A} =$$
$$a,(a),A,A.$$

The only place where the general case is different is the starred inequality, which in general uses $a \leq a$ or $a \leq a^A$, etc. ♠

Lemma 0.2 *If $A > a > 1$ and $A \leq a^{a-1}$, then $P(A,n) < P(a,n+1)$.*

Proof: We apply Lemma 0.1 iteratively: $P(A,n) < (a)P(A,n) =$

$$(a), A, ..., A \leq a, (a), A, ..., A \leq ... \leq a, a, ..., a, (a) = P(a, n+1).$$

♠

The first two inequalities are easy from Lemma 0.2. For the first one take $(a, A) = (10, 1000)$ and for the second one take $(a, A) = (5, 10)$. The third case is trickier. Let's tweak Lemma 0.1.

Lemma 0.3 *Suppose $A \geq 4$, $a \geq 2$, $A > a$, $\epsilon \in [-1, 1]$, and $A \leq a^{a+\epsilon}$, then*

$$(a+\epsilon+1), A, A, ..., A \leq a, (a+\epsilon+1/2), A..., A. \tag{3}$$

as long as there is at least one A on both sides of the inequality.

Proof: Set $b = a + \epsilon + 1/2$ and $c = a + \epsilon + 1$. We will illustrate the idea of the proof with a (tighter) concrete example.

$$(c), A, A \leq (c), a^{c-1}, A = c \times (a^{c-1})^A = c \times (a^A)^{c-1} \leq^*$$
$$\sqrt{a^A} \times (a^A)^{c-1} = (a^A)^{c-1/2} = (a^A)^b = a^{b \times A} = a, (b), A.$$

This time the starred inequality only works when there is an A on the right hand side of the inequality. The extreme case is $a + 2 \leq \sqrt{a^A}$ with $a = 2$ and $A = 4$. ♠

Now let's take $A = 5$ and $a = 2$ and $\epsilon = 1/2$. We have $5 < 2^{5/2}$, so Lemma 0.3 applies. We have

$$P(5,5) < (7/2), 5, 5, 5, 5, 5 < 2, (3), 5, 5, 5, 5 <$$
$$2, (7/2), 5, 5, 5, 5 < 2, 2, (3), 5, 5, 5 <$$
$$2, 2, (7/2), 5, 5, 5 < 2, 2, 2, (3), 5, 5 <$$
$$2, 2, 2, (7/2), 5, 5 < 2, 2, 2, 2, (3), 5 =$$
$$2, 2, 2, 2, 15 < 2, 2, 2, 2, 16 = P(2, 7).$$

This completes the proof.

Remarks: (i) Same argument shows $P(5, n) < P(2, n+2)$ for all n.
(ii) The inequality seems delicate, because $P(2, 7) < 2, 2, 4, 6, 6$.