

Teaching School Mathematics: Pre-Algebra

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Index

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Symbols

N (whole numbers) 3
Q (rational numbers) 146
R (real numbers) 96
Z (integers) 146
 a^n (a to the n -th power) 224
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 $n!$ (n factorial) 55
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