Problem Sets for
Mathematical Circle Diaries, Year 2
Complete Curriculum for Grades 6 to 8.

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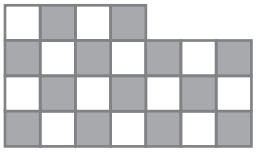
Below you will find complete problems sets from “Mathematical Circle Diaries, Year 2”.

This resource, which is available for free download, comes as a courtesy of the AMS and MSRI. Use it to make your lessons easier to prepare and more fun to teach.

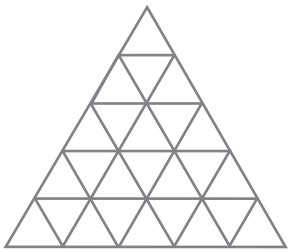
Included is the entire collection of problem sets except for Olympiads and other competitions.

Problem 1. The chess queen baked a cake that looks like a 4×6 chessboard. The chess king sneaked in and ate a 1×3 piece. The queen was left with the cake that is shown in the picture. However, the queen managed to cut the cake into 3 rectangular pieces, which she rearranged into a 5×5 square in such a way that the chess pattern was preserved. How did she do it? (All cuts should follow grid lines. Cake pieces can be rotated but cannot be flipped over.)



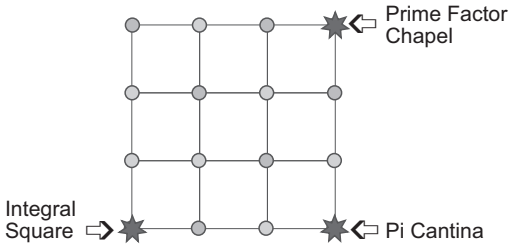
Problem 2. After a successful raid, pirates Archie, Billie, Cindy, and Daniel divided their loot of 70 gold coins. Each pirate received at least 1 coin. Archie got more coins than anyone else. Bill and Cindy together received 45 coins. How big was Daniel's share?

Problem 3. The Triangular Castle (see the picture below) has triangular rooms and a door in each of its interior walls. A tour guide wants to trace a route through this castle that would not visit into any room more than once. What is the maximum number of rooms one can visit on such a tour? (A tour can start and end in any two rooms.)



Problem 4. The difference between 2 numbers is equal to one-half of their sum. What is the ratio of the first to the second number?

Problem 5. Downtown MathHattan has 4 streets going east-west, and 4 avenues going north-south, with a small plaza at each intersection. All roads can be walked in both directions.



(a) Can you walk from the Integral Square to the Pi Cantina, visiting each plaza exactly once?

(b) Can you walk from the Integral Square to the Prime Factor Chapel, visiting each plaza exactly once?

For each part, either present the route or explain why it doesn't exist.

Problem 6. Mischievous Mike has changed two digits in Masha's homework problem, and her exercise now reads:

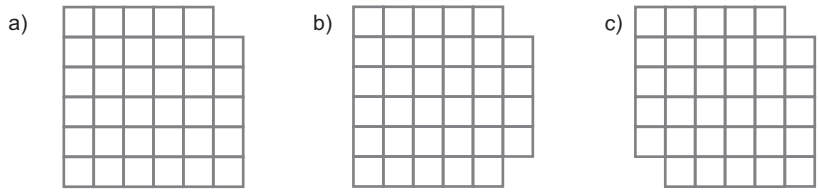
$$4 \times 5 \times 4 \times 5 \times 4 = 2247$$

Help Masha to find the changed digits and to restore the equality.

Problem 7. There are 10 Rainbow Dragons living on the slopes of the Spring Mountain; each dragon has a specific color (red, blue, yellow, etc.) It is known that if you pick any 4 dragons, 2 of them will share a color. Prove that at least 4 of those dragons are of the same color.

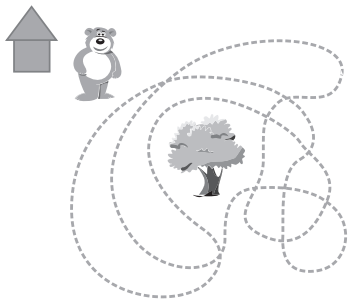
For Teachers: Remember that the horizontal divider indicates a section with more difficult problems. You may decide to skip these problems when working with a younger group.

Problem 1. You start off with a 6×6 chessboard. In part (a), one corner of it is missing. In parts (b) and (c), each chessboard is missing 2 corners. Which of these boards can you tile with dominoes? (A domino is a 2×1 tile. "To tile" means to cover completely and without overlaps.)



Problem 2. 25 beetles sit on 25 squares of a 5×5 chessboard. On a blow of a whistle, each beetle relocates to an adjacent square (a square above, or below, or to the right, or to the left). Prove that at least one of the squares will end up being empty.

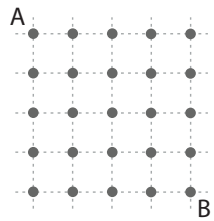
Problem 3. Little Bear came out of his house and found strange footprints going around his favorite tree (see picture). He walked around the tree several times, studying the tracks and sometimes jumping over them to get a better look. In the end, the bear returned to the house. Could he have jumped over the mystery footprints 7 times? (Little Bear never jumps over the places where the track intersects itself.)



Problem 4. Someone has put a rectangular playing card face-down on the table. You can flip the card over one of its edges as many times as you want. Can you end up with the card sitting face-up on the same spot?

Problem 1. Connect points A and B with a path made of 4 segments so that:

- all 4 segments have the same length,
- neighboring segments are not collinear,
- each segment starts and ends at a marked point, but does not pass through any other marked point.



Problem 2. The company "Wheeler Dealer" has a board of directors that consists of the CEO and three vice presidents. The board meets once a month and votes on new salaries for all board members. The list of salaries is prepared by the CEO, who herself does not vote. Each vice president is greedy and votes for the proposal only if his or her own salary goes up. The new list of salaries is approved only if the majority votes for it.

Is it possible for the CEO to come up with a way to grow her salary 10 times in one year, while the vice president's salaries would become 10 times smaller?

Problem 3. A grasshopper jumps along the road, right or left. He jumps 1 inch, then he jumps 2 inches and so forth; his last jump was 10 inches long. Could the grasshopper, after making these 10 jumps, land exactly where he started?

Problem 4. Karin is pretty sure that if an area of one rectangle is bigger than the area of another one, and if the perimeter of the first one is bigger than the perimeter of the second one, then the first one would definitely fit into the second one. Is she right?

Problem 5. There are 10 red, 8 blue, 8 green, and 4 yellow pencils in a box.

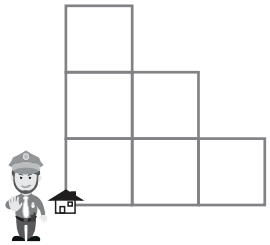
(a) What is the biggest number of pencils you should take, without looking, so that you leave at least 6 blue pencils?

(b) What is the biggest number of pencils you should take, without looking, so that you leave at least 1 pencil of each color?

(c) What is the smallest number of pencils you should take, without looking, so that you leave not more than 6 blue pencils?

Problem 6. Fred Allstar's car is equipped with a high-precision cruise control. Fred was driving his car at 55 mph when he resolved to test it. He decided that he would change his speed by 1 mph after each minute, either accelerating or slowing down. Can Fred come to a complete stop after 100 minutes of driving in this fashion?

Problem 7. A policeman lives in the corner house of a town. (See the town map below.) Every night he has to patrol the streets of his hometown. The policeman wants to plan his patrol route so that he walks every street (maybe more than once) and returns home. What is the length of the shortest possible route?



Problem 8. There are 8 jars of strawberry jam, 7 jars of apricot jam and 5 jars of sour cherry jam in the pantry. What is the biggest number of jars you could take without looking in such a way as to leave at least 4 jars of jam of one kind, and at least 3 jars of another kind?

Problem 9. Six monks entered a temple leaving their shoes by the doors. (The shoe sizes of all monks were different.) The monks were leaving the temple one by one, at night. Some of them, instead of putting their own shoes on, were putting bigger shoes on. What was the greatest possible number of monks that had to leave barefoot? (A monk would leave barefoot if all the shoes that are left are too small.)

Problem 1. You have three boxes of chocolates. The first box has 6 chocolates less than the second and the third together, and the second box has 10 less than the first and the third together. How many chocolates are there in the third box?

Problem 2. You start with 3 piles of candy, with 2, 3, and 4 pieces. In a single turn, you can choose two piles and then add 1 piece of candy to both. By repeating this operation a number of times, could you end up with 3 piles that contain 200, 300, and 400 pieces candy?

Problem 3. A broken calculator can only do several operations: multiply by 2, divide by 2, multiply by 3, divide by 3, multiply by 5, and divide by 5. Using this calculator any number of times, could you start with the number 12 and end up with 49?

Problem 4. The numbers 1 through 12 are written on a board. You can erase any two of these numbers (call them a and b) and replace them with the number $a+b-1$. After 11 such operations, there will be just one number left. What could this number be?

Problem 5. Thee are 6 trees grow along Park Lane, and a bird is sitting in each of them. Every time a car passes, exactly 2 birds fly up and go to the next tree in either direction. Can all birds gather in the same tree?

Problem 6. If a magician puts 1 dove into his hat, he pulls out 2 rabbits and 2 flowers from it. If the magician puts 1 rabbit in, he pulls out 2 flowers and 2 doves. If he puts 1 flower in, he pulls out 1 rabbit and 3 doves. The magician starts with 1 rabbit. Could he end up with the same number of rabbits, doves, and flowers after performing his hat trick several times?

Problem 7. Two identical cups—one filled with coffee, and another with milk, have been placed on the table. You take a spoonful of milk from the milk cup and put it into the coffee cup. Next, you take a spoonful of the mixture from the coffee cup and put it into the milk cup. Do you have more milk in the coffee cup now or more coffee in the milk cup?

Problem 1. Recently, 10 rich guys from Seattle married 10 rich girls from Portland. The total net worth of each new family was 10 million dollars. Prove that if the girls' total worth was less than 70 million, then the guys' total worth was at least 30 million.

Problem 2. Twenty-five extraterrestrials from 8 different planets came to the Intergalactic Congress. Prove that at least 4 of them are from the same planet.

Problem 3. There are 5 magicians standing in a row. Altogether, these 5 magicians know 300 different powerful spells. Prove that there are two magicians next to each other who, combined, know at least 100 powerful spells.

Problem 4. Romeo and Juliet quarreled. Juliet started walking east, and Romeo – west. In 4 minutes, Romeo turned back and started running after Juliet. In two minutes, he reaches the spot where they parted in 2 minutes. If Romeo runs 3 times as fast as Juliet walks, how long will it take him to reach Juliet from the moment he turned back?

Problem 5. A 5×9 rectangle is cut into 10 smaller rectangles. (All cuts follow gridlines.)

(a) Prove that at least 2 of these rectangles have the same area.

(b) Prove that at least 2 of these rectangles are congruent.

Problem 6. There are 31 gnomes, 31 elves, and 30 humans are standing side by side in line. It is known that no elf stands next to a gnome. Prove that at least 3 gnomes or at least 3 elves are standing next to each other.

Problem 7. The game of Trick-a-Troll is played with 10 players and a deck of 20 cards: 2 through 10 and an ace of spades, and 2 through 10 and an ace of clubs. Each player gets 1 club and 1 spade, and adds their cards (aces count as 1). Prove that there will be at least 2 players with sums that end in the same digit.

Problem 8. Each square of an infinite grid is occupied by one of four creatures: a frog, a bunny, a hamster, or a mouse. It is known that every 2×2 square contains all four creatures. Prove that is it possible to choose a row or a column that contains just two types of creatures.

Problem 1. My house number has five digits, and the sum of these digits is equal to my friend Winnie's age. When I told this to Winnie, he immediately figured out what my house number is. How old is my friend Winnie?

Problem 2. $R + RR = BOW$. What is the last digit of

$$F \times A \times I \times N \times T \times I \times N \times G\ ?$$

(Identical letters stand for the same digits; different letters stand for different digits.)

Problem 3. How many 9-digit numbers are there such that going from left to right the digits of this number get smaller?

Problem 4. Shmerlin the magician found the door to the Cave of Wisdom. The door is guarded by Drago the Math Dragon, and also locked with a 3-digit lock. Drago agrees to put Shmerlin to the test: Shmerlin has to choose three integer numbers, x, y, and z, and the dragon will tell him the value of $A \times x + B \times y + C \times z$, where A, B, and C are the three secret digits that open the lock. After that, Shmerlin should come up with a guess of the secret digits. If the guess is correct, Drago will let the magician into the cave. Otherwise, Shmerlin will perish. Does Shmerlin have a way to succeed?

Problem 5. A six-digit number starts from the digit 1. If this digit is relocated to the rightmost position, the number becomes 3 times bigger. What is the number?

Problem 6. Two 1×1 squares are marked in a 10×10 grid. Prove that one can always cut the grid into two identical pieces such that the two marked squares would belong to different pieces. The cuts have to follow the grid lines.

Problem 7. Solve: $HE \times HE = SHE$

(Identical letters stand for the same digits; different letters stand for different digits.)

Problem 1. Olivia, a straight-A student, wrote a number that is composed of all the integer numbers from 1 to 500: $1234567\ldots10111213\ldots499500$. Her friend, Mischievous Mike, erased the first 500 digits from this number. What is the first digit of the resulting number?

Problem 2.

(a) Find the biggest 6-digit integer number such that each digit, except for the two on the left, is equal to the sum of its two left neighbors.

(b) Find the biggest integer number such that each digit, except for the first two, is equal to the sum of its two left neighbors. (Compared to part (a), we removed the "6-digit number" restriction.)

Problem 3. In the 100-digit number $12345678901234\ldots7890$, Ben crosses out all digits that are located at the odd-numbered positions. In the 50-digit number that is left he crosses out all digits that are located at the odd-numbered positions. He keeps doing so until he ends up with a single-digit number. What is this number?

Problem 4. The number x is a three-digit number, and y is the same number written backward. Could it be possible that the sum of x and y is a number that is written using odd digits only?

Problem 5. Which 500 consecutive positive integer numbers do you need to write down to get 2006 digits?

Problem 1. Convert the following numbers from binary (base-2) to decimal (base-10):

(a) 10_2
(b) 101_2
(c) 111_2
(d) 1000_2
(e) 1101_2
(f) 10000000_2
(g) 1111111_2

Problem 2. Convert from decimal to binary:

(a) 3
(b) 8
(c) 15
(d) 32
(e) 31
(f) 40
(g) 53

Problem 3.

(a) Which binary number is 1 bigger than 100000000_2? Which is 1 smaller?

(b) Which binary number is 1 smaller than 1111111111_2? Which is 2 smaller? Which is 2 bigger?

Problem 4.

(a) How can one tell if a binary number is even or odd?

(b) How can one tell if a binary number is divisible by 4?

(c) If a binary number ends on 10, what is the remainder when this number is divided by 4?

Problem 5. Joe the Gold Miner has thirty golden nuggets worth $1, 2, ..., 30$ dollars. Which five of these nuggets should he take with him if he wants to be able to pay any integer amount ranging from 1 dollar to 30 dollars?

Problem 6. Julia is buying a new laptop. She pays \$260 of her own money, and her dad, sister, and brother are helping her with the rest of the sum. Her sister gave her $1/8$ of the price of the laptop. Her brother gave her $1/4$ of the price, and her dad gave her $1/2$ of the price of the laptop. What is the price of the new laptop?

Problem 7. You have a sack of flour, a 1 gram weight and a box of light plastic bags. Can you measure 1000 grams of flour in 10 weighings on a balance scale?

Problem 8. Five pirates – Archie the Captain, Bob the First Mate, Chad the Gunner, Don the Rigger, and Eamon the Cook – got a chest of gold coins. Archie took half of all the coins and half of a coin; Bob received half of the remaining coins and half of a coin; Chad's share was half of the remaining coins and half of a coin, and Don got half of the remaining coins and half of a coin. Finally, when Eamon took half of what was left and half of a coin, there was no gold left. How many gold coins did the pirates originally have? (The pirates did not cut any coins in half.)

Problem 9. Merlin the Sorcerer has built his castle next to the dwelling place of the mighty Dragon Shmok. Every morning with the first rays of the rising sun the Dragon flies out of his lair, heading toward one of the four nearby towns, which are located to the north, south, east, and west of the castle. Merlin's task is to warn the townspeople where to expect the Shmok today.

The castle has three towers topped with magic spheres; one is made out of ruby, the other of emerald, and the third one is made out of sapphire. When the magic spheres are lit up, their radiance is seen from the farthest corners of the land. So it has been agreed that by these lights Marlin sends his signal, using some code to indicate where the Dragon is heading.

The spheres' lights, however, tend to flicker randomly at night, and the morning may find any combination of them turned on or off. And, because of his old age, Merlin will have the strength to climb only one tower to manipulate its light (to turn it on or off).

Thus, starting from a random combination, Merlin can change the state of no more than one sphere. After that, the townspeople should be able to read where the Dragon is heading by seeing which lights are on.

Can you help Merlin come up with such code?

Problem 1. Add these binary numbers using the column addition algorithm:

(a) $1000_2 + 101_2$
(b) $11011_2 + 111111_2$
(c) $1011100_2 + 10101_2$
(d) $11111011_2 + 111111111_2$

Problem 2.

(a) Double these binary numbers: 1001_2, 11111111101_2
(b) Write a binary number that is 4 times bigger than 111000111_2
(c) Write a binary number that is 64 times bigger than 111_2

Problem 3. Convert these numbers to binary. (Use the conversion algorithm of your choice.)

(a) 80
(b) 152
(c) 401
(d) 522
(e) 1023

Problem 4. The first water lily in the Lily Pond blossomed on June 1st. Since this day, the number of flowers has doubled every day. On June 15th, the flowers covered the pond completely. If two flowers (instead of one) started blooming on June 1st, and the number of flowers doubled every day, when would the pond become completely covered in flowers?

Problem 5. To get to the Stone of Wisdom, a young wizard must solve a puzzle. She is given a 7×7 square that she has to cut into 9 smaller rectangular pieces. Using these pieces, she should be able to tile any integer rectangle that fits into the 7×7 square: 1×1, 1×2, 1×3, ... , 1×7, 2×2, 2×3, ... 7×6, and 7×7. Solve this puzzle. (All cuts should follow grid lines. "To tile" means to cover without gaps and overlaps.)

Problem 6. Brother Fox and Brother Rabbit are planting beets and carrots in their vegetable patch. The patch has 20 rows, with spots for 7 plants in each row. They plant as follows: Brother Fox chooses the next seedling (beet or carrot), and Brother Rabbit chooses a free spot where to plant it. They keep doing this until they fill all the spots.

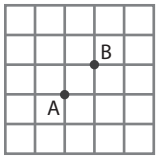
Brother Fox and Brother Rabbit agree to divide the crops as follows: Brother Fox can take the yield from as many rows as he wants, as long as all rows he claims to himself are planted according to different patterns.

For example, if 4 rows are planted as in the picture on the right, Brother Fox gets the yield from 3 rows. ("B" and "C" stand for "beet" and "carrot.")

B	B	B	B	B	B	C
C	B	B	B	B	B	B
B	B	B	B	B	B	C
B	C	B	C	B	C	B

What is the largest number of rows Brother Rabbit can always secure for himself no matter how smartly Brother Fox chooses what to plant?

Problem 7. In the picture, you can see the map of the Old City, with streets heading east-west and north-south. At each intersection of two streets, there is a small plaza. A tourist would like to take a walk from the train terminal (A) to his hotel (B). He would like to take the longest route possible without visiting any of the plazas twice. Draw such a route and prove that a longer route cannot be found.



Problem 1. Two brothers went mushroom-picking together. In the forest, they split up. There was a road going through the forest, and the brothers crossed it from time to time. By the end of the day, the older brother crossed the road 3 times as often as the younger brother. Did the brothers end up on the same side or different sides of the road?

Problem 2. Said the devil to a lazy man: "Every time you cross this enchanted bridge, your money will double. For this advice, you will have to pay me 40 coins every time you cross it." The lazy man crossed the bridge three times and ended up having no money at all. How many coins did he have before the deal?

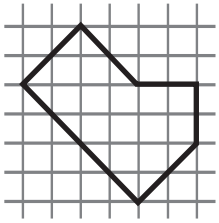
Problem 3. Emily had 24 dolls, 26 stuffed animals, and 25 cars. Eventually, she decided to give these toys to her younger sister Rachel. To make it more fun, she started to present Rachel with two toys at a time, once a week: a doll and a car, or a doll and an animal, or car and an animal. Whenever little Rachel received her gift of 2 toys, she would give Emily a toy in return. If she got a doll and a car, she would give back a stuffed animal; if she got a doll and an animal, she would give back a car; if she got a car and an animal, she would present a doll. (Initially, Rachel had enough toys to keep this exchange going.) Finally, Emily was left with a single toy.

(a) For how many days did this exchange take place?
(b) What was Emilie's last toy?

Problem 4. The cubic die has numbers from 1 to 6 written on its six faces. The first time Sasha tossed this die, the sum of the numbers displayed on the four side faces was equal to 12. The second time, it was equal to 15. What number is written on the face that is opposite to the face with number 3?

Problem 5. The road between the villages of Holy and Smoke has mileposts placed at every mile. One side of each post shows the distance to Holy and the other—the distance to Smoke. Peter noticed that the sum of all digits on each milepost is equal to 13. How far is Holy from Smoke?

Problem 1. Cut the figure on the picture into three equal parts. (You can cut along the grid lines and diagonals of small squares.)



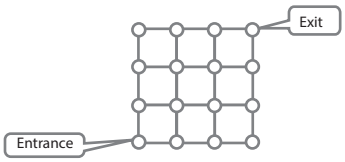
Problem 2. WizardLand Middle School offers a new elective this year: an astrology class. Fifteen students have registered for this course. Prove that at least 2 of these students were born under the same zodiac sign. (There are 12 zodiac signs in total.)

Problem 3. Eight knights took part in a 3-contest tournament. They competed in archery, sword fighting, and lance throwing. For each contest, a knight was awarded 0, 1 or 2 points. Prove that at least two of these knights earned the same total number of points.

Problem 4. In a group of 12 politicians, some are supporting each other, some are not. Prove that it is possible to find two politicians in this group who have the same number of supporters within the group. (Note that "to support" is a mutual relation: if A supports B, then B supports A.)

Problem 5. With a red marker, Margareta marked three points with integer coordinates on a number line. With a blue marker, Angelina marked a midpoint for every pair of red points. Prove that at least 1 of the blue points has an integer coordinate.

Problem 6. The halls of the Haunted Labyrinth can be walked right and upward only. Each room of the Labyrinth (marked by a circle) is inhabited by a magical creature—an elf, a leprechaun, or a fairy. If you pass through a room where an elf lives he'll give you 3 coins, if you pass through a fairy's room you'll get 6 coins, and a leprechaun will give you nothing. Twelve kids enter the labyrinth. Prove that at least two will exit on the other side with the same number of coins.



Problem 7. A team of three math circle students—Ashley, Betty, and Cindy—takes part in the "Best Logician" championship.

Here is their challenge:

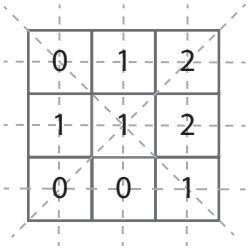
- They enter an entirely dark room with a box of 2 red, 2 yellow and 3 green hats. Each of them randomly chooses a hat, put it on, and step out

of the room. Now, each student can see the color of her friends' hats but is unable to see her hat.

- Next, Ashley will be asked to guess the hat color that she definitely ***is not*** wearing. She will have a choice to guess the color or to stay silent. After that, it will be Betty's turn, and then Cindy's. If the first guess is correct, the team wins; otherwise, the team loses.

When Ashley is asked about the color she does not have, she chooses to stay silent. Next, Betty decides to stay silent as well. Now, it is Cindy's turn. What should her reply be?

Problem 1. Alexander randomly filled a 3×3 table with numbers 0, 1, and 2 (one number per every cell). Then he counted all the row, column and diagonal sums in this table. Prove that at least two of these sums are equal.



Problem 2. The city of Seattle has more than five million inhabitants. Show that two of these people must have the same number of hairs on their heads if it is known that no person has more than one million hairs on his or her head.

Problem 3. Prove that out of any 11 numbers, 2 can be found such that their difference is a multiple of 10.

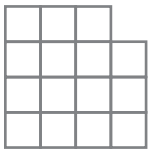
Problem 4. Today, 70 students took 3 tests each—in Math, English and Social Sciences. The tests are scored on a scale from 1 to 4. Is it true that in this group there always be two students whose results are identical? (Simplified version: 20 students and 2 tests.)

Problem 5. Alice took a red marker and marked 5 points with integer coordinates on a coordinate plane. Miriam took a blue marker and marked a midpoint for each pair of red points. Prove that at least 1 of the blue points has integer coordinates.

Problem 6. The 7th-grade math circle student Emilio wrote a computer program for his house robot, Basil. Starting from 1, Basil should keep writing bigger and bigger numbers formed by 1s: 1, 11, 111, etc.. The program terminates when Basil writes a number that is a multiple of 19. Prove that the program will terminate in fewer than 20 steps.

Problem 1. Place 6 positive numbers (not all equal) around the circle in such a way that every number is a product of its 2 neighbors.

Problem 2. Divide the figure into three parts of the same shape and size. (All cuts should follow gridlines.)



Problem 3. A 10-mile by 10-mile square of Martian land is inhabited by 100 settlers from Earth. Each settler owns a plot that has a shape of a 1-mile by 1-mile square. More than a quarter of these settlers come from Seattle. Prove that at least 2 settlers from Seattle are neighbors. (Neighbors are those settlers whose plots share a side or a vertex.)

Problem 4. Prove that if you pick 50 different numbers from the integers 1 to 98, then 2 of them must add up to 99.

Problem 5. Sam and Brendon are playing Battleship on an 8×8 grid. Sam hides a single 4×1 battleship somewhere on the board. Brendon can pick squares on the grid and fire upon them. What is the smallest number of shots Brendon has to fire to guarantee at least one hit on the battleship? (In Battleship, a *hit* means a shot fired to a square that belongs to a ship.)

Problem 6. There are 25 flies sitting on a table. It is known that out of any 3 of them, 2 can be found that are less than 1 foot apart. Prove that at least of 13 these flies can be swatted with a single strike of a fly swatter with a hoop of radius 1 foot.

Problem 7. Intelligent cacti from the planet Karellia are happy if they are planted at least 2 meters apart from each other. The Seattle zoo has an exhibit of Karellian cacti. The exhibit has a circular shape 20 meters in diameter, and 20 cacti have been planted there in such a way that they all are happy. Prove that is it possible to plant one more Karellian cactus in such a way that all cacti will be happy.

Problem 1. Cowboy Joe fired 15 shots into a 4×4 carpet and made 15 holes in it. Is it always possible to cut a 1x1 square rug out of the ruined carpet that has no holes in it? (Bullet holes are tiny, point-like, but they don't look good.)

Problem 2. A scorpion cannot tolerate another scorpion if they are less than 2 meters apart. How many angry scorpions can cohabit on a wireframe cube with side length 1 meters? (The scorpions measure the distance between two points along the edges of the cube.)

Problem 1. Mr. and Mrs. Jones have six kids – 3 boys and 3 girls. Today, a photographer is taking pictures of the family.

(a) In how many ways can the daughters be seated in a row for a photo shot?

(b) In how many ways can the kids be seated in a row so that all the girls are on the left and all the boys are on the right?

(c) In how many ways can Jones' kids be seated in a row so that girls and boys alternate?

(d) All the girls must be sitting together, all the boys must be sitting together as well, and parents must be either together in the center, or on both sides.

Problem 2. In how many ways can you match 40 wizards and 40 brooms in such a way that every wizard gets a broom?

Problem 3.

(a) A palindromic number is a number that reads the same backward and forward. (For example, 13531 is palindromic.) How many 5-digit numbers are palindromic?

(b) How many 5-digit numbers are palindromic and consist of distinct digits?

(c) How many 5-digit numbers consist of distinct digits and end on 2?

(d) How many 5-digit numbers are odd and consist of distinct digits?

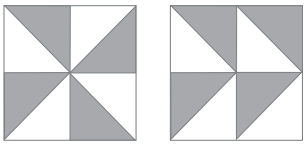
Problem 4. One Mathematician has a spouse and three kids, and all members of that family celebrate their birthdays on the same day. Once, the Mathematician said:

— When our first child was born, the sum of ages in my family was equal to 45. A year ago, when our third child was born, it was equal to 70. And now the sum of ages of our three kids is equal to 14.

Determine the ages of the Mathematician's kids.

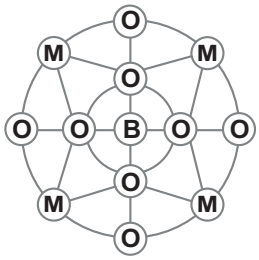
Problem 5.

(a) A 2×2 square is divided into four 1×1 squares. We would like to cover it with 8 right triangles (two triangles per square) that are either black or white. A "right" covering is the one where two neighboring triangles never have the same color. (Two examples of "right" coverings are presented below.) How many "right" coverings are there?



(b) Answer the same question for a 3×3 square.

Problem 1. In how many different ways can you trace the word BOOM on the drawing below?



Problem 2. An ogre has 25 prisoners in his dungeon.

(a) In how many ways can he choose 1 for lunch and 1 for dinner?

(b) The ogre promised his bride to free some of his 25 prisoners. In how many ways can he choose 2 prisoners to set them free?

Problem 3. In how many different ways can you trace the word BOOGIE in the table below? (You can move from a cell to a cell if they share a side or a corner.)

E	I	G	I	E
I	G	O	G	I
G	O	B	O	G
I	G	O	G	I
E	I	G	I	E

Problem 4. In how many ways can you arrange digits 1, 2, 3, 4, 5, 6, 7 into a 4-digit number so that

(a) All digits are different and the number starts with an even digit?

(b) All digits are different and the number ends on an odd digit?

Problem 5. You are in the robot-selling business on the Tau Ceti planet, where you own a small store. You just received a big shipment of house helper robots and nanny robots. You also have one medical robot and one supervisor robot – you would like to sell them as well. Your store window display has space for 6 robots. In how many different ways can you arrange this display if

(a) You plan to display household and nanny robots only?

(b) You plan to display five household and nanny robots, and the medical robot?

(c) You plan to display five household and nanny robots, and one robot of another type?

Problem 6. Mary, who wrote an online computer game, plans to assign every user a unique password.

(a) Her current plan is to have passwords that are 8 symbols long, and are all made from lowercase letters a, b, and c. How many different passwords can Mary generate?

(b) Her friend Brandon suggests the modification: to use 8-symbol passwords made from letters a, b, and c, with exactly one capital letter. How many times more passwords can Mary generate now?

(c) Finally, Mary decides that her passwords will be 9 symbols long, made from one digit (0 to 9), and from letters a, b, and c, with exactly one capital letter. How many times more passwords can she generate compared to the original plan?

Problem 1. Is it possible that a fraction with the numerator smaller than denominator is equal to another fraction with the numerator greater than denominator?

Problem 2. Tom's dad built a 9 board-long fence, which Tom's mother painted white. Tom, who has 3 different cans of paint—red, green and blue—would like to decorate the fence.

(a) If he paints every second board (boards 2, 4, ...), in how many ways can he do it?

(b) If he paints every second board, and if exactly one of the boards should be red, in how many ways can he do it?

(c) If he paints every board, if exactly three boards should be red, and if the fence should be symmetrical, in how many ways can he do it?

Problem 3. The SunnyHill Middle School girls' gymnastics team has 20 students, with Masha and Sasha being the only 7th-graders. How many different ways are there to choose 8 girls to participate in a meet if at least one of the 7th-graders has to be included?

Problem 4. Someone chose 10 points on a plane such that no three points are on the same line. How many triangles with vertices in these points are there?

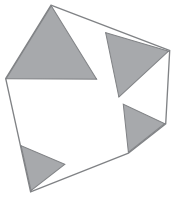
Problem 5.

(a) In how many different ways can 13 girls be seated in a row?

(b) In how many different ways can 13 girls be seated in a row if 3 of these girls - Ashley, Betty, and Cindy - would like to sit next to each other? (They could sit next to each other in any order.)

(c) In how many different ways can 13 girls be seated in a row if 2 of these girls—Masha and Dasha—do not want to sit next to each other?

Problem 6. Several equilateral triangles have been made out of wood and fixed on a wall. We then fitted a piece of string around them. (See picture.) Prove that if the string forms a triangle, then that triangle is equilateral.



Problem 1. Solve the encrypted problem: $(AA + AA + 1) \times A = AAA$. (The same letters stand for same digits.)

Problem 2. Can you come up with two convex quadrilaterals such that: the first one is located inside the second one, and the sum of the diagonals of the first one is bigger than the sum of the diagonals of the second one? (A quadrilateral is *convex* if all of its angles are less than 180°.)

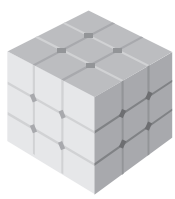
Problem 3. Two travelers started from A to B on foot. The first traveler walked half of the distance at 4 miles/hour, and the second half at 6 miles/hour. The second traveler walked half of the time at 4 miles/hour, and the rest of the time at 6 miles/hour. What was the average speed of both travelers? Who will arrive at B first? (The average speed is calculated as total distance divided by the total time used for travel.)

Problem 4. A row of numbers 1, 2, . . . , 100 is written on the board. Each minute, a professor erases 3 numbers next to each other and replaces them with numbers that are one bigger. Can the professor make all these numbers to be equal to 1,000?

Problem 5. Roberto has 13 foam bricks of size 1×1×2.

(a) Can he glue them into a solid 3×3×3 cube without a 1×1×1 corner?

(b) Can he glue them into a 3×3×3 cube with a 1×1×1 hole in the middle?



Problem 6. Barons, dukes, and counts at Prince Lemon's court started dueling each other. Each duel resulted in a fatality, and, strangely enough, barons would only kill dukes, dukes would only kill counts, and counts killed only barons. It is also known that no one won in more than one duel. In the end, only Baron Orange was left alive and victorious. What was the title of the first victim if originally Prince Lemon had 100 courtiers?

Problem 7. Several jewelry boxes contain 2000 pearls altogether. Prove that you can remove some pearls and some boxes so that all remaining boxes contain the same number of pearls, and the total number of remaining pearls is at least 100.

Problem 1. Three karate masters come to the martial art school where 20 students are apprenticing in karate. The school teacher must choose 3 students to pair with masters for individual sessions. In how many ways can these 3 pairs be chosen?

Problem 2. There are 20 kids and 3 counselors in Camp Colman, located on the left bank of the river. In how many ways can they choose a group of 10 to visit Camp Orka, which is located on the other bank if at least one counselor should stay in Camp Colman?

Problem 3. We would like to count 6-digit numbers with at least 1 even digit. How many such numbers are there?

Problem 4. In how many ways can you arrange the digits 1, 2, 3, 4, 5 into a 5-digit number such that:

(a) All the even digits are placed next to each other.

(b) The even digits are separated from each other by at least 1 odd digit.

Problem 5. In a $1,000 \times 1,000$ grid-aligned square, several border-to-border lines have been traced along the grid lines. The rectangles of the resulting pattern have been colored in checkerboard order. Prove that the total area of all black rectangles is an even number.

Problem 6. Several segments have been marked on a line. The left half of each segment have been colored red, the right half – blue. It turns out that all the left halves form one continuous red segment, and all the right halves form one continuous blue segment. The red segment is 20 cm longer than the blue one. Prove that out of the original segments it is possible to find two such that one will be at least 40 cm longer than the other.

Problem 1. Strangestun Air Force has 10 spy planes.

(a) On Monday, 4 of these 10 planes should fly a secret mission to Fingalia. In how many ways can these 4 planes be chosen?

(b) On Tuesday, 4 of these 10 planes should fly a secret mission to Tartaria, 4 planes – to Rosalia, and 2 planes - to Santinia. In how many ways can these planes be chosen and assigned to the missions?

Problem 2.

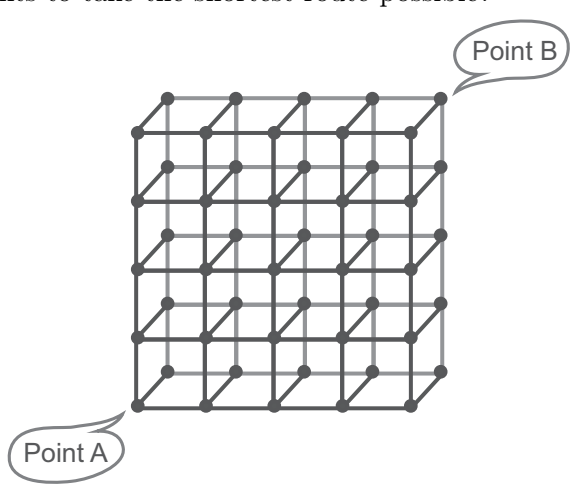
(a) How many different patterns can you make from 10 precious stones – 4 sapphires and 6 rubies? (A pattern is a distinct sequence of stones.)

(b) How many different patterns can you make from 4 sapphires, 4 rubies, and 2 emeralds?

(c) How many different patterns can you make from 4 sapphires, 3 emeralds, 4 rubies, and 1 onyx?

Problem 3. Ashley, Betty, and Cindy were working on a set of problems. To add some zest, they agreed that the kid who is the first to solve a problem receives 4 pieces of candy, the second child receives 2 pieces of candy, and the third kid receives 1 candy. As a result, each child solved all problems, and there were no ties. The kids claim that each of them received 20 candies. Prove that they are mistaken.

Problem 4. Tim the Ant lives on a 3D cubical grid (lattice) with dimensions $4 \times 4 \times 1$ (see picture). In how many ways can Tim crawl from the low left front corner (Point A) to the upper right back corner (Point B) of the lattice if he wants to take the shortest route possible?



Problem 5.

(a) A coin is tossed 20 times. How many different outputs (sequences of heads and tails) can you get?

(b) A hoppy rook stands at the leftmost square of a 20-squares-long strip. In a single hop, the rook can jump to the right over any number of squares. In how many different ways can the rook hop to the last square of

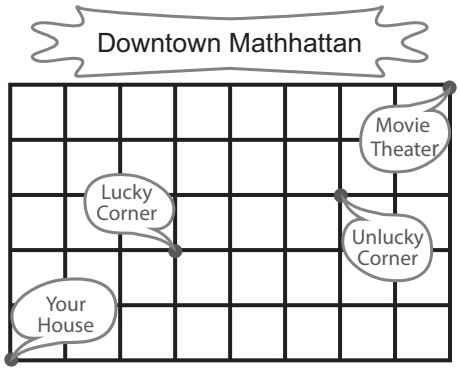
Problem 1. Downtown MathHattan has a grid pattern – it has 6 east-west streets and 9 north-south streets. All streets are one-way: a car can go north or east.

(a) How many different routes are there from your house to the movie theater?

(b) How many different routes are there from your house to the movie theater that pass through the Lucky Corner?

(c) How many different routes are there from your house to the movie theater that avoid Unlucky Corner?

(d) How many different routes are there from your house to the movie theater that pass through the Lucky Corner and avoid Unlucky Corner?



Problem 2.

(a) In how many ways can you place 7 different coins into 3 pockets?

(b) In how many ways can you place 7 identical coins into 3 pockets?

the strip? (Examples: the rook can get there in one 19-square-long hop, or in one 2-square-long hop followed by one 15-squares hop, and by one 2-squares hop, and so on ...)

Problem 6. Let us call a counting number "cute" if it is smaller than any other counting number with the same sum of digits. What is the 200th cute number?

0.1. Take-Home Problem Set

Problem 1. Can a 3×3 magic square be constructed from the first 9 prime numbers? Either present such a square or explain why such a square is impossible.

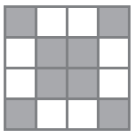
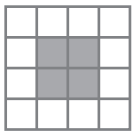
Problem 2. Prove that in any 3×3 magic square the sum of the numbers in the corner cells is equal to the sum of the numbers in the side cells. (In the picture, the side cells are marked by dots, and the corner cells – by crosses.)

x	o	x
o		o
x	o	x

Problem 3. Prove that in a 4×4 magic square

(a) The sum of the numbers in the 4 central and 4 corner cells is equal to twice the magic constant of this square. (In the picture on the left, these cells are shaded.)

(b) The sum of the numbers in the central 2×2 square is equal to the magic constant of this square. (In the picture on the right, these cells are shaded.)

a)  b) 

Problem 4. Ten daughters of King Dodon wanted to learn new skills. They enrolled in the "Perfect Princess" Academy where they signed up for three classes—cooking, poetry, and martial arts. Each princess either passed or failed each of these classes. Prove that at least two princesses learned the same set of skills.

Problem 5.

(a) What angle does the minute hand of a clock swipe in a minute?

(b) What is the angle between the hour and minute hands at 5 minutes past midnight?

(c) What is the first time after midnight when the hour and minute hands will be pointing in opposite directions?

(d) What is the first time after midnight when the hour and minute hands will be pointing in the same direction?

Problem 6. Knights of the Round Table were eating pies with raisins. Merlin, the Magician, made sure that each knight ate either twice as many raisins, or 10 raisins less than his right neighbor. Prove that the knights could not have eaten exactly 1001 raisins.

Problem 7. The 4×4 magic square was filled with numbers from 1 to 16. Some of these numbers were erased. Restore the square.

4	15		5
		8	
7			

Problem 1. Can you decorate an 8×8 cake with chocolate roses in such a way that any 2×2 piece would have exactly 2 roses on it, and any 3×1 piece would have exactly one rose? Either draw such a cake or explain why this is not possible.

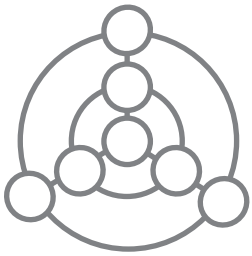
Problem 2. All the pirates of the Black Pearl ship wear tricorn hats. Each hat is decorated with gold tassels: one tassel at the front corner, and two and three tassels at each of the two back corners. Every evening, the pirates stack their hats neatly into a three-corner tower. (The front corners of the hats can, possibly, face different directions.) One night, when Captain Barbarossa was unable to sleep, he added up all the tassels for every corner of the hat tower. There were 25 tassels total at each corner. After a short consideration, the captain figured out that at least one tassel was lost. Is he correct?

Problem 3.

(a) Arrange all 7 integer numbers from 1 to 7 in such a way that the 3 numbers on each of the 3 lines would add up to 12 and the 3 numbers on each of the 2 concentric circles would add up to 12 as well.

(b) Could you find another solution such that the number in the center is different?

(c) Would you be able to solve this puzzle for a number other than 12?



Problem 4. Sixty people were riding a tram. Out of this group, there were several conductors and several fake conductors (people who were impersonating conductors), several inspectors and several fake inspectors, and perhaps some regular passengers as well. The total number of impersonators (fake conductors and fake inspectors) was 4 times smaller than the total number of real conductors and inspectors. The total number of inspectors (real and fake ones) was 7 times greater than the total number of conductors (real and fake ones). How many of the tram riders were regular passengers?

Problem 5. Several films were nominated for the "Best Math Movie" award. Each of the 10 judges secretly picked the top movie of their choice. It is known that out of any 4 judges, at least 2 voted for the same film. Prove that there exists a film that was picked by at least 4 judges.

Problem 6. An 8×8 chessboard has 30 diagonals total (15 in each direction). Is it possible to place several chess pieces on this chessboard in such a way that the total number of pieces on each diagonal would be odd? (Hint: the black/white coloring of the board helps solve this problem.)

Problem 1. Francesca, Isabella, and Lorenzo played chess together. Each child played 10 rounds.

(a) What was the total number of rounds?

(b) Is it possible that Lorenzo played more rounds with Isabella than with Francesca?

Problem 2. Thirteen happy toddlers came to the Sunny Bunny Childcare this morning. Every toddler brought in three toys: a doll, a stuffed animal, and a car. From time to time, a pair of toddlers exchanged a toy with each other. Could it happen that each toddler went home with all toys of the same kind: either all dolls, all animals, or all cars?

Problem 3. There are 36 warrior tomcats standing in a 6×6 square formation. Each cat has several daggers strapped to his belt. Is it possible for the total number of daggers in each row to be more than 50 and for the total number of daggers in every column to be less than 50?

Problem 4. Farmer John is very proud of his pigs.

(a) John has recently noticed that out of his 8 best pigs each 4, put together, weigh at least 1,000 pounds. Prove that all 8 pigs weigh at least 2,000 pounds

(b) John has sold some pigs, and he now has only 6 out of which each 4, put together, weigh at least 1,000 pounds. Prove that all 6 pigs weigh at least 1,500 pounds.

Problem 5. A dark wizard assembled a battalion of orcs and goblins. During the first day together, each orc quarreled with 10 goblins and 5 orcs, and each goblin quarreled with 9 orcs and 6 goblins. Were there more orcs or goblins in this battalion? (Each quarrel involves 2 creatures, and is 2-sided.)

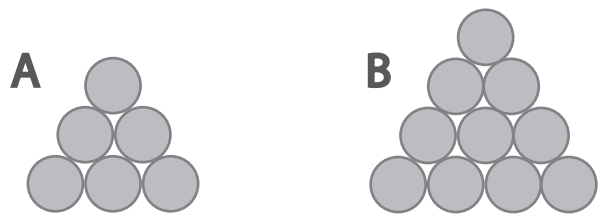
Problem 6. Mrs. Smith has some 10-dollar bills and 20-dollar bills. If she uses all her 10-dollar bills, she is 60 dollars short of buying 4 raspberry pies. If she uses all her 20-dollar bills, she is 60 dollars short of buying 5 raspberry pies. If she uses all her 10-dollar bills and 20-dollar bills, she is 60 dollars short of buying 6 raspberry pies. What is the price of one raspberry pie?

Problem 7. Several graduate students registered for several math courses each. Altogether, 7 courses were offered: mathematical logic, proofing methods, combinatorics, number theory, graph theory, cryptography, and strategic games. It turned out that each student took an odd number of courses, and each course had an odd number of participants. Was the total number of students odd or even?

Problem 8.

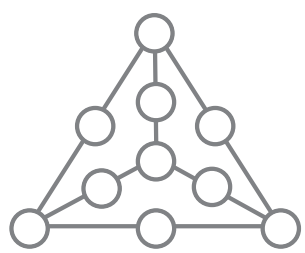
(a) Six coins are placed together to form a triangle (see picture A). It is known that the total value of any 3 coins that are all bordering each other (making a triangle) is equal to 100 shmollars. What is the total value of all 6 coins? (The coins are identical in size, but they may be of different values.)

(b) Look at picture B. Under the same conditions as in (a), what could the possible total value of all 10 coins be? (Either calculate this total or give examples of 2 different totals.)



Problem 9. Last weekend, 10 presidential candidates were holding 2-day rallies, with the goal of receiving endorsements. On day 1, the candidates received a different number of endorsements each, ranging from 1 to 10. The same outcome took place on day 2. Also, in 2 days, no 2 politicians got the same total number of endorsements. Prove that 1 of these 10 candidates is exactly 10 endorsements ahead of another.

Problem 10. Can you place all 10 numbers 0, 1, ..., 9 in the circles in such a way that the sums of three numbers along each segment would be the same?



Problem 11. Out of 7 integer numbers, the sum of any 6 is a multiple of 5. Prove that each number is a multiple of 5.

Problem 12. Monique tiled a grid-aligned rectangle with 2×1 dominoes. She noticed that every grid line cuts through a number of dominoes that is divisible by 4. Prove that at least one side of this rectangle is divisible by 4.

Problem 13. A 4×4 table is filled with numbers as shown in the picture. Each number is assigned a "+" or a "−" sign so that there are exactly 2 pluses and 2 minuses in each row and each column. Prove that the sum of all numbers taken with their assigned signs will always be equal to 0.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Problem 14. Yesterday, 27 British gentleman got together. Each of them belongs to no more than 2 clubs. Moreover, any 2 of these gentlemen belong to the same club. Prove that at least 18 of these gentlemen are members of the same club.

Problem 1. The inscription in an ancient math textbook reads: "$3^{333333} + 1$ is a prime number." Could this inscription be correct?

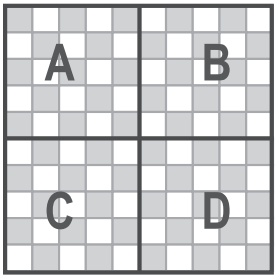
Problem 2. What is the smallest integer number N such that $N!$ is divisible by 990? ($N!$ is a product of all integer numbers from 1 to N.)

Problem 3. Is 100! is divisible by 2^{100}?

Problem 4. Solve $HE \times HE = SHE$. (The same letters stand for the same digits; different letters stand for different digits.)

Problem 5. An absent-minded mathematician believes that he can place 99 integer numbers in a circle so that for each pair of neighbors the ratio of the bigger and smaller numbers in the pair is a prime number. Prove that the mathematician is mistaken.

Problem 6. Someone placed 8 rooks on a chess board so that no two rooks attack each other. Prove that if you cut the chessboard into four 4×4 squares as shown in the picture, then the squares A and D will contain the same number of rooks.



Problem 1. The product of two natural numbers, neither of which ends with 0, is equal to 1000. Find the sum of these numbers.

Problem 2. Prove that a number is divisible by 8 only when the number formed by its last three digits is divisible by 8.

Problem 3. Several hundred years ago, the famous mathematician Leonhard Euler published a very interesting formula: $n^2 + n + 41$. If you start trying this formula on first positive integers $n = 1$, 2, 3, 4, and so on – the values that it produces would all be prime. For example, for $n = 20$ the answer is 461, which is a prime number. For $n = 21$ the answer is 503, which is also prime, etc. Could it be true that for any positive integer n this formula would produce a prime number?

Problem 4. Prove that out of any 7 perfect squares you can always find 2 such that their difference is divisible by 10.

Problem 5. I am thinking of a 3-digit prime number. Its last digit is equal to the sum of 2 other digits. What could my number's last digit be?

Problem 6. Prove that if $n! + 1$ is divisible by $n + 1$, then $n + 1$ is a prime number.

Problem 1. Rena and Simon bought several firecrackers each. Simon paid 12 shmollars more than Rina; Rina paid an odd number of shmollars. What could a price of the firecracker be? (A firecracker costs a whole number of shmollars.)

Problem 2. There are horses and cows on a farm. The number of horses is half of the number of cows and 10 more. The number of cows is 20 more than the number of horses. How many horses and how many cows are there on the farm?

Problem 3. Michael has a wooden triangle with one of its angles being equal to 40 degrees. Using this triangle, how can he measure the following:

(a) a 160-degree angle?

(b) a 20-degree angle?

(The sizes of the other angles are unknown.)

Problem 4.

(a) King Haggard has a velvet pouch filled with diamonds. He can divide these diamonds into 3 equal piles, 4 equal piles, or 5 equal piles. How many diamonds does he have if it is known that his collection contains less than 100 diamonds in total?

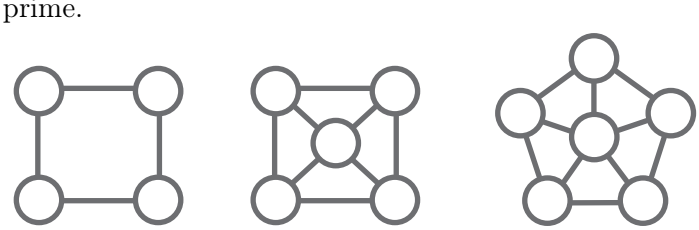
(b) King Haggard has a stash of gold coins. He is one coin short of being able to divide these coins into 4 equal piles, or 5 equal piles, or 6 equal piles, or 7 equal piles. How many coins does he have if he has fewer than 500?

Problem 5. What is the smallest integer number n such that 1000! is not divisible by 38^n?

Problem 6. For each shape below, place integer numbers at the nodes in such a way that:

- If two nodes are connected by an edge, then the numbers at these nodes are not relatively prime, and

- If two nodes are not connected, then the numbers at these nodes are relatively prime.



Problem 7. Suppose that p is a prime number.

(a) How many numbers that are less than p are relatively prime to it?

(b) How many numbers that are less than p^2 are relatively prime to it?

Problem 1.

(a) How many common factors do numbers 907 and 908 have?

(b) How many common factors do numbers 56785 and 56789 have?

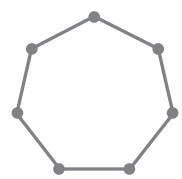
Problem 2. I am thinking of 5 non-prime numbers, and every pair of them are relatively prime. Prove that at least one of my numbers is greater than 100.

Problem 1. Here are phrases in Swahili with their English translations:
- atakupenda — He will love you.
- nitawapiga — I will beat them.
- atatupenda — He will love us.
- anakupiga — He beats you.
- nitampenda — I will love him.
- unawasumbua — You annoy them.

Translate the following into Swahili:
- You will love them.
- I annoy him.

Problem 2.

(a) Some vertices of a regular heptagon are colored black; others — white. Prove that is it possible to choose 3 vertices of the same color that make an isosceles triangle.



(b) Is the same true for an octagon?

Problem 3. Robbie the Robot calculated 15! and wrote this number on the whiteboard. Naughty Nick replaced some of the digits in this number by asterisks, leaving this expression:

$15! = 130 * 674368 * **$.

Find the missing digits without actually calculating 15!.

Problem 4. Prove that a number is divisible by 4 only when the number formed by its last two digits is divisible by 4.

Problem 5. Prove that a number composed of digits 6 and 2 can never be a perfect square.

Problem 6. Can you find 100 consecutive natural numbers that would not contain a single prime number?

Problem 7. There are 101 wise men are standing in a circle. Each one of them either believes that the Earth is flat or that the Earth is round. Every five minutes, all at once, wise men voice out their opinions on this matter. Immediately after that, each wise man whose both neighbors disagree with him changes his opinion. Prove that after several iterations the wise men will stop changing their points of view.

Problem 1. Construct a table of remainders when divided by 3 for the difference of two numbers, a and b. (Remember that remainders cannot be negative.)

$-$		b		
		0	1	2
a	0			
	1			
	2			

Problem 2. Can the sum of three consecutive counting numbers be prime?

Problem 3. Prove that $37^{73} - 1$ is divisible by 3.

Problem 4. Don Pedro has a purse full of 3-shmollar coins. With some of these coins, he bought 1 sombrero hat and received change, which was 1-shmollar coin. Next day, he bought 7 sombrero hats, paying with 3-shmollar coins as well. Prove that this day his change was a 1-shmollar coin again.

Problem 5. Find the remainder of $16^{10} - 2^{101} \times 7^{22}$ when divided by 3.

Problem 6. Prove that the numbers $12n + 1$ and $12n + 7$ are relatively prime.

Problem 7. Clarissa is tearing her test (1 piece of paper) into small pieces. She picks up a piece and tears it into either 4 pieces or 10 pieces. Can she end up with 2 million pieces?

Problem 8. Three brothers harvested their wheat and arranged the bundles in a neat square. (For example 6×6, 8×8, or maybe 20×20, etc. — we don't know how large). They wanted to divide the wheat fairly so that each would get the same number of bundles, but could not. Then their father pointed out that two more bundles were left out in the field. Will the brothers be able to divide their harvest now?

Problem 9. There are 25 bugs sit on the squares of a 5×5 board, one bug per square. When I clap my hands, each bug jumps to a square *diagonally* from where it was before. Prove that after I clap my hands, at least 5 squares on the board will be empty.

Problem 1. Tim had more hazelnuts than Tom. If Tim gave Tom as many hazelnuts as Tom already had, Tim and Tom would have the same number of hazelnuts. Instead, Tim gave Tom only a few hazelnuts (no more than five) and divided his remaining hazelnuts equally between 3 squirrels. How many hazelnuts did Tim give to Tom?

Problem 2. Prove that if a and ax have the same remainders when divided by 7, and $0 < a < 7$, then x has remainder 1.

Problem 3. Basil, a math geek, noticed that if he added up any 7 of his friends' telephone numbers, he would always get a number that ends with a 9. If Basil adds together phone numbers of all of his 11 friends, what will be the last digit of such sum?

Problem 4. For how many pairs of numbers x and y between 1 and 100, $x^2 + y^2$ is divisible by 7?

Problem 5. Astronaut A is in a broken moon rover that is 18 kilometers away from the main base. The air supply in the rover will last for another 3 hours. Also, A has one oxygen tank that he can use while at the rover or when walking. This tank has a 1-hour supply of oxygen.

Astronaut B is at the base. He has plenty of oxygen tanks that last 2 hours each. When walking, he cannot carry more than 2 tanks; one of these he should be using himself. Astronauts regularly communicate via portable radio. They both can walk at the speed of 6 kilometers per hour. Is there a way for astronaut B to save A and to get to the base himself?

Problem 6. Seven robbers are dividing a bag of coins of various denominations. It turned out that the sum could not be divided equally between them, but if any coin is set aside, the rest could be divided so that every robber would get an equal part. Prove that the bag cannot contain 100 coins.

Problem 1. Our office has 9 computers. My manager wants to link them into a network so that every computer would be connected with exactly 4 computers by network cables.

(a) How many cables will we need? Can you draw such a configuration?

(b) The manager decided to connect every computer with exactly 5 other computers. Is this a possible configuration?

Problem 2. There are 15 planets that circle the star called Aldebaran. Some of these planets have signed trade agreements with each other. Could it be possible that:

- 4 of these planets have exactly 4 trade partners each,
- 8 of these planets have exactly 5 trade partners each,
- the rest of these planets have exactly 3 trade partners each?

Problem 3. During a chess tournament some people played 5 games and some people played 6. Prove that the number of people who played 5 games is even.

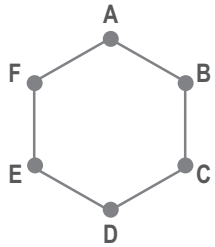
Problem 4. Eight people came to a party, some of them shook hands. Is it possible that 6 of them shook hands with 6 different people at the party, and two shook hands with only 2 each?

Problem 5. Martian amoebas gemmate (they multiply by division). When a red amoeba divides, it splits itself into 5 blue amoebas; when a blue amoeba divides, it splits into 7 red amoebas. When my spaceship left Mars, I put one amoeba into a jar. Upon arriving on Earth, the customs officer found 100 amoebas in that jar. Why does he keep searching my ship, insisting that some amoebas must have escaped?

Problem 6. The 99 greatest scientists of Mars and Venus are seated evenly around a circular table. If any scientist sees two colleagues from their own planet sitting an equal number of seats to their left and right, they wave to them. For example, if you are from Mars and the scientists sitting two seats to your left and right are also from Mars, you will wave to them. Prove that at least one of the 99 scientists will be waving, no matter how they are seated around the table.

Problem 1. Connect 6 points with curved lines such that the lines would not intersect and each point would have 4 lines ending in it.

Problem 2. The six towns in the country of Soupstop are connected by six roads as shown. Several flying saucers landed on each of the roads. The captain of each saucer sent an alien to each of the two nearest towns to steal a fresh bowl of soup (everyone knows that flying saucers use chicken soup as their fuel). As a result, a bowls of soup were stolen in A, b bowls went missing in B, C lost c bowls, D lost d bowls and in E the aliens snatched e bowls of soup. How many bowls of chicken soup were abducted in F?



Problem 3. In a graph of 40 vertices, the degree of each vertex is at least 20. Prove that the graph is connected.

Problem 4.

(a) Michael, the computer technician, likes number 5. Therefore, he connected 10 computers into a network so that each computer is connected to exactly 5 others. Draw an example of such a network.

(b) Now, Michael plans to connect 30 computers into a network with every computer connected to 5 others. How many cables would he need?

(c) Finally, Michael is planning to come up with a network that uses 56 cables. (Still, every computer should be connected to 5 others.) Will he be able to succeed?

Problem 5. Every person from the Knights and Liars Island said two phrases:

- "I have an odd number of knight friends,"
- "I have an even number of liar friends."

Is the total number of Islanders odd or even?

Problem 6. The remote country of Booroodoo has two airlines, Royal and Republican. Each Booroodoo airport is served by 4 flights, two of which are Royal and two are Republican. Out of all these flights, only two are international. Prove that these two flights belong to the same airline.

Problem 1. Can you come up with 99 positive integer numbers such that their sum and their product are both equal to 99?

Problem 2.

(a) Can you draw a graph with vertices of degrees 8, 6, 5, 4, 4, 3, 2, 2?
(b) Can you draw a graph with vertices of degrees 7, 7, 6, 5, 4, 2, 2, 1?
(c) Can you draw a graph with vertices of degrees 6, 6, 6, 5, 5, 3, 2, 2?
For each part, either show how or explain why this is not possible.

Problem 3. In a graph, every vertex is either blue or green. Every green vertex is connected by an edge with 9 blue and 6 green vertices, and every blue vertex is connected by an edge with 5 blue and 10 green vertices. Does this graph have more green or more blue vertices?

Problem 4. In the network of 101 computers, each is connected with exactly k computers. What are the possible values of k?

Problem 5. A ship hold is a rectangular-shaped room. Through the hole in the hull, the water started to leak into the hold. The water pump, which was immediately turned on, was not powerful enough. The water level was rising steadily; it reached the height of 20 cm in 10 minutes. At this moment, the second pump, identical to the first one, was turned on. In 10 minutes, the water level went down to 10 cm. At the same moment, the leak was fixed. How long will it take for the two pumps to pump all the water out?

Problem 6. Captain Flint has 60 pirates in his crew. Each pirate dislikes exactly one other pirate. Prove that the captain can split the pirates into 3 groups such that within each group no one dislikes no one else. ("Dislike" is not necessarily symmetrical. If pirate A dislikes pirate B, it does not mean that B dislikes A.)

Problem 7. On the first day of a chess tournament, each of the 10 participants played 1 game. Next day, each player played another game, with a different player. Prove that after the second day you can still find 5 players that did not play each other.

Problem 8. In Galaxy M31, transportation between planets is carried out by two companies: Royal Shuttle Service, and Spacehound Shuttles. Every pair of M31 planets is connected by a single shuttle route, served either by Royal or by Spacehound.

Prove that a traveler can visit all planets of M31 Galaxy using the services of only one of these companies. (Transfers are allowed.)