Supplementary Material for Chapter 2

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2.10. Continuous or convergence stability

We already know from §2.6 that when a discrete population game has more than one ESS, initial conditions determine which ESS the population adopts. In continuous population games, however, another issue arises. Suppose that a population of drivers in Crossroads II (§2) is at the ESS defined by (2.31), i.e., the universally adopted critical lateness above which a driver initially goes is

$$v^* = \begin{cases} 0 & \text{if } \delta \leq \frac{1}{2}(1-\eta)\tau \\ \frac{\delta + \frac{1}{2}(\eta - 1)\tau}{\delta + \frac{1}{2}\eta\tau + \epsilon} & \text{if } \delta > \frac{1}{2}(1-\eta)\tau, \end{cases} \tag{2.78}$$

where δ, ϵ, η and τ denote impetuosity, ditheriness, discount factor and junction transit time, respectively; and that there is a sudden shift in the value of any of these four parameters—e.g., perhaps η rises because everyone listens on his car radio to advice from a lifestyle guru. Then the population is no longer at its ESS:[1] instead it remains at v defined by using the *old* parameter values in (2.78), whereas

[1]Unless $\delta - \frac{1}{2}(1-\eta)\tau$ is negative both before and after the parameter shift.

the ESS v^* requires the new values. What happens now—will the population of drivers converge on this new ESS?

The answer depends on whether v^* now lies in the interior of $[0, 1]$ or on its boundary. For this particular game, the only possible boundary ESS is $v^* = 0$; however, to make our analysis apply more broadly to any population game with strategy set $[0, 1]$, let a boundary ESS be denoted by b ($= 0$ or 1), and let h and k be infinitesimally small. Then from Taylor's theorem, to a mutant strategy $u = v + h$, the perturbed population strategy $v = b + k$ yields the reward

$$\begin{aligned} f(u,v) &= f(b+h+k, b+k) \\ &= f(b+k, b+k) + h\, \tfrac{\partial f}{\partial u}\big|_{u=v=b+k} \\ &= f(b+k, b+k) + h\, \tfrac{\partial f}{\partial u}\big|_{u=v=b} \quad + \ o(\max\{h,k\}) \end{aligned} \tag{2.79}$$

(where $o(\xi)$ denotes terms so small that they can be divided by ξ and the result still tends to zero as $\xi \to 0$). We assume that h and k are positive if $b = 0$ but negative if $b = 1$, to ensure that $u \in [0, 1]$. If $b = 0$ is a strong ESS then $\frac{\partial f}{\partial u}\big|_{u=v=b} < 0$ (because f achieves its maximum on the boundary); whereas if $b = 1$ is a strong ESS then $\frac{\partial f}{\partial u}\big|_{u=v=b} > 0$. So $f(b+h+k, b+k)$ is invariably lower than $f(b+k, b+k)$: mutant strategies that would move the population further from the boundary are disfavored, while those that would move it back (with h having the sign of $b - \frac{1}{2}$) are favored. Thus a non-b population invariably converges to a b population.

Matters are less simple when $0 < v^* < 1$ and $(v^*, v^*) \in R$ requires

$$\tfrac{\partial f}{\partial u}\big|_{u=v=v^*} = 0 \tag{2.80}$$

(if f is sufficiently differentiable, which we assume). Then

$$\begin{aligned} \tfrac{\partial f}{\partial u}\big|_{u=v^*+k, v=v^*+k} \ &= \ \tfrac{\partial f}{\partial u}\big|_{u=v^*, v=v^*} \ + \\ & k\, \tfrac{\partial^2 f}{\partial u^2}\big|_{u=v^*, v=v^*} \ + \ k\, \tfrac{\partial^2 f}{\partial u \partial v}\big|_{u=v^*, v=v^*} \ + \ o(k) \end{aligned} \tag{2.81}$$

from Taylor's theorem; and, to a mutant strategy $u = v + h$, the perturbed population strategy $v = v^* + k$ yields the reward

$$\begin{aligned} f(u,v) &= f(v^*+h+k, v^*+k) \\ &= f(v^*+k, v^*+k) \ + \ h\, \tfrac{\partial f}{\partial u}\big|_{u=v^*+k, v=v^*+k} \\ &= f(v^*+k, v^*+k) \ + \ hk\big(\tfrac{\partial^2 f}{\partial u^2} \ + \ \tfrac{\partial^2 f}{\partial u \partial v}\big)\big|_{u=v=v^*} \end{aligned} \tag{2.82}$$

plus terms that are negligibly small. For mutant strategies to have the effect of moving the population from its current threshold $v = v^* + k$ to the ESS v^*, those with $h > 0$ must be favored when $k < 0$ and those with $h < 0$ must be favored when $k > 0$; in other words, we require $f(u, v)$ to exceed $f(v, v)$ when $hk < 0$. Thus a non-v^* population converges to a v^* population when

$$\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial u \partial v}\right)\Big|_{u=v=v^*} < 0. \tag{2.83}$$

If (2.83) holds then the (interior) ESS v^* is said to be *continuously stable* [**1**].

Clearly, such stability is a very desirable property; and when a continuous population game has more than one ESS, continuous stability yields a criterion for distinguishing among them, although we will not meet such games until Chapter 6. Meanwhile, you can readily check that Crossroads II has a continuously stable interior ESS when the drivers are not slow; see Exercise 2.26.

Additional exercises for Chapter 2

26. Verify that (2.30) is a continuously stable interior ESS for Crossroads II.

27. A large population of local manufacturers of home-brewed beer is interacting in triads because each local market is large enough only for three firms to compete. Each firm advertises its product on local television once a day, either in the morning or at night. If more than one firm in a local market (= triad) advertises at the same time, then none of the firms in that triad experiences a profit increase; whereas if one firm advertises when the others do not, then its profit increases by $\$\alpha$ per day if it advertises in the morning or by $\$\nu\alpha$ per day if it advertises at night. All firms use a mixed strategy: advertising in the morning with probability u (and at night with probability $1 - u$) is defined to be strategy u. Obtain an expression for the reward to a mutant u-strategist in a population of v-strategists, and show that $v^* = \frac{1}{1+\sqrt{\nu}}$ is a continuously stable ESS.[2]

28. Read [**3**] and critique its argument that no strategy in the prisoner's dilemma is an ESS. What is the flaw in its §6?

[2] This exercise was suggested by [**2**, p. 72].

Biblio raphy

[1] I. Eshel, *Evolutionary and continuous stability*, Journal of Theoretical Biology **103** (1983), 99–111.

[2] H. Gintis, *Game Theory Evolving*, Princeton University Press, Princeton, New Jersey, 2000.

[3] L. Marinoff, *The inapplicability of evolutionarily stable strategy to the prisoner's dilemma*, British Journal for the Philosophy of Science **41** (1990), 461–472.