More on the localization of a supersymmetric integral

The change of variables formula given in the display on Page 153 line 11 is incorrect, and in fact Z' does *not* vanish as incorrectly claimed in display (104). It turns out that the claimed method only works for supersymmetry transformations δ satisfying $\delta^2 = 0$, which does not hold in this situation since, for example,

$$\delta^2(\psi_1) = \delta(\delta(\psi_1)) = \delta(h'(x)\epsilon_2) = h''(x)\delta(x)\epsilon_2 = h''(x)(\epsilon_1\psi_1 + \epsilon_2\psi_2)\epsilon_2 \neq 0.$$

Nevertheless, the conclusion (106) of the localization method is correct, an important point in the treatment. It requires a lengthier argument, which is supplied here.

The argument can be repaired by replacing h by λh with λ a real parameter and letting λ approach infinity. This changes the action (100) to

$$S_\lambda(x, \psi_1, \psi_2) = \frac{\lambda^2 h'(x)^2}{2} - \lambda h''(x)\psi_1\psi_2,$$

which changes the partition function from Z to

$$Z(\lambda) = \int \lambda h''(x) \exp(-\lambda^2 h'(x)^2/2)\, dx.$$

By the deformation invariance explained on page 155, it follows that $Z(\lambda)$ is independent of λ, i.e., coincides with the original Z. In particular,

$$Z = \lim_{\lambda\to\infty} Z(\lambda).$$

We will reach (106) by calculating $\lim_{\lambda\to\infty} Z(\lambda)$.

Note also that the replacement of h by λh does not alter the location of the critical points of h.

For any interval $[a, b]$ not containing any critical points of h, including the possibility that $a = -\infty$ or $b = \infty$, we have

$$\lim_{\lambda\to\infty} \int_a^b \lambda h''(x) \exp(-\lambda^2 h'(x)^2/2)\, dx = \int_a^b \lim_{\lambda\to\infty} \lambda h''(x) \exp(-\lambda^2 h'(x)^2/2)\, dx = 0,$$

due to the vanishing of the second limit which follows from the rapid decay of the exponential.

Denoting the critical points as $x_{c_1} < x_{c_2} < \ldots < x_{c_k}$, choosing any positive $\epsilon < \min\{(x_{c_{i+1}} - x_{c_i})/2\}$, and denoting the integrand of $Z(\lambda)$ by $H_\lambda(x) = \lambda h''(x) \exp(-\lambda^2 h'(x)^2/2)$ for ease of notation, we get for $Z(\lambda)$ by splitting up the domain of integration the expression

$$\int_{-\infty}^{x_{c_1}-\epsilon} H_\lambda(x)\,dx+\sum_{i=1}^{k}\int_{x_{c_i}-\epsilon}^{x_{c_i}+\epsilon} H_\lambda(x)\,dx+\sum_{i=1}^{k-1}\int_{x_{c_i}+\epsilon}^{x_{c_{i+1}}-\epsilon} H_\lambda(x)\,dx+\int_{x_{c_k}+\epsilon}^{\infty} H_\lambda(x)\,dx.$$

It follows immediately that

$$Z = \lim_{\lambda\to\infty} Z(\lambda) = \lim_{\lambda\to\infty} \sum_{i=1}^{k} \int_{x_{c_i}-\epsilon}^{x_{c_i}+\epsilon} \lambda h''(x) \exp(-\lambda^2 h'(x)^2/2)\,dx$$

because the limits of all of the other terms in the expansion of $Z(\lambda)$ vanish as we have seen before, since the intervals that $H_\lambda(x)$ is being integrated over contain no critical points.

By Taylor's Theorem with remainder, we now expand h near a critical point x_c as a truncated power series

$$h(x) = h(x_c) + \frac{h''(x_c)}{2}(x - x_c)^2 + O(x - x_c)^3,$$

where $O(x-x_c)^3$ refers to a function of "order $(x-x_c)^3$", which more precisely means a function $f(x)$ such that $|f(x)| \le M|(x - x_c)^3|$ as x approaches x_c, for some contant M. We have

$$\int_{x_c-\epsilon}^{x_c+\epsilon} \lambda h''(x) \exp(-\lambda^2 h'(x)^2/2)\,dx =$$

$$\int_{x_c-\epsilon}^{x_c+\epsilon} \lambda\left(h''(x_c) + O(x - x_c)\right) \exp\left(-\lambda^2\left(\frac{h''(x_c)(x - x_c)^2}{2} + O(x - x_c)^3\right)\right) dx.$$

We next make the change of variables $y = \lambda(x - x_c)$, and we get

$$\int_{-\lambda\epsilon}^{\lambda\epsilon} \left(h''(x_c) + O\left(\frac{y}{\lambda}\right)\right) \exp\left(-\frac{h''(x_c)^2y^2}{2} + \lambda^2\left(O\left(\frac{y}{\lambda}\right)^3\right)\right) dy$$

Taking the limit as λ approaches infinity, the terms $O(y/\lambda)$ and $O((y/\lambda)^3)$ approach 0 and we are left with

$$\int_{-\infty}^{\infty} h''(x_c)\exp\left(-\frac{h''(x_c)^2y^2}{2}\,dy\right).$$

This is precisely the second line of (105) after another change of variables $y = x - x_c$.[1] From the above expression of Z as a sum of integrals in the neighborhood of critical points, we are led to (106), completing the argument.

Thanks to Kentaro Hori for assistance with the correction.

[1] At the risk of introducing some confusion, we have inconsistently introduced the substitution variable x here which has already been used, in order to match (105).