Invariant Theory, Corrections

Thanks to everybody who sent me comments, corrections, and typos!

Page 20, Line -10

Thus $a = b = 0$, but then T is not invertible.

Proposition 2.24

It is assumed, but not explicitely stated that the ring R is graded and the elements $f_1, \ldots, f_m$ are homogeneous.

Page 34, Proof of Proposition 2.24

In the first part of the proof ((1) $\Rightarrow$ (2)) it is not shown that the f_i's form an irredundant generating set for the ideal I. This follows easily for degree reasons.

Page 48, Line -11

Replace the display by

$$(^+_-a, ^+_-b) \quad \text{and} \quad (^+_-b, ^+_-a).$$

Page 57, Line -1

Replace the last line of the display by

$$-\begin{cases} 0 & \text{if } m-n \text{ is odd,} \\ \binom{m-n}{\frac{m-n}{2}}(x_1^2x_2^2)^{\frac{m-n}{2}} & \text{if } m-n \text{ is even,} \end{cases}$$

Page 62, Exercise 17

Replace display by

$$f_k = x^{2^{n-1}-k}y^k + x^k y^{2^{n-1}-k} \in \mathbb{C}[x,y].$$

Page 63, Line -1

The last n-tuple in the display has to look like

$$(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}).$$

Page 68, Line 8

The last n-tuple in the display has to look like

$$(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}).$$

Page 68, Lines 11-12,14,17,20

In contrast, here the $v_{\sigma^{-1}(i)}$'s have to be replaced by $v_{\sigma(i)}$.

Page 81, Line 6

$$o(\mathbf{x}^I) = x_1x_2^3x_3 + x_1^3x_2x_4.$$

Page 81, Lines 10-12

$$\begin{aligned} o(x_1x_2^2x_3)s_1 &= (x_1x_2^2x_3 + x_1^2x_2x_4)(x_1 + \cdots + x_4) \\ &= x_1^2x_2^2x_3 + x_1^3x_2x_4 + x_1x_2^3x_3 + x_1^2x_2^2x_4 \\ &\quad + x_1x_2^2x_3^2 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1^2x_2x_4^2. \end{aligned}$$

Page 81, Line 17

$$x_1^3x_2x_3 \quad \text{and} \quad x_1x_2^3x_4$$

Page 101, Example 5.3

The image of the representation ρ lies of course in $\mathsf{GL}(2,\mathbb{C})$.

Page 110, Proposition 5.17

The coefficient in the display is not $\det(g)$ but $\det(\rho(g)^{-1})$.

Page 111, Lines 3,5, and 15

All the $\rho(g)$'s need to be replaced by $\rho(g)^{-1}$.

Page 115, Exercise 20 (iii)

The degree of the Hessian is $n(\deg(f) - 2)$.

Page 118, Line -12
Replace $(-1)^{i+1}x_2$ by $(-1)^i x_2$.

Page 119, Line 6
Replace ω^{12} by ω^{-3}.

Page 120, Line 6
Replace $\eta_g \pi^G$ by $\eta_G \pi^G$.

Page 123, Line -9
Replace $i = 1, 2$ by $j = 1, 2$.

Page 128, Line -9
$t_0 = 1$ and $t_d = 0$.

Page 136, Exercise 12
Since the representation of Q_8 this exercise refers to lives over the complex numbers, also the ring of invariants is an algebra over $\mathbb{C}$.

Page 141, Footnote
This paper has appeared in *Involve* 1 (2008), 159-165.

Lemma 7.6
We need to diagonalize the matrix R in order to conclude that $\lambda_1 = \cdots = \lambda_{n-1} = 0$.

Example 7.12[Groups of Type $G(m, p, 2)$]
The following should be a better discussion of the example than the one given in the book:
Let $m = pq$, $m, p, q \in \mathbb{N}$, and ω a primitive mth root of unity. The subgroup $G(m, p, 2) \leq \mathsf{GL}(2,\ \mathbb{C})$ is generated by the transposition

$$\mathsf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the matrices

$$\mathsf{R}(a_1, a_2) = \begin{bmatrix} \omega^{a_1} & 0 \\ 0 & \omega^{a_2} \end{bmatrix}$$

for all $a_1 + a_2 \equiv 0 \bmod p$. We observe that this group consists of all matrices of the form $\mathsf{R}(a_1, a_2)$ and

$$\mathsf{S}(a_1, a_2) = \begin{bmatrix} 0 & \omega^{a_1} \\ \omega^{a_2} & 0 \end{bmatrix},$$

where $a_1 + a_2 \equiv 0 \mod p$. We have m choices for a_1. Once an a_1 is choosen, we have q choices for a_2. Hence the order of $G(m, p, 2)$

is $2qm$. We claim this group is a pseudoreflection group. The transposition T is a reflection of order 2 fixing the line $\mathbf{e}_1 + \mathbf{e}_2$, while the matrices $\mathsf{R}(a_1, a_2)$ are in general not, because they fix only the origin. However, the group can be generated by T, $\mathsf{R}(p, 0)$, $\mathsf{R}(0, p)$, and $\mathsf{S}(a, -a)$. We note that $\mathsf{R}(p, 0)$ fixes $\mathbf{e}_2$, the element $\mathsf{R}(0, p)$ fixes $\mathbf{e}_1$, and finally, $\mathsf{S}(a, -a)$ fixes $\mathbf{e}_1 + \omega^{-a}\mathbf{e}_2$. Furthermore, the invariants are

$$\mathbb{C}[x, y]^{G(m,p,2)} = \mathbb{C}[x^m + y^m, (xy)^q];$$

cf. Exercise 5 in this chapter. Note that for the special case $q = 1$ we obtain the dihedral groups.

Page 153, Exercise 11

The matrix M_2 must be

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and M_3 is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Page 160, Lines 10 and 13

The upper bound on the integrals slipped down. They are supposed to read as

$$\int_{-1}^{1} h(t)\mathrm{d}t \quad \text{resp.} \quad \int_{-1}^{1} L_{n,i}(t)\mathrm{d}t.$$

Page 171, Line 8

We need to assume that the M_i's are submodules in some big module M.

Page 186, Line 8

Replace the display by:

$$(a + b\mathbf{i})^2 - 2a(a + b\mathbf{i}) + (a^2 + b^2) = 0$$

for $a,\ b \in \mathbb{R}$.

Page 192, Line -13

Replace $\mathbb{F}[V][t]$ by $\mathbb{F}[V][X]$.

Page 194, Line 5

Assume without loss of generality that $s_0 \neq 0$.

Page 196, Line 11

Replace $\mathfrak{p}_1 \supset \mathfrak{p}_2$ by $\mathfrak{p}_1 \subset \mathfrak{p}_2$.

Page 197, Line -8

Replace $\mathfrak{q}' \subseteq \mathbb{F}[V]^G$ by $\mathfrak{q}' \subseteq \mathbb{F}[V]$.

Page 197, Line -4

$g_0\mathfrak{q} \subseteq \mathfrak{q}'$

Page 202, Lines 7-8

.. then we can find elements $r_i \in I$, $i \in \mathbb{N}$, such that $(r_1, r_2, r_3, \dots) \subseteq I$.

Page 204, Line 7

Replace M by M_n.

Page 207, Line 14

Replace display by

$$r = \sum_I t_I \mathbf{x}^I.$$

Page 210, Line 1

$(\mathfrak{p}'_{i_0})^{\mathrm{e}} = (\mathfrak{p}'_{i_0+1})^{\mathrm{e}}$

Page 210, Line 3

$\mathfrak{p}_{i_0} = (\mathfrak{p}'_{i_0})^{\mathrm{e}}$ and $\mathfrak{p}_{i_0+1} = ((\mathfrak{p}'_{i_0})^{\mathrm{e}}, x_n)$.

Page 212, Line -5

Replace the display by

$$\mathbb{FF}(A) = \mathbb{F}(V)^G.$$

Page 227, Example 11.6

Reference should be to Lemma 11.1 and not 11.4.

Page 235, Line 6

Since there is no term in degree one, ... (The word "no" is missing.)

Page 238, Line 12

$f_3^2 = f_1^2 f_2 - 4f_2^2$

Page 240, Line 14

Delete $y_1^2 + y_2^2$.

Page 244, Exercise 11

Assume that the matrices $\mathsf{A}, \mathsf{B}, \mathsf{C}$, and D commute. Then $\det(\mathsf{M}) = \det(\mathsf{AD} - \mathsf{BC})$.

Page 251, Line -12

The elements in $V(J)$ are $n+1$-tuple, i.e., $(\alpha_1, \dots, \alpha_{n+1})$.

Page 254, Line 10

The given argument for non-invertible matrices $[H_{ij}]_{ij}$ requires the choice of a suitable basis.

Page 255, Proof of Proposition 12.17

Add in Line 4: A fortiori π^G is an $\mathbb{F}[h_1, \dots, h_n]$-module epimorphism splitting the inclusion.

Page 255, Proof of Proposition 12.17, Line 6

Replace $\mathbb{F}[V]^G$ by $\mathbb{F}[h_1, \dots, h_n]$.

Page 255, Proof of Proposition 12.17

Replace the last 12 lines of the proof (starting at "We obtain a decomposition ...") by the following:

Set $A = \mathbb{F}[h_1, \dots, h_n]$, and $\mathfrak{m} \subset A$ the augmentation ideal in A. By Proposition 8.13 we obtain that $\mathbb{F}[V]^G/\mathfrak{m}\mathbb{F}[V]^G$ is a finite dimensional $\mathbb{F}$-vector space. Thus $A \otimes_{\mathbb{F}} (\mathbb{F}[V]^G/\mathfrak{m}\mathbb{F}[V]^G)$ is a free A-module, where A acts by multiplication on the first factor. Moreover the canonical A-module map

$$A \otimes_{\mathbb{F}} (\mathbb{F}[V]^G/\mathfrak{m}\mathbb{F}[V]^G) \longrightarrow \mathbb{F}[V]^G, a \otimes f \mapsto af$$

is surjective, again by Proposition 8.13. We claim this map is also injective, i.e., $\mathbb{F}[V]^G$ is isomorphic to a free A-module, and hence free.

To that end note that we already know that $\mathbb{F}[V]$ is a free A-module. Thus

$$\mathbb{F}[V] = \bigoplus_{i=1}^{m} Af_i$$

for suitable $f_i \in \mathbb{F}[V]$. So every polynomial can written as an A-linear combination of the f_i's in a unique way. In particular, for every invariant polynomial, say $f \in \mathbb{F}[V]^G$, we find

$$f = \sum_{i=1}^{m} a_i f_i$$

for uniquely determined $a_i \in A$. Thus the map

$$\mathbb{F}[V]^G \longrightarrow A \otimes_{\mathbb{F}} (\mathbb{F}[V]/\mathfrak{m}\mathbb{F}[V]), f \mapsto \sum_{i=1}^{m} a_i \otimes f_i$$

is a well-defined A-module homomorphism. The composition of maps

$$\mathbb{F}[V]^G \longrightarrow A\otimes_{\mathbb{F}}(\mathbb{F}[V]/\mathfrak{m}\mathbb{F}[V]) \xrightarrow{\mathrm{id}\otimes\pi^G} A\otimes_{\mathbb{F}}(\mathbb{F}[V]^G/\mathfrak{m}\mathbb{F}[V]^G) \xrightarrow{\mu} \mathbb{F}[V]^G$$

is the identity: Let $f \in \mathbb{F}[V]^G$. Then

$$f \mapsto \sum_{i=1}^{m} a_i \otimes f_i \mapsto \sum_{i=1}^{m} a_i \otimes \pi^G(f_i) \mapsto \sum_{i=1}^{m} a_i \pi^G(f_i) = f.$$

In other words the short exact sequence of A-modules

$$0 \longrightarrow \mathrm{Ker}(\mu) \longrightarrow A \otimes_{\mathbb{F}} (\mathbb{F}[V]^G/\mathfrak{m}\mathbb{F}[V]^G) \xrightarrow{\mu} \mathbb{F}[V]^G \longrightarrow 0$$

splits and

$$A \otimes_{\mathbb{F}} (\mathbb{F}[V]^G/\mathfrak{m}\mathbb{F}[V]^G) = \mathrm{Ker}(\mu) \oplus \mathbb{F}[V]^G$$

as A-modules. Thus $\mathrm{Ker}(\mu)/\mathfrak{m}\mathrm{Ker}(\mu) = 0$. So Proposition 8.13 gives us that $\mathrm{Ker}(\mu) = 0$ and hence

$$A \otimes_{\mathbb{F}} (\mathbb{F}[V]^G/\mathfrak{m}\mathbb{F}[V]^G) \cong \mathbb{F}[V]^G$$

is free over A.

Page 255, Line -9

Replace the coefficients $\mathbb{C}$ by $\mathbb{F}$.

Page 259, Proof of Proposition 12.23

There is nothing wrong with the proof. However, it seems to be confusing. So, let me explain: By Dade's Condition ($\star$) the linear forms $g_1x_1, \ldots, g_nx_n \in V^*$ are linearly independent. Thus they span the entire vector space V^*. Since they are homogeneous the ideal they generate is the augmentation ideal. In the next to last line of this proof observe that

$$\bigcap_{j=1}^{n}(\bigcup_{g_j\in G} \mathrm{Ker}(g_jx_j)) = \bigcup_{g_j\in G}(\bigcap_{j=1}^{n} \mathrm{Ker}(g_jx_j))$$

Again, since the g_jx_j's are linearly independent the intersection $\bigcap_{j=1}^{n} \mathrm{Ker}(g_jx_j)$ vanishes.

Page 259, Line -5

$\mathbb{C}$ needs to be $\mathbb{F}$.

Page 264, Line 8

$\deg(A) = \frac{1}{d_1 \cdots d_n}$

Page 268, Example 13.1

This example is riddled with typos, so we do it again:

Consider the real 2×2-matrix

$$\mathsf{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

This is a pseudoreflection of order 2 and fixes the subspace spanned by $\mathbf{e}_1 + (1 - \sqrt{2})\mathbf{e}_2$. Thus

$$\mathrm{Im}(\mathsf{I} - \mathsf{R}) = \mathrm{span}_{\mathbb{R}}((\sqrt{2} - 1)\mathbf{e}_1 + \mathbf{e}_2)$$

and we might pick $\mathbf{v}_{\mathsf{R}} = (\sqrt{2} - 1)\mathbf{e}_1 + \mathbf{e}_2$. Since

$$\mathsf{R}(\mathbf{e}_i^{\mathrm{t}}) = \begin{cases} \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2) & \text{for } i = 1, \\ \frac{1}{\sqrt{2}}(-\mathbf{e}_1 - \mathbf{e}_2) & \text{for } i = 2, \end{cases}$$

we obtain

$$\mathbf{e}_1 - \Delta_{\mathsf{R}}(\mathbf{e}_1)((\sqrt{2} - 1)\mathbf{e}_1 + \mathbf{e}_2) = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2),$$
$$\mathbf{e}_2 - \Delta_{\mathsf{R}}(\mathbf{e}_2)((\sqrt{2} - 1)\mathbf{e}_1 + \mathbf{e}_2) = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 - \mathbf{e}_2).$$

Therefore we find

$$\Delta_{\mathsf{R}}(\mathbf{e}_i) = \begin{cases} \frac{1}{\sqrt{2}} & for \mathrm{i{=}1} \\ \frac{1}{\sqrt{2}(\sqrt{2}-1)} & for \mathrm{i{=}2}, \end{cases}$$

hence in general:

$$\Delta_{\mathsf{R}}(\alpha\mathbf{e}_1 + \beta\mathbf{e}_2) = \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}(\sqrt{2} - 1)}.$$

Page 268, Line -1

Thus the right hand side of the display is $\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}(\sqrt{2}-1)}$.

Page 285, Line 7

Replace $p(\mathbf{d})$ by $p_k(\mathbf{d})$.

Pages 273 and 274, Proof of Theorem 13.8

We redo these two pages:

because our f_i's are homogeneous. Hence we obtain

$$\begin{aligned}
\deg(f_i)f_i &= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} x_j \\
&= \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} + \sum_{\alpha=1,\neq i}^{m} \frac{\partial f_\alpha}{\partial x_j} - \sum_{\alpha=1,\neq i}^{m} \frac{\partial f_\alpha}{\partial x_j} \right) x_j \\
&= \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} + \sum_{\alpha=1,\neq i}^{m} \frac{\partial f_\alpha}{\partial x_j} \right) x_j - \sum_{j=1}^{n} \left(\sum_{\alpha=1,\neq i}^{m} \frac{\partial f_\alpha}{\partial x_j} \right) x_j \\
&= \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} + \sum_{\alpha=1,\neq i}^{m} \frac{\partial f_\alpha}{\partial x_j} \right) x_j - \sum_{\alpha=1,\neq i}^{m} \left(\sum_{j=1}^{n} \frac{\partial f_\alpha}{\partial x_j} x_j \right) \\
&= \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} + \sum_{\alpha=1,\neq i}^{m} \frac{\partial f_\alpha}{\partial x_j} \right) x_j - \sum_{\alpha=1,\neq i}^{m} (\deg(f_\alpha) f_\alpha) .
\end{aligned}$$

Let us assume for the moment that the coefficients of the first sum are in I. Then we can proceed as follows. First we rewrite them as

$$A_j = \sum_{\alpha=1}^{m} \frac{\partial f_\alpha}{\partial x_j} = F_{j1} f_1 + \cdots + F_{jm} f_m$$

for some $F_{j1}, \ldots, F_{jm} \in \mathbb{F}[V]$. Then we plug this into our expression above and obtain

$$\begin{aligned}
\deg(f_i)f_i &= \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} + \sum_{\alpha=1,\neq i}^{m} \frac{\partial f_\alpha}{\partial x_j} \right) x_j - \sum_{\alpha=1,\neq i}^{m} (\deg(f_\alpha) f_\alpha) \\
&= \sum_{j=1}^{n} (F_{j1} f_1 + \cdots + F_{jm} f_m) \, x_j - \sum_{\alpha=1,\neq i}^{m} (\deg(f_\alpha) f_\alpha) .
\end{aligned}$$

Collecting terms gives us

$$\deg(f_i)f_i = \left(\sum_{j=1}^{n} F_{ji} x_j \right) f_i - \sum_{\alpha=1,\neq i}^{m} (\deg(f_\alpha) f_\alpha) .$$

Solving for f_i then gives

$$f_i = \lambda \sum_{\beta \neq i} H_\beta f_\beta \tag{✠}$$

for $\lambda = \frac{1}{\deg(f_i)} \in \mathbb{F}^\times$ and some $H_\beta \in \mathbb{F}[V]$. We apply the averaging operator π^G to (✠) and obtain

$$f_i = \frac{1}{\deg(f_i)} \sum_{\beta \neq i} \pi^G(H_\beta) f_\beta.$$

Thus the set $f_1, \ldots, f_m$ is not a minimal generating set of $\mathbb{F}[V]^G$.

Finally we need to show that $\frac{\partial f_i}{\partial x_j} \in I$ for all i, j. Denote by

$$p_j = \frac{\partial p}{\partial X_j}|_{(f_1,\ldots,f_m)} \in \mathbb{F}[V], \quad j = 1, \ldots, m,$$

the partial derivatives of the polynomial p evaluated at $f_1, \ldots, f_m$. Let $J = (p_1, \ldots, p_m) \subseteq \mathbb{F}[V]$ be the ideal generated by the p_j's. Without loss of generality we assume that the set

$$\{p_1, \ldots, p_k\} \subseteq \{p_1, \ldots, p_m\}$$

is minimal such that $J = (p_1, \ldots, p_k)$. Since $\mathbb{F}[V] = \mathbb{F}[V]^G h_1 \oplus \cdots \oplus \mathbb{F}[V]^G h_l$ for some suitable $h_1, \ldots, h_l$ we have that

$$\frac{\partial f_i}{\partial x_j} = H_{i,j,1} h_1 + \cdots + H_{i,j,l} h_l$$

for some $H_{i,j,1}, \ldots, H_{i,j,l} \in \mathbb{F}[V]^G$. We need to show that $\deg(H_{i,j,k}) > 0$ for $k = 1, \ldots, l$. Since $p(f_1, \ldots, f_m) = 0$ we obtain from the chain rule that

$$\begin{aligned} 0 = \frac{\partial}{\partial X_j}(p(f_1, \ldots, f_m)) &= \sum_{i=1}^m p_i \frac{\partial f_i}{\partial x_j} \\ &= \sum_{i=1}^m p_i \left(H_{i,j,1} h_1 + \cdots + H_{i,j,l} h_l\right) = \sum_{k=1}^l \sum_{i=1}^m (p_i H_{i,j,k}) h_k \end{aligned}$$

for $j = 1, \ldots, m$. Since the h_k's are independent over $\mathbb{F}[V]^G$ the coefficients of this equation must be zero:

$$\sum_{i=1}^m p_i H_{i,j,k} = 0$$

for all i, j, k's. However, if one of the $H_{i,j,k}$'s were a constant, then the p_i's would not form a minimal generating set of I. Thus the $H_{i,j,k}$ have positive degree and $\frac{\partial f_i}{\partial x_j} \in I$.

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