Appendix D

Solutions to Selected Exercises

Exercise 1.5.6 part (8) Prove that f is a regulated function on $I = [a, b]$ if and only if both the limits

$$\lim_{x \to c+} f(x) \quad \text{and} \quad \lim_{x \to c-} f(x)$$

exist for every $c \in (a, b)$. (See section VII.6 of Dieudonné [**D**]).

Proof. Suppose that f is a regulated function defined on $[a, b]$. Given ε there exists a step function g such that $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$. Let $A = \lim_{x \to c+} g(x)$ so

$$A - \varepsilon \leq \liminf_{x \to c+} f(x) \leq \limsup_{x \to c+} f(x) \leq A + \varepsilon.$$

Hence

$$\limsup_{x \to c+} f(x) - \liminf_{x \to c+} f(x) < 2\varepsilon.$$

Since this holds for arbitrary positive ε we conclude $\limsup_{x \to c+} f(x) = \liminf_{x \to c+} f(x)$ so $\lim_{x \to c+} f(x)$ exists. The argument for $\lim_{x \to c-} f(x)$ is the same.

Conversely suppose the limits

$$L_y^+ = \lim_{x \to y+} f(x) \quad \text{and} \quad L_y^- = \lim_{x \to y-} f(x)$$

exist for every $y \in (a, b)$. Define $f(x)$ for $x \notin [a, b]$ by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. Suppose $\varepsilon > 0$ is given. Then for each $y \in [a, b]$ there is a $\delta_y > 0$ such that $|L_y^- - f(x)| < \varepsilon$ for $x \in (y - \delta_y, y)$ and $|L_y^+ - f(x)| < \varepsilon$ for $x \in (y, y + \delta_y)$.

We define a step function g_y by

$$g_y(x) = \begin{cases} L_y^-, & \text{if } x \in (y - \delta_y, y) \\ L_y^+, & \text{if } x \in (y, y + \delta_y) \\ f(y), & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Notice that for all x in the open interval $U_y = (y - \delta_y, y + \delta_y)$ we have $|g_y(x) - f(x)| < \varepsilon$.

The collection $\{U_y\}$ for $y \in [a, b]$ is an open covering of the compact interval $[a, b]$. Hence there is a finite subcovering $\{U_{y_i}\}_{i=1}^n$. Let $U_i = U_{y_i}$ and $g_i = g_{y_i}$.

Define $\mathfrak{X}_i(x) = 1$ if $x \in U_i$ and 0 otherwise and let

$$N(x) = \sum_{i=1}^n \mathfrak{X}_i(x)$$

so $N(x)$ is the number of U_i's which contain x. Note that $N(x)$ is a step function and $N(x) \geq 1$. Also we define $f_i(x) = f(x)\mathfrak{X}_i(x)$ observe that $|f_i(x) - g_i(x)| < \varepsilon$ for all $x \in [a, b]$ since $f_i = f$ on U_i and both f_i and g_i are 0 outside U_i. Moreover we observe

$$f(x) = \frac{1}{N(x)} \sum_{i=1}^n f_i(x).$$

because there are $N(x)$ values of i where $f_i(x) = f(x)$ and $f_i(x) = 0$ for the others.

Finally we define a step function

$$g(x) = \frac{1}{N(x)} \sum_{i=1}^n g_i(x).$$

Notice that $g(x)$ is just the average of the $N(x)$ values $\{g_i(x)\}$ corresponding to the U_i's containing x. Since each of these values $g_i(x)$ is within ε of $f(x)$ so is their average. More formally

$$\begin{aligned} |f(x) - g(x)| &= \Big| \frac{1}{N(x)} \sum_{i=1}^{n} f_i(x) - \frac{1}{N(x)} \sum_{i=1}^{n} g_i(x) \Big| \\ &= \Big| \frac{1}{N(x)} \sum_{i=1}^{n} (f_i(x) - g_i(x)) \Big| \\ &\leq \frac{1}{N(x)} \sum_{i=1}^{n} |f_i(x) - g_i(x)| \\ &\leq \frac{1}{N(x)}\ N(x)\ \varepsilon = \varepsilon. \end{aligned}$$

Hence for any ε we have found a step function g uniformly within ε of f. So f is regulated. □

Exercise B.2.11 part (8) Outer measure μ^* is not in general additive. Prove however that if $U = \bigcup_{n=1}^{\infty} U_n$ is a countable union of pairwise disjoint open intervals $\{U_n\}$ and A is a bounded subset of $\mathbb{R}$ then

$$\mu^*(A \cap U) = \sum_{n=1}^{\infty} \mu^*(A \cap U_n).$$

Proof. Countable subadditivity of μ^* implies

$$\mu^*(A \cap U) \leq \sum_{n=1}^{\infty} \mu^*(A \cap U_n).$$

Given $\varepsilon > 0$ we choose a countable cover $\{W_j\}_{j=1}^{\infty}$ of $A \cap U$ by pairwise disjoint open intervals W_j such that

$$\sum_{j=1}^{\infty} \operatorname{len}(W_j) < \mu^*(A \cap U) + \varepsilon.$$

It exists by part (4) of Exercise B.2.11.

Let $V_{jn} = U_n \cap W_j$ so $\{V_{jn}\}$ is also a cover of $A \cap U$ by pairwise disjoint open intervals. Since $W_j = \bigcup_n V_{jn}$ we may conclude from part (7) of Exercise B.2.11 that

$$\sum_{n=1}^{\infty} \mu^*(V_{jn}) = \mu^*(W_j)$$

and hence that

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\mu^*(V_{jn}) = \sum_{j=1}^{\infty}\mu^*(W_j) \le \mu^*(A\cap U)+\varepsilon. \tag{D.1.1}$$

But $\{V_{jn}\}_{j=1}^{\infty}$ is a cover of $A\cap U_n$ for each fixed n and hence

$$\mu^*(A\cap U_n) \le \sum_{j=1}^{\infty}\mu^*(V_{jn}).$$

So

$$\mu^*(A\cap U) \le \sum_{n=1}^{\infty}\mu^*(A\cap U_n) \le \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\mu^*(V_{jn}) \le \mu^*(A\cap U)+\varepsilon$$

by inequality (D.1.1). Since this holds for every $\varepsilon > 0$ we conclude

$$\sum_{n=1}^{\infty}\mu^*(A\cap U_n) = \mu^*(A\cap U).$$

□

Exercise B.3.5 Let $\mathcal{S}$ denote the collection of all subsets A of $\mathbb{R}$ such that

$$\mu^*(A\cap J)+\mu^*(A^c\cap J) = \mu^*(J)$$

for every interval $J = [a,b]$ in $\mathbb{R}$.

(1) Prove that if $A \in \mathcal{S}$ and U is an open subset of $\mathbb{R}$. then

$$\mu^*(A\cap U)+\mu^*(A^c\cap U) = \mu^*(U).$$

Proof. We first observe that

$$\mu^*(A\cap U)+\mu^*(A^c\cap U) = \mu^*(U)$$

for every open interval $U = (a,b)$ in $\mathbb{R}$ as a consequence of part (1) of Exercise B.2.11 since $J = [a,b] = U \cup \{a,b\}$.

Let U be an arbitrary open subset of $\mathbb{R}$. Then by Theorem (A.6.3) $U = \bigcup_{n=1}^{\infty} U_n$, a countable union of pairwise disjoint open intervals U_n. By part (8) of Exercise B.2.11

$$\mu^*(A\cap U) = \sum_{n=1}^{\infty}\mu^*(A\cap U_n).$$

Similarly

$$\mu^*(A^c \cap U) = \sum_{n=1}^{\infty} \mu^*(A^c \cap U_n) \text{ and } \mu^*(U) = \sum_{n=1}^{\infty} \mu^*(U_n).$$

Hence

$$\begin{aligned}\mu^*(U) = \sum_{n=1}^{\infty} \mu^*(U_n) &= \sum_{n=1}^{\infty} \mu^*(A \cap U_n) + \mu^*(A^c \cap U_n) \\ &= \sum_{n=1}^{\infty} \mu^*(A \cap U_n) + \sum_{n=1}^{\infty} \mu^*(A^c \cap U_n) \\ &= \mu^*(A \cap U) + \mu^*(A^c \cap U).\end{aligned}$$

□

(2) Use (1) to prove that if $A \in \mathcal{S}$ and X is any bounded subset of $\mathbb{R}$ then

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(X).$$

Proof. Let X be an arbitrary bounded subset of $\mathbb{R}$. By the regularity of outer measure there is a sequence of open sets U_n such that $X \subset U_n$ and $\lim_{n \to \infty} \mu^*(U_n) = \mu^*(X)$. Defining $V_m = \bigcap_{n=1}^{m} U_n$ we observe that V_m is open, $V_{m+1} \subset V_m$ and $\lim_{m \to \infty} \mu^*(V_m) = \mu^*(X)$.

If $A \in \mathcal{S}$ monotonicity implies $\mu^*(A \cap V_m) \geq \mu^*(A \cap X)$ and $\mu^*(A^c \cap V_m) \geq \mu^*(A^c \cap X)$ for each $m \in \mathbb{N}$. So

$$\begin{aligned}\mu^*(X) &= \lim_{m \to \infty} \mu^*(V_m) \\ &= \lim_{m \to \infty} \mu^*(A \cap V_m) + \mu^*(A^c \cap V_m) \text{ by part (1)} \\ &\geq \mu^*(A \cap X) + \mu^*(A^c \cap X).\end{aligned}$$

The fact that $\mu^*(X) \leq \mu^*(A \cap X) + \mu^*(A^c \cap X)$ follows from subadditivity of outer measure. Hence $\mu^*(X) = \mu^*(A \cap X) + \mu^*(A^c \cap X)$. □

Exercise 4.2.6 part (5) *Egorov's theorem*: If $\{f_n : I \to \mathbb{R}\}$ is a sequence of measurable functions converging pointwise to $f : [0,1] \to \mathbb{R}$, prove that for any $\varepsilon > 0$ there is a set $A \subset I$ with $\mu(A) < \varepsilon$ such that $\{f_n\}$ converges uniformly to f on A^c. This is sometimes referred to as the third of Littlewood's three principles.

Proof. Consider the sets

$$E(n,m) = \bigcup_{k=n}^{\infty} \left\{ x \;\middle|\; |f_k(x) - f(x)| \geq \frac{1}{m} \right\}.$$

Since

$$\lim_{k \to \infty} f_k(x) = f(x)$$

for each x, it follows that $|f_k(x) - f(x)| < 1/m$ for all sufficiently large k and hence $\bigcap_{k=1}^{\infty} E(k,m) = \emptyset$.

The sets $E(k,m)$ satisfy $E(k+1,m) \subset E(k,m)$ so

$$\lim_{k \to \infty} \mu(E(k,m)) = 0.$$

Thus for each m there is n_m such that $\mu(E(n_m, m)) < \dfrac{\varepsilon}{2^m}$.

Let $A = \bigcup_{m=1}^{\infty} E(n_m, m)$. Then

$$\mu(A) = \mu\big(\bigcup_{m=1}^{\infty} E(n_m, m) \big) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon.$$

If $x \in A^c$, then $x \notin E(n_m, m)$ for every $m > 0$ so $|f_k(x) - f(x)| < 1/m$ for all $k > n_m$. This implies that f_k converges uniformly to f on A^c. $\square$