4.6. Continuous Direct Sums of Representations

The discrete direct sums of representations discussed above are sufficient for dealing with compact groups, but for noncompact groups we need the notion of direct integral. That notion extends the idea of direct sum in the way that integration extends the idea of addition.

The following result describes L^p direct integrals of Hilbert spaces and shows that the concept makes sense. There are several distinct ingredients in this recipe, and each has a special rôle. The basic ingredient is the family $\{H_y\}_{y\in Y}$ of Hilbert spaces to be summed. The next ingredient is the measure space[5] structure $(Y, \mathcal{M}, \tau)$, which provides the framework for the summation. The third and crucial ingredient is the family $\{s_\alpha\}_{\alpha\in A}$ of "vector fields" $y \mapsto s_\alpha(y) \in H_y$. First, it tells us which of the "vector fields" $y \mapsto s(y) \in H_y$ in the sum of the $\{H_y\}$ will be measurable. Second, it tells us which of those measurable fields will be L^p. Third, it gives us the global L^p norm. Fourth, in the unitary case it defines the global inner product underlying the L^2 norm. These rôles understood, the construction is straightforward. Here is the formal definition.

Definition 4.6.1. Let $(Y, \mathcal{M}, \tau)$ be a measure space. For each $y \in Y$ let H_y be a separable Hilbert space. Fix a countable family $\{s_\alpha\}_{\alpha\in A}$ of maps $Y \to \bigcup_{y\in Y} H_y$ such that

(i) $s_\alpha(y) \in H_y$ a.e. $(Y, \mathcal{M}, \tau)$, for all $\alpha \in A$,

(ii) $y \mapsto \langle s_\alpha(y), s_\beta(y)\rangle_{H_y}$ belongs to $L^1(Y, \tau)$, for all $\alpha, \beta \in A$, and

(iii) H_y is the closed span of $\{s_\alpha(y)\}_{\alpha\in A}$ a.e. (Y, τ).

Then the **(Hilbert space) direct integral** defined by the measure space $(Y, \mathcal{M}, \tau)$, the family $\{H_y \mid y \in Y\}$ of Hilbert spaces, and the family $\{s_\alpha\}_{\alpha\in A}$, is the vector space

$$\mathcal{H}^2 = \int_Y H_y \, d\tau(y) : \text{all maps } s : Y \to \bigcup_{y\in Y} H_y \text{ such that}$$

(i) $s(y) \in H_y$ a.e. (Y, τ),

(ii) $y \mapsto \langle s(y), s_\alpha(y)\rangle_{H_y}$ is measurable, for each $\alpha \in A$, and

(iii) $y \mapsto \langle s(y), s_\alpha(y)\rangle_{H_y}$ belongs to $L^1(Y, \tau)$, for all $\alpha \in A$

with inner product $\langle s, s'\rangle = \int_Y \langle s(y), s'(y)\rangle_{H_y} \, d\tau(y)$. ♢

Lemma 4.6.2. *The inner product of* Definition 4.6.1 *is well defined, and the direct integral* $\mathcal{H}^2$ *is a separable Hilbert space.*

Definition 4.6.3. Let $1 \leqq p \leqq \infty$. Fix a measure space $(Y, \mathcal{M}, \tau)$, a family $\{H_y \mid y \in Y\}$ of separable Hilbert spaces, and a family $\{s_\alpha\}_{\alpha\in A}$ as in Definition

[5] By **measure space** we mean the usual: a set Y, a σ–algebra $\mathcal{M}$ of subsets of Y, and a σ–additive function $\tau : \mathcal{M} \to \mathbb{R}^+ \cup \{\infty\}$. Then τ is the measure, the sets $M \in \mathcal{M}$ are *measurable*, and $M \in \mathcal{M}$ has measure $\tau(M)$. We require $(Y, \mathcal{M}, \tau)$ to be complete: if $M \in \mathcal{M}$ with $\tau(M) = 0$, and if $M' \subset M$, then $M' \in \mathcal{M}$ with $\tau(M') = 0$.

We give $C_c^{-\infty}(G)$ the weakest topology for which the maps $T \mapsto T(f)$, where $f \in C^\omega(G)$, are continuous. Item (5.) shows that this topology is Hausdorff. The next item is an observation of Godement [**Go**].

LEMMA 8.3.12. $\{E \in C^{-\infty}(G) \mid \mathrm{Supp}\,(E) = \{1\}\}$ *is dense in* $C_c^{-\infty}(G)$.

This comes right out of the Hahn–Banach Theorem. Suppose $f \in C^\omega(G)$ such that $E(f) = 0$ whenever $E \in C^{-\infty}(G)$ with $\mathrm{Supp}\,(E) = \{1\}$. Then $Df(1) = 0$ for all left–invariant differential operators D on G, so $f = 0$.

LEMMA 8.3.13. *Let* $S \in C^{-\infty}(G)$. *Then the operators* $T \mapsto T*S$ *and* $T \mapsto S*T$ *on* $C_c^{-\infty}(G)$ *are continuous.*

For example $(T * S)(f) = T(f * S')$ for some $S' \in C^{-\infty}(G)$, and $f * S'$ is C^ω by Lemma 8.3.10.

PROOF OF THEOREM 8.3.1 By Lemma 8.3.13, $T \mapsto \delta_K * T * \delta_K$ is continuous on $C_c^{-\infty}(G)$. By Lemma 8.3.8 its image is the subalgebra $\mathcal{A}(K\backslash G/K)$ of $C_c^{-\infty}(K\backslash G/K)$ corresponding to $\mathcal{D}(G, K)$. Thus by Lemma 8.3.12 $\mathcal{A}(K\backslash G/K)$ is dense in $C_c^{-\infty}(K\backslash G/K)$. If $\mathcal{D}(G, K)$ is commutative now $C_c^{-\infty}(K\backslash G/K)$ is commutative, and then Lemma 8.3.6 says that (G, K) is a Gelfand pair. □

The proof of Theorem 8.3.3 combines results of Gelfand [**Ge1**], Godement [**Go**] and Helgason [**H1**]. Gelfand found the differential equations for the spherical functions and Godement developed their properties and related them to work of Harish–Chandra [**Ha1**]. Helgason characterized the solutions to these differential equations by a functional equation

$$0 \neq \varphi \in C(G) \text{ and } \varphi(x)\varphi(y) = \int_K \varphi(xky)\, d\mu_K(k) \text{ for } x, y \in G, \tag{8.3.14}$$

which is that of Theorem 8.2.6 for spherical functions. See Proposition 8.3.15 just below. This characterization proves Theorem 8.3.3.

PROPOSITION 8.3.15. *Suppose that* $\mathcal{D}(G, K)$ *is commutative and* $0 \neq \varphi \in C(G/K)$. *Then* φ *satisfies* (8.3.14) *if and only if* (i) $\varphi \in C^\infty(G/K)$, (ii) $\varphi(1K) = 1$, *and* (iii) φ *is a joint eigenfunction of* $\mathcal{D}(G, K)$.

PROOF. We follow Helgason's argument [**H1**, Chapter X, Proposition 3.2]. Identify φ with its lift to G. Suppose first that φ is C^∞ and is a joint eigenfunction of $\mathcal{D}(G, K)$, and that $\varphi(1) = 1$. The manifold G/K is C^ω, as is any G–invariant riemannian metric, so the Laplace–Beltrami operator Δ for any such metric is C^ω. As $\Delta \in \mathcal{D}(G, K)$ and Δ is elliptic, now φ is C^ω by elliptic regularity.

Fix $x \in G$ and define $h(y) = \int_K \varphi(xky) d\mu_K(k)$. Then $h \in C^\omega(K\backslash G/K)$. If $D \in \mathcal{D}(G, K)$, say $D\varphi = \chi(D)\varphi$, then $(Dh)(y) = \int_K (D\varphi)(xky)\, d\mu_K(k) = \chi(D)h(y)$. Thus $[D(\varphi(1)h - h(1)\varphi)](1) = 0$. The map $\widetilde{D} \mapsto \int_K (\widetilde{D} \cdot r(k)) d\mu_K(k)$ sends the algebra $\mathcal{D}(G)$ of left–invariant differential operators on G, onto $\mathcal{D}(G, K)$. Now $[\widetilde{D}(\varphi(1)h - h(1)\varphi)](1) = 0$ for every $\widetilde{D} \in \mathcal{D}(G)$, where we have pulled the functions φ and h back to G. Thus all derivatives of $\varphi(1)h - h(1)\varphi$ vanish at 1. Since φ is C^ω, so is the function $\varphi(1)h - h(1)\varphi$, while its Taylor series expansion at 1 is identically zero. Thus $\varphi(1)h(y) = h(1)\varphi(y)$ for all $y \in G$. Since $\varphi(1) = 1$ and $h(1) = \varphi(x)$ now $\int_K \varphi(xky) d\mu_K(k) = \varphi(x)\varphi(y)$. Thus φ satisfies (8.3.14).

metric, carried to $\mathfrak{m}$. Then $\mathfrak{v}$ is the sum of the orthocomplement to $[\mathfrak{n},\mathfrak{n}]$ in $\mathfrak{n}$ (which generates $\mathfrak{n}$) and the orthocomplement to $\mathfrak{n}$ in $\mathfrak{m}$, and all these summands are $\mathrm{Ad}_G(K)$–stable.

The standard expression of the Levi–Cività connection on a homogeneous riemannian manifold shows that $t \mapsto \exp(t\xi)(x_0)$ is a geodesic if and only if $\langle [\xi,\eta]_{\mathfrak{m}}, \xi_{\mathfrak{m}}\rangle = 0$ for all $\eta \in \mathfrak{m}$. See [**Ko-V**]. Let $p : \mathfrak{m} \to [\mathfrak{n},\mathfrak{n}]$ be the projection with kernel $\mathfrak{v}$. Let $\xi \in \mathfrak{g}$ such that $t \mapsto \exp(t\xi)(x_0)$ is a geodesic, and $\xi = \xi_{\mathfrak{k}} + \xi_{\mathfrak{m}}$ according to $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, such that $\xi_{\mathfrak{m}} \in \mathfrak{v}$. Let $\zeta \in [\mathfrak{n},\mathfrak{n}]$ and compute

$$0 = \langle [\xi,\zeta]_{\mathfrak{m}}, \zeta\rangle = \langle [\xi,\zeta]_{[\mathfrak{n},\mathfrak{n}]}, \zeta\rangle = \langle [\xi_{\mathfrak{k}},\zeta],\zeta\rangle + \langle p(\mathrm{ad}(\xi_{\mathfrak{m}})(\zeta)),\zeta\rangle.$$

As $\mathrm{ad}(\xi_{\mathfrak{k}})$ is skew–symmetric on $\mathfrak{m}$ we have $\langle [\xi_{\mathfrak{k}},\zeta],\zeta\rangle = 0$, so $\langle p(\mathrm{ad}(\xi_{\mathfrak{m}})(\zeta)),\zeta\rangle = 0$. Thus $p\cdot \mathrm{ad}(\beta)|_{[\mathfrak{n},\mathfrak{n}]}$ is skew–symmetric for every $\beta \in \mathfrak{v}$. But $p\cdot\mathrm{ad}(\beta)|_{[\mathfrak{n},\mathfrak{n}]}$ is nilpotent for every $\beta \in \mathfrak{n}$, so now $p\cdot\mathrm{ad}(\mathfrak{n}\cap\mathfrak{v})|_{[\mathfrak{n},\mathfrak{n}]} = 0$. In other words $[(\mathfrak{n}\cap\mathfrak{v}), [\mathfrak{n},\mathfrak{n}]] = 0$. Since $\mathfrak{n}\cap\mathfrak{v}$ generates $\mathfrak{n}$ now $[\mathfrak{n},\mathfrak{n}]$ is central in $\mathfrak{n}$. Thus $\mathfrak{n}$ is abelian or 2–step nilpotent. □

Combining Propositions 13.1.8 and 13.1.9 we have

THEOREM 13.1.10. *Let (M, ds^2) be a connected and simply connected weakly symmetric riemannian manifold, let $G = \mathbf{I}(M, ds^2)^0$, and let N be the nilradical of G. Then N is abelian or 2–step nilpotent.*

13.2. The Case Where N is a Heisenberg Group

In this section we look at the cases where (M, ds^2) is a connected and simply connected weakly symmetric riemannian nilmanifold. Those $M = G/K$ were the first cases of nonsymmetric commutative pairs (G,K) where G is not reductive. Our treatment depends on the paper [**B-J-R1**] of Benson, Jenkins and Ratcliff.

The standard **Heisenberg group** H_n of real dimension $2n+1$ is the group $H_{n,0;\mathbb{C}}$ of Sections 2.10 and 4.10. There

$$H_{p,q;\mathbb{F}} : \text{ real vector space } \mathrm{Im}\,\mathbb{F} + \mathbb{F}^{p,q} \text{ with group composition}$$
$$(z,w)(z',w') = (z + z' + \mathrm{Im}\,h(w,w'), w + w').$$

where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, h is a hermitian form of signature (p,q) on $\mathbb{F}^{p+q}$, and Im denotes imaginary component.

The automorphism group $\mathrm{Aut}(H_n) = (\mathbb{R}^* \times Sp(n;\mathbb{R}))/\{\pm(1,1)\}$. The $\mathbb{R}^*$ factor acts by $a : (z,w) \mapsto (a^2, az)$. The $Sp(n;\mathbb{R})$ factor is the automorphism group of the antisymmetric bilinear form $\omega(v,w) := -i\ \mathrm{Im}\,h(v,w)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$, and it acts by $g : (z,w) \mapsto (z, gw)$. (Here we need the $-i$ factor in ω because $\mathrm{Im}\,h(u,v) \in i\mathbb{R}$ and we want ω to be real–valued.) The maximal compact subgroup of $\mathrm{Aut}(H_n)$ is the usual complex unitary group $U(n)$, and it acts by $k : (z,w) \mapsto (z, kw)$. This leads to the family of pairs

(13.2.1) $\qquad (G,K)$ where K is a closed subgroup of $U(n)$ and $G = H_n \rtimes K$.

THEOREM 13.2.2. (Carcano [**Ca**]) *Let K be a closed subgroup of $U(n)$ acting naturally on $\mathbb{C}^n$. Then $(H_n \rtimes K, K)$ is a Gelfand pair if and only if the representation of K on $\mathbb{C}^n$ is "multiplicity free" in the sense that any irreducible representation of K occurs at most once in the representation of K (as a subgroup of $U(n)$) on polynomials on $\mathbb{C}^n$.*

PROOF. The matrix $a = \operatorname{diag}\{a_1, a_2, \ldots, a_n\} \in T_n$ sends the monomial

$$z^m := z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$$

to $a_1^{-m_1} a_2^{-m_2} \ldots a_n^{-m_n} z^n$. Thus the representations of T_n on the various $z^n\mathbb{C}$ are inequivalent. Now Theorem 13.2.2 says that $(H_n \rtimes T_n, T_n)$ is a Gelfand pair. In other words the convolution algebra $L^1(T_n \backslash H_n \rtimes T_n / T_n)$ is commutative. Thus its subalgebra $L^1(K \backslash H_n \rtimes K/K)$ is commutative and $(H_n \rtimes K, K)$ is a Gelfand pair. □

Remark. One can prove Corollary 13.2.3 without reference to the machinery of Theorem 13.2.2 as follows. By direct computation one checks that the double cosets $U(1)xU(1)$ in $H_1 \rtimes U(1)$ commute. Let $\widetilde{H}$ denote $H_1 \times \cdots \times H_1$ (n factors) and let $\widetilde{U}$ denote $U(1) \times \cdots \times U(1)$. Now the double cosets $\widetilde{U}x\widetilde{U}$ commute in $\widetilde{H} \rtimes \widetilde{U}$. Note that H_n is a central quotient of $\widetilde{H}$ and that $\widetilde{U} = T_n$. Thus the double cosets $T_n x T_n$ commute in $H_n \rtimes T_n$, and $(H_n \rtimes T_n, T_n)$ is a Gelfand pair. The argument now goes as in the proof above of Corollary 13.2.3.

Example. Let $K = U(n_1) \times \cdots \times U(n_r)$ where $n = n_1 + \cdots + n_r$ and the n_ℓ are positive integers. Then K is a closed subgroup of $U(n)$ that contains the maximal torus T_n. Thus $(H_n \rtimes K, K)$ is a Gelfand pair.

We combine Theorem 13.2.2 with Kač' classification of connected irreducible complex linear groups that are multiplicity–free on the polynomial ring. The table is taken from [**Ka**, Theorem 3], except that we list the compact group K as well as its complexification $K_{\mathbb{C}}$.

THEOREM 13.2.4. ([**B-J-R1**, Theorem 4.6]) *Let K be a closed connected subgroup of $U(n)$ acting irreducibly on $\mathbb{C}^n$. Then the following are equivalent.*

1. *(G, K) is a Gelfand pair where G is the semidirect product group $H_n \rtimes K$.*

2. *The representation of $K_{\mathbb{C}}$ on $\mathbb{C}^n$ is "multiplicity free" in the sense that any irreducible representation of $K_{\mathbb{C}}$ occurs at most once as a summand of the corresponding representation on the ring of polynomials on $\mathbb{C}^n$.*

3. *The representation of K on $\mathbb{C}^n$ is equivalent to one of the following.*

(13.2.5)

"Multiplicity Free" Irreducible Representations of K and $K_{\mathbb{C}}$ on $\mathbb{C}^n$				
	Group K	Group $K_{\mathbb{C}}$	Acting on	Conditions on n
1	$SU(n)$	$SL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
2	$U(n)$	$GL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 1$
3	$Sp(m)$	$Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
4	$U(1) \times Sp(m)$	$\mathbb{C}^* \times Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
5	$U(1) \times SO(n)$	$\mathbb{C}^* \times SO(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
6	$U(m)$	$GL(m;\mathbb{C})$	$S^2(\mathbb{C}^m)$	$m \geqq 2,\ n = \frac{1}{2}m(m+1)$

continued on next page

annoyance by writing "multiplicity free" (with quotes to indicate that the term is not precise) when the term really refers to the corresponding representation on the polynomial ring of the representation space. Thus Table 13.2.5 gives the classification of finite dimensional irreducible "multiplicity free" representations of compact or complex connected Lie groups. ♢

Consider the case where K fails to be irreducible on $\mathbb{C}^n$. The underlying real symplectic structure of $(\mathbb{C}^n, h)$ is $(\mathbb{R}^{2n}, \omega)$ where $\omega(u, v) = -i \operatorname{Im} h(u, v)$ as above. Suppose that $\mathbb{R}^{2n} = U \oplus V$ where U and V are K–invariant real subspaces, and that (G, K) is a Gelfand pair where $G = H_n \rtimes K$. We can assume that $U \perp V$ relative to the real bilinear form $\operatorname{Re} h(u, v)$. If $u \in U$ and $v \in V$ then

$$(0, u, 1)(0, v, 1) \in K(0, v, 1)(0, u, 1)K,$$

in other words there exist $k, k' \in K$ with

$$(0, u, 1)(0, v, 1) = (0, 0, k)(0, v, 1)(0, u, 1)(0, 0, k').$$

Thus

$$\begin{aligned}(\operatorname{Im} h(u, v), u + v, 1) &= (0, kv, k)(0, u, k') \\ &= (0, kv, 1)(0, ku, kk') = (\operatorname{Im} h(kv, ku), kv + ku, kk').\end{aligned}$$

Now $\operatorname{Im} h(u, v) = \operatorname{Im} h(kv, ku) = \operatorname{Im} h(v, u) = -\operatorname{Im} h(u, v)$. We have just seen that $\operatorname{Im} h(u, v) = 0$. We already had $\operatorname{Re} h(u, v) = 0$. Now $h(U, V) = 0$, in particular $U = V^\perp$ and $V = U^\perp$ relative to h. Thus U and V are h–orthogonal complex subspaces of $\mathbb{C}^n$ and ω is nondegenerate on each of them.

Now we have a decomposition $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_\ell$ where the V_i are K–irreducible complex subspaces that are mutually orthogonal relative to the hermitian form h. Then K acts on V_i as a closed subgroup K_i of the unitary group $U(V_i)$. Let $n_i = \dim V_i$ so we have the Heisenberg group H_{n_i} and the pair (G_i, K_i) given by $G_i = H_{n_i} \rtimes K_i$. Denote $\widetilde{H_n} = H_{n_1} \times \cdots \times H_{n_\ell}$, $\widetilde{K} = K_1 \times \cdots \times K_\ell$, and $\widetilde{G} = G_1 \times \cdots \times G_\ell$. Then $\widetilde{G} = \widetilde{H_n} \rtimes \widetilde{K}$, K is a quotient $\widetilde{K}/\Xi$ where Ξ is a closed normal subgroup of K, H_n is a quotient $\widetilde{H_n}/Y$ where Y is a closed connected subgroup of codimension 1 in the center of $\widetilde{H_n}$, G is the quotient $\widetilde{G}/Y\Xi$, and $\widetilde{G}/Y = H_n \rtimes \widetilde{K}$.

If each (G_i, K_i) is a Gelfand pair then their product $(\widetilde{G}, \widetilde{K})$ is a Gelfand pair. The converse fails, as seen by the following example of Benson and Ratcliff.

Example 13.2.13. Let $V = V_1 \oplus V_2$ where $V_1 = \mathbb{C}^2$ and $V_2 = S^2(\mathbb{C}^2)$, so $V = \mathbb{C}^5$. Let $K = U(2)$ acting on V diagonally. If we view V_2 as symmetric 2×2 complex matrices then $k : (u, v) \mapsto (ku, kvk^{tr})$. Now $(G_1, K_1) = (H_2 \rtimes U(2), U(2))$ and $(G_2, K_2) = (H_3 \rtimes U(2), U(2))$ are Gelfand pairs.

The action τ_1 of K on $V_1 = \mathbb{C}^2$ has diagram $\overset{1}{\circ}\ \ \overset{1}{\times}$, so its action on $\mathbb{C}[V_1]$ is $\sum_{a \geqq 0} S^a(\tau_1)$, and $S^a(\tau_1)$ has diagram $\overset{a}{\circ}\ \ \overset{a}{\times}$. Here the circle refers to $SU(2)$ and the $\times$ refers to the circle center of $U(2)$. The action τ_2 of K on $V_2 = S^2(\mathbb{C}^2)$ is $S^2(\tau_1)$ with diagram $\overset{2}{\circ}\ \ \overset{2}{\times}$, so its action on $\mathbb{C}[V_2]$ is $\sum_{b \geqq 0} S^b(\tau_2)$.

Write $\tau_{\ell,m}$ for the irreducible representation of K with diagram $\overset{\ell}{\circ}\ \overset{m}{\times}$. Thus $S^a(\tau_1) = \tau_{a,a}$ and $S^b(\tau_2)$ is given by

$$S^{2c}(\tau_2) = \sum\nolimits_{0 \leqq i \leqq c} \tau_{4(c-i),4c} \quad \text{and} \quad S^{2c+1}(\tau_2) = \sum\nolimits_{0 \leqq i \leqq c} \tau_{4(c-i)+2,4c+2}.$$

The action of K on $\mathbb{C}[v]$ is

$$\sum\nolimits_{a,b \geqq 0} S^a(\tau_1) \otimes S^b(\tau_2).$$

Since

$$\tau_{\ell,m} \otimes \tau_{s,t} = \tau_{u,m+t} \oplus \tau_{u+2,m+t} \oplus \tau_{u+4,m+t} \oplus \ldots \oplus \tau_{\ell+s,m+t}$$
$$\text{where } u = \max\{\ell, s\} - \min\{\ell, s\}$$

we have

$$S^a(\tau_1) \otimes S^b(\tau_2) = \sum\nolimits_{0 \leqq i \leqq a,\ 0 \leqq a+2b-2i} \tau_{a+2b-2i,a+2b}.$$

In particular $S^2(\tau_1) \otimes S^1(\tau_2)$ and $S^0(\tau_1) \otimes S^2(\tau_2)$ each has $\tau_{4,4}$ as a summand. Thus the representation of K on $\mathbb{C}[v]$ fails to be multiplicity free. In other words, (G, K) is not a Gelfand pair. ♢

13.3. The Chevalley–Vinberg Decomposition

In this section we discuss the Chevalley decomposition for real linear algebraic groups, its analogs for real Lie groups, and the corresponding result for the groups that occur in Gelfand pairs. This last is due to Vinberg and it gives the original proof of a refined form of the 2–step Nilpotent Theorem. That proof is based on the notion of weakly commutative space, and it uses both Poisson geometry and invariant theory.

13.3A. Digression: Chevalley Decompositions. Let $G_{\mathbb{C}} \subset GL(n;\mathbb{C})$ be a complex linear algebraic group. Its **unipotent radical** $U_{\mathbb{C}}$ is the maximal normal subgroup consisting of unipotent linear transformations of $\mathbb{C}^n$. It is automatic that $U_{\mathbb{C}}$ is connected. If $L_{\mathbb{C}}$ is any maximal reductive linear algebraic subgroup of $G_{\mathbb{C}}$ then $G_{\mathbb{C}}$ is the semidirect product $U_{\mathbb{C}} \rtimes L_{\mathbb{C}}$; that is the **Chevalley decomposition**, and $L_{\mathbb{C}}$ is called a **reductive component** of $G_{\mathbb{C}}$. Similarly, if $G \subset GL(n;\mathbb{R})$ is a real linear algebraic group, its unipotent radical U is defined as in the complex case, and there is a Chevalley semidirect product decomposition $G = U \rtimes L$ whenever L is a maximal reductive linear algebraic subgroup of G. The same holds for real linear algebraic groups.

The corresponding result for Lie groups is considerably more complicated.

A subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is **reductive in** $\mathfrak{g}$ if the adjoint action of $\mathfrak{l}$ on $\mathfrak{g}$ is completely reducible, in other words if every $\mathrm{ad}(\mathfrak{l})$–invariant subspace of $\mathfrak{g}$ has an $\mathrm{ad}(\mathfrak{l})$–invariant complement. Thus a Lie algebra $\mathfrak{l}$ is reductive if and only if it is reductive in itself, and if $\mathfrak{l}$ is reductive in $\mathfrak{g}$ then in particular it is reductive. If $\mathfrak{l}$ is maximal among the subalgebras of $\mathfrak{g}$ reductive in $\mathfrak{g}$ then it is a **(reductive) Levi subalgebra** of $\mathfrak{g}$. A variation on (12.4.1) points out the distinction between *reductive subalgebra of* $\mathfrak{g}$ and *subalgebra that is reductive in* $\mathfrak{g}$. Let $n = m + 2r$ and

$$(13.3.1) \quad \mathfrak{g} = \left\{ \xi \in \mathfrak{gl}(n;\mathbb{R}) \;\middle|\; \xi = \left(\begin{smallmatrix} a & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{smallmatrix}\right); a \in \mathfrak{gl}(m;\mathbb{R}); u, v \in \mathbb{R}^{m \times r}; w \in \mathbb{R}^{r \times r} \right\}.$$

acting effectively on $M = G/K$ with compact isotropy, M is simply connected, and some nilpotent subgroup N of G is transitive on M. Then $G = N \rtimes K$ and N is its nilpotent radical.) We will discuss square integrable representations of N and, in the square integrable cases, describe the $(N \rtimes K, K)$–spherical representations and the $(N \rtimes K, K)$–spherical functions.

13.4A. The Irreducible Case — Classification. Theorem 13.1.1 says that N is abelian or 2–step nilpotent. We first consider the case where $(N \rtimes K, K)$ is **irreducible** in the sense that $[\mathfrak{n}, \mathfrak{n}]$ (which must be central) is the center of $\mathfrak{n}$ and K acts irreducibly on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$.

Let (G, K) be a Gelfand pair and Z_G^0 the identity component of the center of G. If Z is a closed connected $\mathrm{Ad}(K)$–invariant subgroup of Z_G^0, then $(G/Z, K/(K \cap Z))$ is a Gelfand pair and is called a **central reduction** of (G, K). The pair (G, K) is called **maximal** if it is not a nontrivial central reduction. Here is a table of all the groups K and algebras $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$, for irreducible maximal Gelfand pairs $(N \rtimes K, K)$ where N is a connected simply connected nilpotent Lie group. As toward the end of Section 2.10, here $\mathrm{Im}\,\mathbb{F}^{s\times s}$ is the space of skew hermitian $s \times s$ matrices over $\mathbb{F}$, $\mathrm{Re}\,\mathbb{F}^{s\times s}$ is the space of hermitian $s \times s$ matrices over $\mathbb{F}$; $\mathrm{Im}\,\mathbb{F}_0^{s\times s}$ and $\mathrm{Re}\,\mathbb{F}_0^{s\times s}$ are those of trace 0. Similarly $\mathrm{Skew}\,\mathbb{F}^{s\times s}$ and $\mathrm{Sym}\,\mathbb{F}^{s\times s}$ are the antisymmetric and the symmetric $s \times s$ matrices over $\mathbb{F}$. The Lie algebra structure $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ is explained just below the table.

(13.4.1)

Maximal Irreducible Nilpotent Gelfand Pairs $(N \rtimes K, K)$ ([**V1**], [**V2**])

	Group K	$\mathfrak{v}$	$\mathfrak{z}$	$U(1)$	max
1	$SO(n)$	$\mathbb{R}^n$	$\mathrm{Skew}\,\mathbb{R}^{n\times n} = \mathfrak{so}(n)$		
2	$Spin(7)$	$\mathbb{R}^8 = \mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
3	G_2	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
4	$U(1) \cdot SO(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		$n \neq 4$
5	$(U(1)\cdot)SU(n)$	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n \oplus \mathrm{Im}\,\mathbb{C}$	n odd	
6	$SU(n), n$ odd	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$		
7	$SU(n), n$ odd	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		
8	$U(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{n\times n} = \mathfrak{u}(n)$		
9	$(U(1)\cdot)Sp(n)$	$\mathbb{H}^n$	$\mathrm{Re}\,\mathbb{H}_0^{n\times n} \oplus \mathrm{Im}\,\mathbb{H}$		
10	$U(n)$	$S^2\mathbb{C}^n$	$\mathbb{R}$		
11	$(U(1)\cdot)SU(n), n \geqq 3$	$\Lambda^2\mathbb{C}^n$	$\mathbb{R}$	n even	
12	$U(1) \cdot Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$		
13	$U(1) \cdot Spin(9)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
14	$(U(1)\cdot)Spin(10)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
15	$U(1) \cdot G_2$	$\mathbb{C}^7$	$\mathbb{R}$		
16	$U(1) \cdot E_6$	$\mathbb{C}^{27}$	$\mathbb{R}$		
17	$Sp(1) \times Sp(n)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} = \mathfrak{sp}(1)$		$n \geqq 2$
18	$Sp(2) \times Sp(n)$	$\mathbb{H}^{2\times n}$	$\mathrm{Im}\,\mathbb{H}^{2\times 2} = \mathfrak{sp}(2)$		
19	$(U(1)\cdot)SU(m) \times SU(n)$ $m, n \geqq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	$\mathbb{R}$	$m = n$	
20	$(U(1)\cdot)SU(2) \times SU(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$	$n = 2$	
21	$(U(1)\cdot)Sp(2) \times SU(n)$	$\mathbb{H}^2 \otimes \mathbb{C}^n$	$\mathbb{R}$	$n \leqq 4$	$n \geqq 3$
22	$U(2) \times Sp(n)$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
23	$U(3) \times Sp(n)$	$\mathbb{C}^3 \otimes \mathbb{H}^n$	$\mathbb{R}$		$n \geqq 2$

Often one can replace K by a smaller group in such a way that (G, K) continues to be a Gelfand pair. For example, in Table 13.4.1, Item 2, where N is the octonionic

CHAPTER 15

Classification of Commutative Spaces

We now summarize the last two chapters of Yakimova's thesis [**Y3**]. There are three parts here. The first is a commutativity criterion for a pair (G, K), $G = N \rtimes L$ with $K \subset L$, as before; it leads to a reduction of the classification to the cases where G is reductive and the nilmanifold cases $L = K$. The second indicates a recipe for dropping the requirements that (G, K) be $Sp(1)$–saturated or that it be principal. The third is a discussion of just which commutative pairs are weakly symmetric.

15.1. The Classification Criterion

The basic result for classifying the Gelfand pairs (G, K) such that G is neither reductive nor nilpotent is the following. It is the first big step in combining the reductive and the nilpotent classifications. As usual, if a group F acts on a space X and $x \in X$ then F_x denotes the F–stabilizer of x.

THEOREM 15.1.1. (Yakimova [**Y3**]; or see [**Y4**]) *Let $G = N \rtimes L$ where N is a connected simply connected nilpotent Lie group and L is a reductive Lie group. Let K be a compact subgroup of L. Then (G, K) is a Gelfand pair if and only if the following conditions all hold.*

1. $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$.
2. *If $\gamma \in \mathfrak{n}^*$ then (L_γ, K_γ) is a Gelfand pair.*
3. *If $\beta \in (\mathfrak{l}/\mathfrak{k})^*$ then $(N \rtimes K_\beta, K_\beta)$ is a Gelfand pair.*

In this section we set up the criteria for a pair (G, K) to be a maximal, indecomposable, principal, $Sp(1)$–saturated Gelfand pair. These conditions will be dropped in later sections. As before, G is a connected Lie group and K is a compact subgroup such that $M = G/K$ is simply connected and G acts effectively on M. First we recall the definitions of some of the relevant terms and give the definitions of the others.

Let (G, K) be a Gelfand pair and Z_G^0 the identity component of the center of G. If Z is a closed connected $\mathrm{Ad}(K)$–invariant subgroup of Z_G^0, then $(G/Z, K/(K \cap Z))$ is a Gelfand pair and is called a **central reduction** of G/K. The pair (G, K) is called **maximal** if it is not a nontrivial central reduction.

A Gelfand pair (G, K) is **decomposable** if up to local isomorphism it is a product of Gelfand pairs. Thus (G, K) is decomposable if there is a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\dim \mathfrak{g}_i > 1$ such that (i) $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{g}_i$, (ii) the corresponding analytic subgroups G_i and K_i are closed in G and (iii) the (G_i, K_i)