We give $C_c^{-\infty}(G)$ the weakest topology for which the maps $T \mapsto T(f)$, where $f \in C^\omega(G)$, are continuous. Item (5.) shows that this topology is Hausdorff. The next item is an observation of Godement [**Go**].

Lemma 8.3.12. $\{E \in C^{-\infty}(G) \mid \mathrm{Supp}\,(E) = \{1\}\}$ *is dense in* $C_c^{-\infty}(G)$.

This comes right out of the Hahn–Banach Theorem. Suppose $f \in C^\omega(G)$ such that $E(f) = 0$ whenever $E \in C^{-\infty}(G)$ with $\mathrm{Supp}\,(E) = \{1\}$. Then $Df(1) = 0$ for all left–invariant differential operators D on G, so $f = 0$.

Lemma 8.3.13. *Let* $S \in C^{-\infty}(G)$. *Then the operators* $T \mapsto T{*}S$ *and* $T \mapsto S{*}T$ *on* $C_c^{-\infty}(G)$ *are continuous.*

For example $(T * S)(f) = T(f * S')$ for some $S' \in C^{-\infty}(G)$, and $f * S'$ is C^ω by Lemma 8.3.10.

Proof of Theorem 8.3.1 By Lemma 8.3.13, $T \mapsto \delta_K * T * \delta_K$ is continuous on $C_c^{-\infty}(G)$. By Lemma 8.3.8 its image is the subalgebra $\mathcal{A}(K\backslash G/K)$ of $C_c^{-\infty}(K\backslash G/K)$ corresponding to $\mathcal{D}(G,K)$. Thus by Lemma 8.3.12 $\mathcal{A}(K\backslash G/K)$ is dense in $C_c^{-\infty}(K\backslash G/K)$. If $\mathcal{D}(G,K)$ is commutative now $C_c^{-\infty}(K\backslash G/K)$ is commutative, and then Lemma 8.3.6 says that (G,K) is a Gelfand pair. □

The proof of Theorem 8.3.3 combines results of Gelfand [**Ge1**], Godement [**Go**] and Helgason [**H1**]. Gelfand found the differential equations for the spherical functions and Godement developed their properties and related them to work of Harish–Chandra [**Ha1**]. In [**H1**, Chapter 10, Corollary 7.4], Helgason characterized the solutions to these differential equations by a functional equation

$$\varphi(x)\varphi_0(y) = \int_K \varphi(xky)\, d\mu_K(k) \text{ where } \varphi_0(y) = \int_K \varphi(ky)\, d\mu_K(k), \tag{8.3.14}$$

based on the following result, which extends Theorem 8.3.3.

Proposition 8.3.15. *Suppose that* $\mathcal{D}(G,K)$ *is commutative. Let* $\varphi \in C(G/K)$ *with* $\varphi(1K) = 1$. *Then* φ *satisfies* (8.3.14) *if and only if* (i) $\varphi \in C^\infty(G/K)$ *and* (ii) φ *is a joint eigenfunction of* $\mathcal{D}(G,K)$.

Proof. We follow Helgason [**H1**, Chapter X, Proposition 3.2 and Corollary 7.4]. Identify φ with its lift to G. Suppose first that φ is C^∞ and is a joint eigenfunction of $\mathcal{D}(G,K)$, and that $\varphi(1) = 1$. The manifold G/K is C^ω, as is any G–invariant riemannian metric, so the Laplace–Beltrami operator Δ for any such metric is C^ω. As $\Delta \in \mathcal{D}(G,K)$ and Δ is elliptic, now φ is C^ω by elliptic regularity.

Fix $x \in G$. Define $h(y) = \int_K \varphi(xky)d\mu_K(k)$. Then $h \in C^\omega(K\backslash G/K)$. If $D \in \mathcal{D}(G,K)$, say $D\varphi = \chi(D)\varphi$, then $(Dh)(y) = \int_K (D\varphi)(xky)\, d\mu_K(k) = \chi(D)h(y)$ and $D\varphi_0 = \chi(D)\varphi_0$. Thus $[D(\varphi_0(1)h - h(1)\varphi_0)](1) = 0$. As in [**H1**, equation (3) on page 400] (see Lemma 8.3.16 below) the map $\widetilde{D} \mapsto \int_K (\widetilde{D} \cdot r(k))d\mu_K(k)$ sends the algebra $\mathcal{D}(G)$, of left–invariant differential operators on G, onto $\mathcal{D}(G,K)$. Now $[\widetilde{D}(\varphi_0(1)h - h(1)\varphi_0)](1) = 0$ for $\widetilde{D} \in \mathcal{D}(G)$, where we have pulled the functions φ_0 and h back to G. Thus all derivatives of $\varphi_0(1)h - h(1)\varphi_0$ vanish at 1. Since φ is C^ω, so is the function $\varphi_0(1)h - h(1)\varphi_0$, while its Taylor series expansion at 1 is identically zero. Thus $\varphi_0(1)h(y) = h(1)\varphi_0(y)$ for all $y \in G$. Since $\varphi_0(1) = 1$ and $h(1) = \varphi(x)$ now $\int_K \varphi(xky)d\mu_K(k) = h(y) = \varphi(x)\varphi_0(y)$. Thus φ satisfies (8.3.14).

Conversely suppose that φ satisfies (8.3.14). Choose $u \in C_c^\infty(G)$ such that $\int_G u(y)\varphi_0(y)d\mu_G(y) \neq 0$. Compute

$$\begin{aligned}\varphi(x)\int_G u(y)\varphi_0(y)d\mu_G(y) &= \int_G u(y)\left(\int_K \varphi(xky)d\mu_K(k)\right)d\mu_G(y)\\ &= \int_K\left(\int_G u(y)\varphi(xky)d\mu_G(y)\right)d\mu_K(k)\\ &= \int_K\left(\int_G u(k^{-1}x^{-1}z)\varphi(z)d\mu_G(z)\right)d\mu_K(k)\\ &= \int_G\left(\int_K u(kx^{-1}z)d\mu_K(k)\right)\varphi(z)d\mu_G(z).\end{aligned}$$

That transfers differentiation in x from φ to the C^∞ function u, proving $\varphi \in C^\infty(G)$. Thus also $\varphi_0 \in C^\infty(G)$. Now look again at (8.3.14). Fix $x \in G$ and let $D \in \mathcal{D}(G,K)$. Then $\varphi(x)(D\varphi_0)(y) = \int_K (D\varphi)(xky)d\mu_K(k)$. If $y = 1$ then

$$\varphi(x)(D\varphi)(1) = \int_K (D\varphi)(xk)d\mu_K(k) = D\varphi(x).$$

Thus φ is a joint eigenfunction of $\mathcal{D}(G,K)$. That completes the proof of the converse. Proposition 8.3.15 is proved. □

Proof of Theorem 8.3.3 According to Theorem 8.2.6, a K–bi–invariant continuous function $\varphi : G \to \mathbb{C}$ is spherical if and only if it satisfies (8.3.14). According to Proposition 8.3.15, φ satisfies (8.3.14) if and only if (i) $\varphi \in C^\infty(G)$, (ii) $\varphi(1) = 1$, and (iii) φ is a joint eigenfunction of $\mathcal{D}(G,K)$. □

Lemma 8.3.16. *Identify the space $\mathcal{D}(G)$ of left G–invariant differential operators on G with the universal enveloping algebra $\mathfrak{G}$, so $\mathcal{D}(G,K)$ is identified with the algebra of all $D|_{C^\infty(G/K)}$ as D ranges over the fixed point set of $\mathrm{Ad}_G(K)$ on $\mathfrak{G}$. Let $\pi : \mathcal{D}(G) \to \mathcal{D}(G,K)$ denote the projection $D \mapsto (\int_K (\mathrm{Ad}(k)D)d\mu_K(k))|_{C^\infty(G/K)}$. Then π is surjective, and if $D \in \mathcal{D}(G)$ and $\varphi \in C^\infty(K\backslash G/K)$ then $(D\varphi)(1) = (\pi(D)\varphi)(1)$.*

Proof. If $\Xi \in \mathfrak{G}$ then $\mathrm{Ad}_G(K)\Xi$ lies in a finite dimensional subspace of $\mathfrak{G}$, because K is compact. Thus $\mathfrak{G}$ is the algebraic direct sum of the convolutions $\psi * \mathfrak{G}$ as ψ runs over the normalized characters of irreducible representations of K. In particular π is surjective.

If $f \in C^\infty(K\backslash G/K)$, $\zeta_i \in \mathfrak{g}$ for $1 \leqq i \leqq \ell$, and $k \in K$, then

$$\begin{aligned}((\mathrm{Ad}(k)(\zeta_1 \dots \zeta_\ell))f)(1) &= \tfrac{d}{dt_1}\Big|_{t_1=0} \cdots \tfrac{d}{dt_\ell}\Big|_{t_\ell=0} f(k\exp(t_1\zeta_1)\dots\exp(t_\ell\zeta_\ell)k^{-1})\\ &= \tfrac{d}{dt_1}\Big|_{t_1=0} \cdots \tfrac{d}{dt_\ell}\Big|_{t_\ell=0} f(\exp(t_1\zeta_1)\dots\exp(t_\ell\zeta_\ell))\\ &= (\zeta_1\dots\zeta_\ell)(f)(1)\end{aligned}$$

If $D \in \mathcal{D}(G)$ now $((\mathrm{Ad}(k)D)f)(1) = (Df)(1)$. Taking the integral over K we conclude that $(\pi(D)f)(1) = (Df)(1)$. □

Proof of Theorem 8.3.4 At the end of the proof of Proposition 8.3.15 we saw that any spherical function φ satisfies $(D\varphi)(x) = (D\varphi)(1)\varphi(x)$ for every