Updates and additional information on “Seifert Fiberings”

p. 38, 2.6.4.

See the discussion in this updates (p.373) for a precise definition of lens spaces.

p. 42, Exercise 2.7.10

Last formula $q\hat{x} = h^{-1} \circ \hat{x}$ should change to $q\hat{x} = h^{-1} \bullet \hat{x}$

p.50, Proof of Lemma 3.1.11.

We do not need $'S^1$ since $\pi_1(M)$ is torsion free and so ev_*^x is either injective or trivial. So use S^1 itself instead of $'S^1$ throughout the proof.

p.51, Line 5. .... commute. (insert) [See Corollary 2.3.6.] Thus, ...

p. 52, after Corollary 3.1.17

Theorem 3.1.16 and Corollary 3.1.17 are also valid for closed aspherical cohomology manifolds over $\mathbb{Z}$. In fact, many places in the book, results for manifolds remain valid when manifold is replaced by cohomology manifold because many of the properties of manifolds are solely cohomological in nature, and consequently are also enjoyed by cohomology manifolds.

p. 53, Add to Remark 3.1.19.

In a recent preprint : arXiv:1108, 2321v1 [math.GT] Aug 11 2011, Sylvian Cappell, Shmuel Weinberger, and Min Yan show that in any dimension ≥ 6, there are closed aspherical manifolds (CAM) with fundamental groups having $\mathbb{Z}$ as centers, yet do not possess any non-trivial topological circle actions. Thus their theorem is a negative answer to the first question in 3.1.19 (2). This question was first proposed in [CR 69]. See also the Remark 11.7.5, page 249.

To construct such a manifold M, they form the mapping torus $T(h)$,where h is a self homeomorphism of N. N is a CAM with centerless fundamental group and h_* is a non-conjugation automorphism of $\pi_1(N)$ of order 2. This N is a refinement of a CAM N' constructed by Jonathon Block and Shmuel Weinberger (B-W) , On the generalized Nielsen realization problem, Comment. Math. Helv. 83 (2008) 21-33, for which the Nielsen realization problem fails. (See 11.3 especially p.217 and 11.3.13.). Now if $M = T(h)$ admits a circle action, it is easy to see that the action is homologically injective, 11.6. By 11.6.2, M would fiber equivariantly over the circle and its fiber N would admit a $\mathbb{Z}_2$ involution inducing h_* as an outer automorphism. In (B-W) it is shown that neither N', nor any ANR homology manifold N'' homotopically equivalent to N', admits an involution inducing h_* or a conjugate of h_*.

The major technical arguments are the modification of N' to N and any ANR homology manifold homotopy equivalent to it will have the desired properties to yield a contradiction to the existence of a circle action on M.

The universal covering of N is contractible but may fail to be Euclidean space. However, the universal covering of M is homeomorphic to Euclidean space. This follows from Manifolds covered by Euclidean space, Topology 14 (1975) by Ronnie Lee and Frank Raymond where it is shown that the universal covering of a CAM ,of dimension > 4, whose fundamental group has a non-trivial finitely generated normal abelian subgroup is homeomorphic to Euclidean space..

p. 53, Add to Remark 3.1.20.

Mike Davis, in his book The Geometry and Topology of Coxeter Groups Princeton University Press (2008), presents a detailed account of his and his collaborators' techniques in constructing closed aspherical manifolds that are very unlike those arising from discrete subgroups of Lie groups. This accessible and well written book covers many topics of current interest in aspherical manifold theory.

p. 54 Line 9 of the proof of Proposition 3.1.21.

The fact that $C_{G^*}(\Pi)$ is Abelian comes from the general fact p.103, Corollary 5.5.3.

p. 56 Theorem 3.2.5 (2).

Add an assumption that $\mathrm{Fix}(G, M) \neq \emptyset$. Only then the evaluation map (at a fixed point) $\theta : G \to \mathrm{Aut}(\Phi)$ is defined.

p.56, Add just before Definition 3.2.4

Theorem 3.2.5 states which conclusions of Theorem 3.2.2 still holds when the space M (which is not assumed to be a manifold) is $\mathcal{A}$-admissible. Let us strengthen p-admissible to now mean a space M for which the only periodic homeomorphisms of $\widetilde{M}$ of period a power of a prime p that commute with the covering transformations are elements of the center of $\pi_1(M)$. This is the same as saying that $\mathbb{Z}_{p^n} \times \pi_1(M)$ does not act effectively on $\widetilde{M}$, or equivalently that the p-torsion of the center of $\pi_1(M)$ injects onto the p-torsion of the centralizer of $\pi_1(M)$ in G^*. With this strengthening, the conclusions (2), (3), (4) of Theorem 3.2.2 hold for p-groups G and conclusion (1) of 3.2.2 holds for G compact and connected.

Theorem 3.3.1 (page 59) also has an analogue for the strengthened p-admissible spaces M. Replace M by a connected closed orientable ANR $\mathbb{Z}_p$-cohomology m-manifold. Let $f : M \to K(\Gamma, 1)$ be a map with Γ p-torsion free. Suppose $f^* : H^m(K(\Gamma, 1); \mathbb{Z}_p) \to H^m(M; \mathbb{Z}_p)$ is onto. Then M is strengthened p-admissible and consequently the conclusions of 3.2.2 holds for p-groups as we have just described above. Note also if $\pi_1(M)$ has no p-torsion, then we get the same conclusion without assuming Γ is p-torsion free. When $p = 2$, the fixed set may have $(\mathrm{dim} M) - 1 = m - 1$ dimension. The map $H^m(\mathbb{Z}_2 \backslash M; \mathbb{Z}_2) \to H^m(M; \mathbb{Z}_2) = \mathbb{Z}_2$ is still trivial since $H^m(\mathbb{Z}_2 \backslash M; \mathbb{Z}_2) = 0$ because the m-th local cohomology group of $\mathbb{Z}_2 \backslash M$, with coefficients in $\mathbb{Z}_2$, is trivial at each x in the image of an $m-1$ dimensional component of the fixed set and is $\mathbb{Z}_2$ otherwise. The sheaf of $\mathbb{Z}_2$-local cohomology groups over $\mathbb{Z}_2 \backslash M$ is locally constant away from the $m-1$ dimensional components of the fixed set of dimension $m-1$. Therefore the sheaf fails to have a non trivial section and hence $H^m(\mathbb{Z}_2 \backslash M; \mathbb{Z}_2) = 0$.

p. 59. Line -3 and Line -5

Replace f by f_*

p. 59. Proposition 3.2.14.

Is the converse true? That is: If N is admissible and M covers N, is M admissible?

p. 61, Lemma 3.3.7. Line 8

$\pi_1(X_H, x')$ is an epimorphism. By [**MY57**, Corollary 1] or [**Bre72**, II-6.2],

p. 67 Line -10

The second X' should read $\widetilde{X}$.

p. 67 Line -9

X' should read $\widetilde{X}$.

P 67, Theorem 3.5.2.

Line 2 below diagram, $\pi_1(T^k, x')$ should read $\pi_1(T^k, 1)$.

P 67, Add after Theorem 3.5.2.

The proof of this theorem does not require the finite generation of the center of the fundamental group of X. The splitting was proved earlier in [CR 69 section 7] for aspherical manifolds.

As an illustration where the center is not finitely generated, consider the mapping telescope T of a sequence of circles each being mapped onto the next by a 2-fold covering. The fundamental group of the telescope is the set of rationals of the form $Q' = \{m/2^n : m \text{ and } n \text{ integers}\}$. T is a $K(Q', 1)$-space. There is an obvious injective circle action on the telescope with orbit space the interval $[0, \infty]$. The action is free over $[0, 1)$, of isotropy group $\mathbb{Z}_2$ over $[1, 2)$, and isotropy $\mathbb{Z}_2^n$ over $[n-1, n)$ for each n.

The splitting action $(S^1, T_{\mathbb{Z}}Z) = (S^1, S^1 \times W)$ where W is a tree. To understand this tree W, consider the pre-telescope T_n, i.e., the telescope for the first n maps of the circle. Starting with the identity over 0, we get the pre-telescope over $[0, 1]$. This is a circle action on the Möbius band which lifts to a splitting action where W_1 is an arc with a $\mathbb{Z}_2$ action – a reflection across the middle. The next stage W_2 is homeomorphic to the capital letter H, where $\mathbb{Z}_4 = \mathbb{Z}_{2^2}$ acts freely everywhere except for the crossbar of the H where the isotropy subgroup is $\mathbb{Z}_2$ except for the middle where it is $\mathbb{Z}_4$. One then continues inductively to construct the action of Q' on the tree, $\lim W_n$, with quotient $[0, \infty)$. At each stage one constructs the covering action of $\mathbb{Z}_2^n$ on $S^1 \times W_n$ commuting with the lifted free S^1 action. One can also construct a similar telescope to be a $K(Q, 1)$-space with an injective circle action with properties similar to T.

p. 68 Corollary 3.5.3.

Insert (cf. [CR 69 , 7.2]).

p. 90 Second display (Line 15)

first line of this display: Replace $\mathbb{Z}_2 \times \mathbb{Z}_2$ with H.

p. 91 Line 3 of third paragraph

Replace “the subgroup” by “a subgroup”

p. 103 Line 2–3

Replace “$g(E)$ is a subgroup” by “of $t(E)$”.

p. 105 Line 6

Replace “(β)” by “$\eta(\beta)$”

p. 147, Example 8.3.13 Line 4

change $G \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ to $G \rtimes (\mathbb{Z}_4 \rtimes \mathbb{Z}_2)$.

p. 156 Line 18

Replace "N" by "$\mathcal{Z}(N)$".

p. 169 Line -5

Replace "Q" by "Q_0".

p. 170 Line -6

$\mathrm{Aut}(\Gamma)$ should read $\mathrm{Out}(\Gamma)$.

p. 177 Line -3 and -1

Replace "$(Q\backslash W)$" by "$(Q_0\backslash W)$".

p. 196 Line -4 (in the short exact sequence)

Replace "$\mathcal{T}^k$" by "T^k".

p. 197 Line 6 of Proof 6 (i.e., Line 21)

Replace "over P" by "P over W".

p. 198 Line 2

"bbrk" should read $\mathbb{R}^k$.

p. 200 Line 4 of Proposition 10.4.9

into an $E[P,Q]$, which is not split. That is, no $\theta : E \to E[P,Q]$, which is not split, exists. Moreover, there are, over each ...

p. 201 Line 7

Replace "$\mathfrak{h}^q_{\mathbb{Z}}$" by "$\mathfrak{h}^0_{\mathbb{Z}}$"

p. 201 Line 1 of 10.5

$E_{p,q}$ should read $E^{p,q}$.

p. 203 Line -3

Replace in the last term, "$Q\backslash W$" by "Q"

p. 220. 11.3.13.

The CAM N of Block and Weinberger as discussed in the addenda to 3.1.19 is an admissible CAM manifold which has an admissible extension which can not be realized by an action.

p.225–226, 11.3.29

(Nothing wrong, more explanation) The embedding $\mathbb{Z}^k \xrightarrow{n} \mathbb{Z}^k$ induced by multiplication by n, induces the homomorphism $H^3_\psi(F;\mathbb{Z}^k) \xrightarrow{n} H^3_\psi(F;\mathbb{Z}^k)$ also given by multiplication by n. If we rewrite $H^3_\psi(F;\frac{1}{n}\mathbb{Z}^k)$ as $H^3_\psi(F;\mathbb{Z}^k)$, then α_* is multiplication by n. Then d^i, in the argument on page 226, is the Bocktein differential $d^i : H^i(F;T^k) \longrightarrow H^{i+1}(F;(\mathbb{Z}_n)^k)$ induced from the exact sequence $0 \to (\mathbb{Z}_n)^k \to T^k \to T^k/(\mathbb{Z}_n)^k \cong T^k \to 0$.

p. 226 Line 13 (in the second diagram)

Replace "$(\alpha\psi)$" by "$o(\psi)$"

p. 226 Line -2 from Theorem 11.3.30
Replace "$H^3(G; \mathbb{Z}^3)$" by "$H^3(G; \mathbb{Z}^k)$"

p. 237 Line 3
Replace "X" by "M"

p.237 (after Theorem 11.6.2 (Splitting Theorem) [CR-71])

Remark. The Halperin-Carlsson Torus Conjecture states that if there exists an almost free torus action T^k on an n-dimensional space X, then

$$2^k \leq \sum_{i=0}^{n} \dim_{\mathbb{Q}} H_j(X; \mathbb{Q}).$$

Recently, Y. Kamishima and M. Nakayama [KN] showed the conjecture holds for homologically injective torus actions. This easily follows from the Splitting Theorem [CR71] as formulated in 11.6.2. For, from the splitting $(T^k, T^k \times_\Phi N)$ of the (T^k, X) action, we have the commutative diagram, where N can be chosen to be path-connected:

$$\begin{array}{ccccc}
T^k & \xleftarrow[p_1]{} & T^k \times N & \xrightarrow{p_2} & N \\
\nu_1 \big\downarrow /\Phi & & \nu \big\downarrow /\Phi\backslash & & \nu_2 \big\downarrow /\Phi\backslash \\
T^k/\Phi & \xleftarrow[\overline{p}_1]{N} & T^k \times N & \xrightarrow[\overline{p}_2]{T^k\backslash} & N
\end{array}$$

We see, using the Künneth theorem, that

$$H_i(T^k; \mathbb{Z}) \otimes H_0(N; \mathbb{Z}) \xrightarrow[{p_1}_*]{} H_i(T^k; \mathbb{Z}) \xrightarrow[{\nu_1}_*]{} H_i(T^k/\Phi; \mathbb{Z})$$

is injective which implies $H_i(T^k; \mathbb{Z}) \otimes H_0(N; \mathbb{Z}) \xrightarrow[{p_1}_*]{} H_i(X; \mathbb{Z})$ is also injective for each i. Since $2^k = \sum_{i=1}^{k} \mathrm{rank} H_i(T^k; \mathbb{Z}) \leq \sum_{i=1}^{k} \mathrm{rank} H_i(X; \mathbb{Z})$, the result follows.

As corollaries, we see that the conjecture holds for all closed flat manifolds, all closed non-positively curved Riemannian manifolds, and almost free torus actions on homologically Kählerian manifolds because these torus actions are all homologically injective as stated in 11.6.9.

In [Pu], Volker Puppe shows that the conjecture holds for $k \leq 3$, and for any $k \geq 3$ one has $\sum_{i=1}^{\infty} \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}) \geq 2(k+1)$.

For $k \leq 2$, the conjecture holds by an elementary spectral sequence argument.

Reference.

[KN] Y. Kamishima, M. Nakayama, *Torus Actions and the Halperin-Carlsson Conjecture*, Jun 22 2012 math.GT arXiv:1206.4790v1.
[Pu] Volker Puppe, *Multiplicative aspects of the Halperin-Carlsson Conjecture*, Georgian Math J. 16 (2009), no.2 369–379.

p. 247 11.16.14 Replace Lines 1–3 of second paragraph

The elements $a \in H^1(H\backslash Y; \mathfrak{G})$ with $\mathfrak{G}$ the sheaf of germs of continuous functions of $H\backslash Y$ into G, represent the principal G-bundles (G, X) over $G\backslash X$. The elements $b \in H^1(H\backslash Y; \mathfrak{H})$ with $\mathfrak{H}$ the sheaf of germs of continuous functions of $H\backslash Y$ into H, represent the principal H-bundles over $H\backslash Y$. Since a is represented

p. 247 11.6.15 Line 3
$\mathfrak{H}$ should read H.

p. 247 11.6.15 Line 6
"The set" should read "The cardinality of this set"

p. 248 Line −11
"$\pi_1(M, p(b))$" should read "$\pi_1(M_H, b)$".

p. 249 11.7.5.
See the addenda to 3.1.19 above.

p. 253 Line −12
Delete "is".

p. 253 Line −12 and Line −9 (in the diagram)
"GL$(n, \mathbb{R})$" should read"GL$(s, \mathbb{R})$".

p. 258 Line −12
After the end of 6, add

Let $h : S^1 \times \mathfrak{G}_6^* \to S^1 \times \mathfrak{G}_6$ be a homeomorphism and $\psi^* : \mathbb{Z}_n \to S^1 \times \mathfrak{G}_6^*$, $\psi : \mathbb{Z}_n \to S^1 \times \mathfrak{G}_6$ be embeddings of $\mathbb{Z}_n$ into their respective first S^1 factors. As translational actions, the free $\mathbb{Z}_n$ actions are locally smooth by default. Then the action $h(\psi^*(\mathbb{Z}_n))$ is conjugate in $S^1 \times \mathfrak{G}_6$ to the action of $\psi(\mathbb{Z}_n)$ by an element of $\text{Aff}(S^1 \times \mathfrak{G}_6)$. This conjugation can never be extended to the action of $h(\psi^*(S^1))$.

p. 319 Line 8
the first "m" should read "$-m$"

p. 322 14.8.1, Line 5
change $(\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash\text{SO}(3)/\text{SO}(3)$ to $(\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash\text{SO}(3)/\text{SO}(2)$.

p. 322 14.8.2, Line 2
$g\Gamma$ should read Γg.

p. 326 Line −14
Replace "gcd" by "lcm"

p. 326 Line −6
Replace "G" by "$G = G_1$"

p. 326 last line
G should read G_r

p. 327 Line 7
$p_1^{-1}(\Gamma_1)/G_\infty$ should read $p_1^{-1}(\Gamma_1)\backslash G_\infty$

p. 336 Line 2 and 3
α_j should read a_j.

p. 336 Line 6

"s_i" should read "s_j"

p. 337 Line −1

$$-e(\Sigma(2,3,9)) = \frac{54}{18^2} = \frac{1}{6} = \frac{3}{2} - \frac{4}{3}.$$

p. 338 (ii) Add

We note that M is not homogeneous.

p. 350 Line 1 below diagram

Clearly, $\overline{\nu}^{-1}$ restricted to $D^2 - \partial D^2 =$

p. 350 Line 3 below diagram

M should read M'.

p. 360 Line −7

Replace "$(z_1, \xi^{\overline{x}_i}, rz_2\xi)$" by "$(z_1\xi^{\overline{x}_i}, rz_2\xi)$"

p. 365 Line 5

Replace "alpha" by "α"

p. 366, 15.3.1 second paragraph

Line −4 change $L(r,1)$ to $L(-r,1)$
Line −3 change "positive" to "negative"

(See the update for page 273.)

p. 367, 15.3.2(b) last line should be

space $-L(2m, 2m-1) = L(2m, 1)$

p. 368, second line above (e)

change $L(\beta,1)$ to $L(-\beta,1)$

p. 368, two lines above Exercise 15.3.3.
course by 15.3.1, $L(p,q)$ has some Seifert fibering of the form $\langle (1,0), (m,\beta), (m, r-\beta)\rangle$, where $mr = p$. Note...

p. 369, 15.3.4. In 4th line below the diagram

change $Q\backslash P_{2r}$ to $Q\backslash P_2$

p. 371 Line −16

Replace second "$P_{2\cdot 3^n\cdot k}$" by "$P_{2\cdot 3^n}$"

p. 373, Line 16

change $''e_2(\gamma) = -1$ to $''e_2(\gamma) = 1$

p. 373. *Lens spaces.*

Lens spaces are the quotients of free linear actions by finite abelian groups on spheres. The 3-dimensional lens spaces $L(p,q)$ are defined on p. 373. The lens space $L(p,q)$ admits an S^1-action with a circle of fixed points and one singular E-orbit with slice invariant (μ,ν) where $\mu = p$ and $\nu q \equiv 1 \bmod p$. This is described in 14.5.1 on p. 312. The lens spaces $L(p,q)$ are oriented according to our conventional orientation. For example, $L(p,1)$ is the principal S^1-bundle over S^2 with euler class $-p$. These lens spaces are classified up to orientation preserving homeomorphisms by

$$L(p,q) \approx L(-p,-q) \approx L(p,q') \approx -L(p,-q) \approx -L(p,p-q)$$

where $qq' \equiv 1 \bmod p$ or $q \equiv q' \bmod p$, and $\approx$ denotes orientation preserving homeomorphism.

Using Orlik's formula (as corrected) on p. 368, but with invariants in Chapter 15, we actually obtain, as to be expected, the oriented lens space $L(-p,q) \approx -L(p,q)$. On pp. 366–368, several lens spaces have the wrong sign if we insist on orientation preserving homeomorphisms. In the updates for pages 368-368 we have listed which signs must be changed to give the correct orientations.

p. 374 15.4.1 (2)

$gT_2 = \cdots$ $e(gT_1) = 0$ should read $e(gT_2) = 0$

$gT_3 = \cdots$ $e(gT_1) = 0$ should read $e(gT_3) = 0$

p. 376 Line 5

Lemma 10.1.7 should read Lemma 10.1.8

p.376 Add before 15.6.

Remarks to 15.5.3 and 15.4.2 and cf. 15.7

We wish to determine the groups Π, and $E(P,G)$ which embed in $\mathrm{TOP}_{S^1}(P)$, where $G = Q'/Q$ is the simple group of order 168 which acts on the surface Σ_3 of genus 3, and P is any principal S^1-bundle over Σ_3. That is, we wish to find the injections θ in the diagram 10.1.1 on page 181, where S^1 is T^k, F is a finite cyclic group, and the role of Q is our group G.

Referring to the notation of 15.5.2, $(S^1, M_{\gamma'}) = \{g = 0; (1,0), (2,-1), (3,1), (7,1)\}$ is the unit tangent bundle of the orbifold $Q'\backslash\mathbb{R}^2$, (14.8.3). M_γ is the pullback Seifert fibering induced by the orbifold map $Q\backslash\mathbb{R}^2 \to Q'\backslash\mathbb{R}^2$. The induced Seifert fibering will be the unit tangent bundle of the surface, Σ_3 of genus 3, whose euler class is $2 - 2g = -4 = -\frac{1}{42} \cdot 168$. $\mathbb{Z} = H^2(Q';\mathbb{Z})$ and γ' is the generator, $M_{m\gamma'} = \{g = 0; (1,0), (2,-m), (3,m), (7,m)\}$ has $e(M_{m\gamma'}) = -\frac{1}{42}m = me(M_{\gamma'})$.

The principal circle action on M_γ commutes with the free $G = Q'/Q$ action on the unit tangent bundle of Σ_3 which is induced by the differential of the G action on Σ_3. By dividing by the free central $\mathbb{Z}_m \subset S^1$-action on M_γ, we obtain an induced G-action on the principal bundle $M_{m\gamma}$ whose euler class is $-4m$.

If m is relatively prime to 42, the G-action on $M_{m\gamma}$ is free, and if not relatively prime to 42, the G-action will be branched over $|M_{m\gamma'}|$, where $|X|$ is the topological space underlying the orbifold X, (see line 11 p.358). In the latter case, the reduced Seifert invariants of the induced S^1-action on $|M_{m\gamma'}|$ are determined by 15.2.15. If $\overline{\gamma}'$ is the other generator of $H^2(Q';\mathbb{Z})$, then $e(M_{m\overline{\gamma}'}) = -e(M_{m\gamma'})$. Therefore, it suffices to consider only non-positive euler classes.

Let P_n denote the principal S^1-bundle over Σ_3 with euler class $-n$. Then P_1 is a $\mathbb{Z}_n$-central covering of P_n and M_γ is P_4. The principal S^1-action on P_1 descends to the principal $S^1/\mathbb{Z}_n$-action on P_n since P_n is just obtained by dividing by $\mathbb{Z}_n \subset S^1$. Thus, the lift of the S^1-action on $M_{\gamma'}$ to $M_\gamma = P_4$ is the $S^1/\mathbb{Z}_4$ fiber action on P_4. The evaluation homomorphism $\mathrm{ev}_*^x : \pi_1(S^1/\mathbb{Z}_4) \to \pi_1(P_4)$ is an isomorphism on to the center of $\pi_1(P_4)$, by 11.7.3. Therefore, this circle action cannot be lifted to P_1. However, the 4-fold central covering group S^1 of $S^1/\mathbb{Z}_4$ does *lift* to the principal S^1-action on P_1. For clarity, we will write S^1 for the principal S^1-action on P_1 and $S^1/\mathbb{Z}_n$, the descent of this S^1-action to the principal action on P_n. Since the G-action on Σ_3 lifts to an action of G as bundle automorphisms on P_4, there is a $\mathbb{Z}_4$-centrally extended action of G, call it G^* as bundle automorphisms on P_1. We will show that this G^*-action is not split over G and this G-action on Σ_3 is split over each $E(P_{4n}, G)$ but not split on $E(P_{4n}, G)$ with $n \not\equiv 0 \bmod 4$.

As in 15.4.2, examine the terms of low degrees of the $'E$ and $''E$ spectral sequences. We have

$$0 \to H^1(G; M(\Sigma_3, S^1)) \xrightarrow{e_1} H^1(G; \mathfrak{S}^1) \xrightarrow{e_2} H^0(G; H^1(\Sigma_3; \mathfrak{S}^1)) \xrightarrow{d} H^2(G; M(\Sigma_3, S^1)) \xrightarrow{e_3} H^2(G; \mathfrak{S}^1).$$

This becomes, using the technique of 15.4.2,

$$0 \to H^2(G;\mathbb{Z}) \xrightarrow{e_1} H^2(G;\mathfrak{Z}) \xrightarrow{e_2} H^2(\Sigma_3;\mathbb{Z})^G \xrightarrow{d} H^3(G;\mathbb{Z}) \xrightarrow{e_3} H^3(G;\mathfrak{Z}).$$

(It is important to keep in mind the interpretation of these exact sequences especially 10.3.13).

The group $H^i(G, \mathbb{Z}) = 0$, for $i = 1$ and 2, because G is perfect and $H^3(G; \mathcal{Z}) = 0$ from the horizontal exact sequence on page 204. $H^2(G; \mathcal{Z})$ is identical with $H^2(Q'; \mathbb{Z}) \cong \mathbb{Z}$ by [CR 72b, 3.10] as indicated in 15.4.2. Clearly, $H^2(\Sigma_3; \mathbb{Z})^G$ is also $\mathbb{Z}$ and is generated by the euler class -1 of P_1. Therefore, e_2 is an injection and d is the homomorphism onto the quotient group. $H^2(G; \mathcal{Z})$ corresponds to the principal G-bundles P over Σ_3 for which $E(P, G)$ splits. Therefore P_4 is mapped by e_2 to its euler class -4 in $H^2(\Sigma_3; \mathbb{Z})^G$. But the generator of $H^2(G; \mathcal{Z}) = H^2(Q'; \mathbb{Z}) = H^2(Q'; \mathcal{Z})$ corresponds to $M_{\gamma'} = G\backslash P_4$, and so e_2 is exactly multiplication by 4. This means that the image of d is $\mathbb{Z}_4$.

If a is the generator of $H^2(\Sigma_3; \mathbb{Z})^G$, corresponding to P_1, $d(a)$ is the generator of $\mathbb{Z}_4$. Thus, the central extension $0 \to M(\Sigma_3, S^1) \to E(P_1, G) \to G \to 1$, represented by $d(a)$ is not split. Hence the extension $0 \to \mathbb{Z}_4 \to G^* \to G \to 1$ is also not split. Note that $d(na) \neq 0$ if $n \neq 0 \bmod 4$ and $E(P_{4n}, G)$ is always split.

We observe that $H^i(G; H^1(\Sigma_3; \mathbb{Z})) = 0$ for all $i \geq 0$. Therefore, $H^i(G; M_0(\Sigma_3, S^1)) \to H^i(G; M(\Sigma_3, S^1))$ is an isomorphism for all $i \geq 0$. Consequently, we have analyzed the terms and verified the claims about the two exact sequences.

Using the universal coefficient theorem for cohomology, we have $H^2(G; \mathbb{Z}_s) = H^3(G; \mathbb{Z}) * \mathbb{Z}_s = \mathbb{Z}_4 * \mathbb{Z}_s = \mathbb{Z}_{\gcd(4,s)}$, a subgroup of $\mathbb{Z}_4$ ("$*$" is the *torsion product* in the universal coefficient theorem: $H^2(G; \mathbb{Z}_s) = H^2(G; \mathbb{Z}) \otimes \mathbb{Z}_s \oplus H^3(G; \mathbb{Z}) * \mathbb{Z}_s$). This group is $\mathbb{Z}_4$ if and only if s is a multiple of 4.

Consider the embedding problem addressed in 10.1.1, page 181 for the principal S^1-bundle P_1 over Σ_3 with euler class -1. We have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_s & \longrightarrow & \mathit{\Pi} & \longrightarrow & G & \longrightarrow & 1,\ a \in H^2(G;\mathbb{Z}_s) \cong \mathbb{Z}_s * \mathbb{Z}_4 \\
 & & \big\downarrow i & & \big\downarrow \bar{i} & & \big\downarrow = & & \big\downarrow i \\
0 & \longrightarrow & S^1 & \longrightarrow & S^1 \cdot \mathit{\Pi} = E & \longrightarrow & G & \longrightarrow & 1,\quad i_* a \in H^2(G;S^1) \\
 & & \big\downarrow \ell & & \big\downarrow \theta & & \big\downarrow = & & \big\downarrow \\
0 & \longrightarrow & \mathrm{M}(\Sigma_3, S^1) & \longrightarrow & E(P_1, Q) & \longrightarrow & G & \longrightarrow & 1,\ (\ell \circ i)_* a \in H^2(G;\mathrm{M}(\Sigma_3,S^1)) \cong \mathbb{Z}_4 \\
 & & \big\downarrow = & & \big\downarrow \cap & & \big\downarrow \rho & & \\
0 & \longrightarrow & \mathrm{M}(\Sigma_3, S^1) & \longrightarrow & \mathrm{TOP}_{S^1}(P_1) & \longrightarrow & \mathrm{TOP}(\Sigma_3) & &
\end{array}$$

where ℓ sends S^1 to the constant map $\ell(t)$. Here the groups are viewed as cohomology classes. That is, a, i_*a and $(\ell \circ i)_*a$ are the cohomology classes representing the group extensions on the horizontal lines.

We saw that the cohomology class representing the third horizontal exact sequence is the generator of $\mathbb{Z}_4$ coming from the d image of the generator of $H^2(\Sigma_3;\mathbb{Z})^G$, the euler class of P_1. This means that θ is injective if and only if s is a multiple of 4. In fact, $\mathit{\Pi}$ is G^* when $s = 4$.

We may factor $\mathbb{Z}_{4n} = \mathbb{Z}_s$ to describe $E(P_n, G)$, cf. 15.7. The group $S^1 \cdot G^*$ can be identified with $S^1 \times_{\mathbb{Z}_4} G^*$ which is injected by θ into $E(P_1, G)$, where $\mathit{\Pi} = G^*$ and $s = 4$. Clearly $G^* \backslash P_1$ is homeomorphic to $M_{\gamma'}$. The central subgroup $\mathbb{Z}_{4n} \subset S^1$ can be factored by dividing by $\mathbb{Z}_4$ and then by $\mathbb{Z}_n$ or alternatively by dividing first by $\mathbb{Z}_n$ and then by the quotient $\mathbb{Z}_4$. We have the commutative diagram

$$\begin{array}{ccc}
(\mathit{\Pi}, P_1) & \xrightarrow{\mathbb{Z}_n \backslash} & (\mathit{\Pi}/\mathbb{Z}_n, P_n) \\
\big\downarrow \mathbb{Z}_4 \backslash & & \big\downarrow \mathbb{Z}_4 \backslash \\
\mathit{\Pi}/\mathbb{Z}_4 \backslash P_4 & \xrightarrow{\mathbb{Z}_n \backslash} & (G, P_{4n})
\end{array}$$

It is now clear that $\mathit{\Pi} \backslash P_1$, $\mathit{\Pi}/\mathbb{Z}_n \backslash P_n$, $G \backslash P_{4n}$ are all homeomorphic to $|M_{n\gamma'}|$.

This has been a complete analysis in the spirit of 15.7. As an exercise, do a similar analysis for the Fuchsian group Q' whose signature is $(o; 2, 3, 8)$. The group Q' has a torsion free normal subgroup Q of index 48.

Reference. An important paper was inadvertently left out in the reference.

G. Hamrick and D. Royster, *Flat Riemannian manifolds are boundaries*, Invent. Math. 66(1982), 405–413.

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"admissible group, 217" should read "admissible extension, 217"